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PARTITIONS, I

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The theorems included in this paper are due esssentially to Sylvester and Cayley (the classical theory of "waves"). During the last 100 years, many other proofs of these theorems or their special cases have been published, but almost all of them make use of analytical means. Already Sylvester desired to eliminate analytical means from the Partitions theory. "During the years 1882–84, Sylvester and his pupils at John Hopkins University, published many papers on partitions, in particular on their graphical representation, with the aim, to derive the chief theorems constructively, without the oid of analysis" (Dickons, History of the theory of numbers, Vol II, Preface, page VII).

In 1920, Skolem wrote (Lehrbuch der Kombinatorik, E. Netto, zweite Auflage, Seite 316–320): "So schön die Eulerischen und Sylvesterischen Resultate auch sind, kann man doch, oft mit gutem Erfolg, in ganz elementarer Weise vorgehen". He examined $P(a, b, c, \ldots)\,\mathscr{C}$, the number of partitions of $\mathscr{C}$ into relatively prime elements $a, b, c, \ldots$. In the present paper, I study, by quite elementary methods, the case when $a, b, c, \ldots = 1, 2, \ldots, n$ are quite arbitrary $(P(1, 2, 3, \ldots, n)(c - n) = V_{n,y}(c))$.

If $c > 0$, $n > 0$ are integers, let $A_n(c)$ be the number of solutions of

$$c = x_1 + x_2 + \ldots + x_n\,, \tag{1}$$

x_j integers, $1 \leqq x_1 \leqq x_2 \leqq \ldots \leqq x_n$. E. g. $A_2(c) = [\frac{1}{2}c]$.

Notation. b_n = least common multiple of $1, 2, 3, \ldots, n$; $d_n = (b_{n-1}, n)$ (greatest common divisor). $\lambda_n = \lambda_n(c)$, $v_n = v_n(c)$ are defined as follows:

$$c \equiv \lambda_n \pmod{n}\,, \quad 0 \leqq \lambda_n < n\,; \quad c \equiv v_n \pmod{b_n}\,, \quad 0 \leqq v_n < b_n\,. \tag{2}$$

The main results of this note are contained in the following theorems:

Theorem 1. *For every integer $n > 0$ and for every integer y there is a polynomial*

$$V_{n,y}(x) = A^{(n)}_{1,y}x^{n-1} + A^{(n)}_{2,y}x^{n-2} + \ldots + A^{(n)}_{n,y} \quad (A^{(n)}_{1,y} > 0)\,, \tag{3}$$

so that for all positive $c \equiv y \pmod{b_n}$ *we have*

$$A_n(c) = V_{n,y}(c)\,. \tag{4}$$

Thus, n being given, $V_{n,y}$ depends only on the residue class of y modulo b_n. If a coefficient $A_{k,y}^{(n)}$ is independent of y, we say that this coefficient is "stable" and we denote it with $A_k^{(n)}$.

Theorem 2. *The first* $[\frac{1}{2}(n+1)]$ *coefficient of* $V_{n,y}(x)$ *are stable. If* $n > 1$, *then for* $h = [\frac{1}{2}(n+3)]$ *the coefficient* $A_{h,y}^{(n)}$ *dependens only on the residue class of* c (i.e. of y) *modulo* 2 *and we have* $A_{h,0}^{(n)} - A_{h,1}^{(n)} \neq 0$. (*In addition, we shall indicate explicitly the value of this difference.*)

In 1948, I proved the Theorem 1 and communicated this result in the sessions of the Polish Mathematical Society. A little later I proved that part of Theorem 2 which affirms that the first $[\frac{1}{2}(n+1)]$ coefficient are stable. The result of the present note, with the exception of $A_{h,0}^{(n)} \neq A_{h,1}^{(n)}$, are also an immediate consequence of the results of E . M. WRIGHT [1]. Wright says that his results are essentially due to J. J. Sylvester [2] who gave also a method of calculating the $A_h^{(n)}$ which is practicalbe for small h. Wright indicates an other method of calculating these coefficients. For more detailed references to GLAISHER, RIEGER and GUPTA, GWYTHER and MILLES see Wright [1].

In what concerns the methods employed, Silvester, Glaisher and Wright use generaling functions; on the contrary, I use only the simplest properties of congruences.

Lemma 1. *For* $n > 1$ *we have*

$$A_n(c) = \sum_{i=1}^{g} A_{n-1}(\lambda_n - 1 + in)\,, \tag{5}$$

where $g = g_n = [c/n] = (c - \lambda_n)/n$ (see (2)).[1])

Proof. Let $P_n^{(a)}(c)$ be the number of solutions of (1), for which $x_1 = a$. To every representation $c = a + (a + z_1) + \ldots + (a + z_{n-1})$ $(z_j \geqq 0)$ corresponds the representation

$$c - n(a-1) - 1 = (1 + z_1) + \ldots + (1 + z_{n-1})$$

and so we have

$$A_n(c) = \sum_{a=1}^{g} P_n^{(a)}(c) = A_{n-1}(c-1) + A_{n-1}(c-1-n) +$$
$$+ A_{n-1}(c-1-2n) + \ldots + A_{n-1}\left(c - 1 - \left(\frac{c-\lambda_n}{n} - 1\right)n\right),$$

which is (5).

[1]) We have $g \gtrless 0$. The empty sum $\sum_{1}^{0}$ signifies 0.

Corrolary. *For* $c \geqq n$ *we have* $A_n(c) = A_{n-1}(\lambda_n - 1 + gn) + \sum_{i=1}^{g-1} A_{n-1}(\lambda_n - 1 + in)$, i.e.

$$A_n(c) = A_{n-1}(c - 1) + A_n(c - n)\,. \tag{6}$$

Remark. If we put $A_n(c) = 0$ for integers $c \leqq 0$ then (6) is true for all integers c.

Lemma 2. *Let* A, B *be complex numbers,* $B \neq 0$, ν *an integer,* $\nu \geqq 0$. *Put*

$$S_\nu(l) = \sum_{j=0}^{l} (A + Bj)^\nu \tag{7}$$

for $l = -1, 0, 1, 2, 3, \ldots$ *(The empty sum* $S_\nu(-1)$ *signifier* 0, *and we put* $0^0 = 1$.) *Then*

$$S_\nu(l) = a_0(A + Bl)^{\nu+1} + a_1(A + Bl)^\nu + \ldots + a_{\nu+1}\,, \tag{8}$$

where the coefficient a_k *depend only of* A, B, ν. *We have*

$$a_0 = \frac{1}{B(\nu + 1)}\,, \quad a_1 = \frac{1}{2} \quad (\textit{for } \nu > 0)\,. \tag{9}$$

Proof. We have $S_0(l) = l + 1$; this is true also for $l = -1$. Further (binomial formula)

$$(A + B(j - 1))^{\nu+1} = \sum_{t=0}^{\nu} \binom{\nu + 1}{t} (A + Bj)^t (-B)^{\nu+1-t} + (A + Bj)^{\nu+1}\,.$$

Summing over $0 \leqq j \leqq l$ $(l \geqq 0)$ we get

$$(A - B)^{\nu+1} = \sum_{t=0}^{\nu} \binom{\nu + 1}{t} (-B)^{\nu+1-t} S_t(l) + (A + Bl)^{\nu+1}$$

and this is obviously true also for $l = -1$. The induction is now obvious. (An other proof, with explicit a_K's, follows from the Euler-Maclaurin formula.)

Proof of Theorem 1. The truth of Theorem 1 is obvious for $n = 1, 2$. Let us suppose that the assertion of Theorem 1 is true with $n - 1$ instead of n. In (5) we have

$$A_{n-1}(\lambda_n - 1 + in) = V_{n-1,y_i}(\lambda_n - 1 + in)\,,$$

where

$$y_i \equiv \lambda_n - 1 + in \pmod{b_{n-1}}\,. \tag{10}$$

Obviously $nb_{n-1} = b_n d_n$. We have $y_i \equiv y_j \pmod{b_{n-1}}$ if and only if $ni \equiv nj \pmod{b_{n-1}}$, i.e. $i \equiv j \pmod{b_{n-1}/d_n}$. Now (5) has the form

$$A_n(c) = \sum_{i=1}^{q} V_{n-1,y_i}(\lambda_n - 1 + in)\,. \tag{11}$$

The condition $i \leqq g = [c/n] = (c - \lambda_n)/n$ can be written $\lambda_n - 1 + in \leqq \lambda_n - 1 + c - \lambda_n = c - 1$.

Writing $i = j + kb_{n-1}/d_n$ $(k \geqq 0,\ 1 \leqq j \leqq b_{n-1}/d_n)$ we have $\lambda_n - 1 + in = \lambda_n - 1 + jn + kb_n$, $y_i \equiv y_j \equiv \lambda_n - 1 + jn \pmod{b_{n-1}}$ and (11) gives

$$A_n(c) = \sum_{j=1}^{b_{n-1}/d_n} \sum_k^{(j)} V_{n-1,y_j}(\lambda_n - 1 + jn + kb_n)\,, \tag{12}$$

where the bounds of k in $\sum_k^{(j)}$ are given by $k \geqq 0$ and $\lambda_n - 1 + jn + kb_n \leqq c - 1$, i.e.

$$k \leqq \frac{c - \lambda_n - jn}{b_n} = \frac{c - v_n}{b_n} + \frac{v_n - \lambda_n}{b_n} - \frac{jn}{b_n}\,,$$

$$k \leqq \frac{c - v_n}{b_n} + \gamma$$

where

$$\gamma = \left[\frac{v_n - \lambda_n}{b_n} - \frac{jn}{b_n}\right].$$

Here

$$0 < \frac{jn}{b_n} \leqq \frac{b_{n-1}n}{b_n d_n} = 1\,, \quad 0 \leqq \frac{v_n - \lambda_n}{b_n} < 1\,,$$

and so $-1 \leqq \gamma \leqq 0$. More precisely, $\gamma = 0$ if $jn \leqq v_n - \lambda_n$, $j \leqq (v_n - \lambda_n)/n = [v_n/n]$, $\gamma = -1$ if $j > [v_n/n]$. From (12) we have

$$A_n(c) = \sum_{1 \leqq j \leqq b_{n-1}/d_n} \sum_{k=0}^{l} V_{n-1,y_j}(\lambda_n - 1 + jn + kb_n)\,, \tag{13}$$

where $l = (c - v_n)/b_n \geqq 0$ if $j \leqq [v_n/n]$, $l = (c - v_n)/b_n - 1 \geqq -1$ if $j > [v_n/n]$. We consider now $A_n(c)$ for all $c \equiv y \pmod{b_n}$ y being given. For all these c's we have the same values of $\lambda_n = \lambda_n(c)$, $v_n = v_n(c)$ and so we can chose the same values of $y_j \equiv \lambda_n - 1 + jn \pmod{b_{n-1}}$ in (13). Since $V_{n-1,y_j}(x)$ is a polynomial of degree $n - 2$ we can write every inner sum $\sum_{k=0}^{l}$ in (13) as a sum of terms of the form

$$\text{const} \sum_{k=0}^{l} k^r \quad (r = 0, 1, \ldots, n - 2)\,,$$

the "const" depending only on j. Following (8) we see that $A_n(c)$ is, for all $c \equiv y \pmod{b_n}$, a polynomial in c of degree $\leqq n - 1$. The obvious inequality $n!\, A_n(cn) \geqq c^{n-1}$ shows that $A_{1,y}^{(n)} > 0$.

Remark. We know from Theorem 1 that the $A_{h,y_j}^{(n-1)}$ depend only of the residue class of y_j modulo b_{n-1}. The calculations transforming the sum (11) into the form

(13) can be applied for every $h = 1, 2, \ldots, n - 1$ to the sum

$$T = \sum_{i=1}^{g} A_{h,y_i}^{(n-1)}(\lambda_n - 1 + in)^{n-1-h} \tag{14}$$

and we get (with $y_j \equiv \lambda_n - 1 + jn \pmod{b_{n-1}}$)

$$T = \sum_{1 \leqq j \leqq b_{n-1}/d_n} A_{h,y_j}^{(n-1)} \sum_{k=0}^{l} (\lambda_n - 1 + jn + kb_n)^{n-1-k} = \tag{15}$$
$$= \frac{c^{n-h}}{b_n(n-h)} \sum_{j=1}^{b_{n-1}/d_n} A_{h,y_j}^{(n-1)} + O(c^{n-h-1})$$

for $c \to +\infty$. λ_n being given, all y_j are congruent modulo d_n. Let $x_0 \equiv \lambda_n - 1 \pmod{d_n}$, $0 \leqq x_0 < d_n$. Thus we have $y_j \equiv x_0 + u_j d_n \pmod{b_n}$ (u_j are integers). Evidently the congruences $y_i \equiv y_j \pmod{b_{n-1}}$, $i \equiv j \pmod{b_{n-1}/d_n}$, $u_i \equiv u_j \pmod{b_{n-1}/d_n}$ are equivalent; and so we get from (15).

Lemma 3. *For the sum* (14) *we have*

$$T = \frac{c^{n-h}}{b_n(n-h)} \sum_{u=1}^{b_{n-1}/d_n} A_{h,x_0+ud_n}^{(n-1)} + O(c^{n-h-1}) \tag{16}$$

where $x_0 \equiv c - 1 \pmod{d_n}$, $0 \leqq x_0 < d_n$.

Thus, the sum in (16) *depends only on the residue class of* c *modulo* d_n.

Proof of Theorem 2. For the least values of n, we can verify the truth of the theorem directly. Thus it is sufficient to prove the theorem by induction beginning with $n = 7$.

We suppose that for a given *odd* number $n \geqq 7$ the following is true:

a) The first k coefficient of $V_{n-1,y}$ are stable, where $k = (n-1)/2$.

b) $A_{k+1,y}^{(n-1)}$ depends only on the residue class of y modulo 2 and, putting $A_{k+1,0}^{(n-1)} = \alpha_1$, $A_{k+1,1}^{(n-1)} = \alpha_2$ we have $\alpha_1 \neq \alpha_2$.

c) $A_{k+2,y}^{(n-1)}$ depends only on the residue class of y modulo d_{n-1}.

We shall prove, that a) b) c) imply the following assertion:

I. In $V_{n,y}$ and in $V_{n+1,y}$ the first $k + 1$ coefficients are stable.

II. $A_{k+2,y}^{(n)}$ and $A_{k+2,y}^{(n+1)}$ depend only of the residue class of y modulo 2 and

$$A_{k+2,0}^{(n)} \neq A_{k+2,1}^{(n)}, \quad A_{k+2,0}^{(n+1)} \neq A_{k+2,1}^{(n+1)}.$$

III. $A_{k+3,y}^{(n+1)}$ depends only on the residue class of y mod d_{n+1}. It is obvious that the proof of this induction step suffices to prove. Theorem 2 (the supposition c) being true for $n = 7$). Proof of I, II, III.

From (5) we see that

$$(17)\quad A_n(c) = \sum_{z=1}^{k} A_z^{(n-1)} \sum_i (\lambda_n - 1 + in)^{n-z-1} + \alpha_{1+x} \sum_{i_1} (\lambda_n - 1 + n + 2ni_1)^{n-k-2} +$$

$$+ \alpha_{2-x} \sum_{i_2} (\lambda_n - 1 + 2n + 2ni_2)^{n-k-2} + \sum_{z=k+2}^{n-1} \sum_i A_{z,y_i}^{(n-1)} (\lambda_n - 1 + in)^{n-z-1} .$$

Here $x = 0\,(x = 1)$ for λ_n even (odd) and the limits in $\sum_i, \sum_{i_1}, \sum_{i_2}$ are the following

$$(18)\quad i \geqq 1\,, \quad \lambda_n - 1 + in \leqq c - 1\,, \quad \text{i.e.} \quad i \leqq \frac{c - \lambda_n}{n} = g_n\,,$$

$$(19)\quad i_1 \geqq 0\,, \quad \lambda_n - 1 + n + 2ni_1 \leqq c - 1\,, \quad \text{i.e.} \quad i_1 \leqq \left[\frac{c - \lambda_n - n}{2n}\right],$$

$$(20)\quad i_2 \geqq 0\,, \quad \lambda_n - 1 + 2n + 2ni_1 \leqq c - 1\,, \quad \text{i.e.} \quad i_2 \leqq \left[\frac{c - \lambda_n - 2n}{2n}\right].$$

It follows from Lemma 2 that the first double sum in (17) is of the form $P(\lambda_n - 1 + ng_n) = P(c - 1)$, $P(x)$ being a polynomial of degree $n - 1$ and this can be written as $P_1(c)$ where P_1 is an other polynomial (the coefficients are independent of c).

n being odd, are can have either $\lambda_n \equiv c \pmod 2$ or $\lambda_n \not\equiv c \pmod 2$. The limit in (19), (20) are

$$(21)\quad 0 \leqq i_1 \leqq \frac{c - \lambda_n - 2n}{2n}\,, \quad 0 \leqq i_2 \leqq \frac{c - \lambda_n - 2n}{2n} \quad \text{for} \quad c \equiv \lambda_n \pmod 2\,,$$

$$(22)\quad 0 \leqq i_1 \leqq \frac{c - \lambda_n - n}{2n}\,, \quad 0 \leqq i_2 \leqq \frac{c - \lambda_n - 3n}{2n} \quad \text{for} \quad c \not\equiv \lambda_n \pmod 2\,.$$

Case A. $c \equiv \lambda_n \pmod 2$. Lemma 2 gives

$$\sum_{i_1} = \frac{1}{2n(n - k - 1)} (c - 1 - n)^{n-k-1} + \tfrac{1}{2}(c - 1 - n)^{n-k-2} + O(c^{n-k-3}) =$$

$$= \frac{1}{2n(n - k - 1)} c^{n-k-1} - \frac{1}{2n} c^{n-k-2} + O(c^{n-k-3})\,,$$

$$\sum_{i_2} = \frac{1}{2n(n - k - 1)} (c - 1)^{n-k-1} + \tfrac{1}{2}(c - 1)^{n-k-2} + O(c^{n-k-3}) =$$

$$= \frac{1}{2n(n - k - 1)} c^{n-k-1} + \left(\frac{1}{2} - \frac{1}{2n}\right) c^{n-k-2} + O(c^{n-k-3})\,.$$

Case B. $c \not\equiv \lambda_n$ (mod 2) Lemma 2 gives

$$\sum_{i_1} = \frac{1}{2n(n-k-1)}(c-1)^{n-k-1} + \tfrac{1}{2}(c-1)^{n-k-2} + O(c^{n-k-3}),$$

$$\sum_{i_2} = \frac{1}{2n(n-k-1)}(c-1-n)^{n-k-1} + \tfrac{1}{2}(c-1-n)^{n-k-2} + O(c^{n-k-3}),$$

i.e. we have the same evaluations as in case A, only we must interchange $\sum_{i_1}$ and $\sum_{i_2}$. Now, if $c \equiv 0$ (mod 2), we have in (17) $\alpha_1 \sum_{i_1} + \alpha_2 \sum_{i_2}$ for $\lambda_n \equiv c$ and $\alpha_2 \sum_{i_1} + \alpha_1 \sum_{i_2}$ for $\lambda_n \not\equiv c$ (mod 2); if $c \equiv 1$ (mod 2), we have in (17) $\alpha_2 \sum_{i_1} + \alpha_1 \sum_{i_2}$ for $\lambda_n \equiv c$ and $\alpha_1 \sum_{i_1} + \alpha_2 \sum_{i_2}$ for $\lambda_n \not\equiv c$. Let us consider the sum

$$\alpha_{1+x} \sum_{i_1} + \alpha_{2-x} \sum_{i_2} + \sum_i A^{(n-1)}_{k+2,y_i}(\lambda_n - 1 + in)^{n-k-3}. \tag{23}$$

Following Lemma 3 the last sum has the form

$$\frac{c^{n-k-2}}{b_n(n-k-2)} \sum_{u=1} A_{k+2,x_0+ud_n} + O(c^{n-k-3}), \tag{24}$$

where $x_0 \equiv c - 1$ (mod d_n), $0 \leqq x_0 < d_n$. Following the supposition c), $A^{(n-1)}_{k+2,y}$ depends only on the residue class of y modulo d_{n-1}. But $(d_{n-1}, d_n) = 1$. So, if u runs through all values from 1 to $b_{n-1}/d_n = d_{n-1}(b_{n-1}/d_{n-1}d_n)$, the value of $x_0 + ud_n$ runs $b_{n-1}/d_{n-1}d_n$ times through a complet residue system modulo d_{n-1}, and so the sum in (24) is

$$\frac{b_{n-1}}{d_{n-1}d_n} \sum_{v=1}^{d_{n-1}} A^{(n-1)}_{k+2,v},$$

and so it is independent of c. Denoting with $R_1, R_2 \ldots$ numbers independent of c, we find that the sum in (23) is

$$\begin{aligned} &\frac{\alpha_1 + \alpha_2}{2n(n-k-1)} c^{n-k-1} - \frac{\alpha_1 + \alpha_2}{2n} c^{n-k-2} + \tfrac{1}{2}\alpha_j c^{n-k-2} + \\ &+ \frac{b_{n-1}}{b_n d_{n-1} d_n (n-k-2)} \sum_{v=1}^{d_{n-1}} A^{(n-1)}_{k+2,v} c^{n-k-2} + O(c^{n-k-3}) = \\ &= R_1 c^{n-k-1} + R_2 c^{n-k-2} + \tfrac{1}{2}\alpha_j c^{n-k-2} + O(c^{n-k-3}), \end{aligned} \tag{25}$$

where $j = 2$ for c even, $j = 1$ for c odd.

Considering that the first double sum in (17) is a polynomial $P_1(c)$ the coefficients of which are independent of c we see immediately from (25) that the first $k + 1$ coefficients of $V_{n,y}(x)$ are stable and that

$$(26) \qquad \begin{aligned} A^{(n)}_{k+2,2v} &= R_3 + \tfrac{1}{2}\alpha_2 \\ A_{k+2,2v+1} &= R_3 + \tfrac{1}{2}\alpha_1 \,. \end{aligned}$$

We have now

$$A_{n+1}(c) = \sum_{z=1}^{k+1} A^{(n)}_z \sum_{i=1}^{g_{n+1}} (\lambda_{n+1} - 1 + i(n+1))^{n-z} +$$
$$+ \sum_{i=1}^{g_{n+1}} A^{(n)}_{k+2,y_i}(\lambda_{n+1} - 1 + i(n+1))^{n-k-2} +$$
$$+ \sum_{z=k+3}^{n} \sum_{i=1}^{g_{n+1}} A^{(n)}_{z,y_i}(\lambda_{n+1} - 1 + i(n+1))^{n-z} = S_1 + S_2 + S_3 \,,$$

say. Here is $S_1 = \Pi(c)$, where $\Pi(x)$ is a polynomial of degree n, the coefficients of which are independent of c and so the first $k+1$ coefficients of $V_{n+1,y}(x)$ are stable. The number $n+1$ is even and $\lambda_{n+1} \equiv c \pmod{n+1}$, and so $\lambda_{n+1} \equiv c \pmod 2$. We have $y_i \equiv \lambda_{n+1} - 1 + i(n+1) \pmod{b_{n+1}}$, and so $y_i \equiv c - 1 \pmod 2$. So we have (see (26))

$$S_2 = (R_3 + \tfrac{1}{2}\alpha_x) \sum_{i=1}^{g_{n+1}} (\lambda_{n+1} - 1 + i(n+1))^{n-k-2}$$

where $x = 1$ $(x = 2)$, if c is even (odd). Using Lemma 2 we have

$$(27) \qquad S_2 = (R_3 + \tfrac{1}{2}\alpha_x)\left(\frac{1}{(n+1)(n-k-1)}(c-1)^{n-k-1} + \right.$$
$$\left. + \tfrac{1}{2}(c-1)^{n-k-2}\right) + O(c^{n-k-3}) \,.$$

It follows that

$$(28) \qquad \begin{aligned} A^{(n+1)}_{k+2,2v} &= R_4 + \tfrac{1}{2}\alpha_1 \cdot \frac{1}{(n+1)(n-k-1)} \\ A^{(n+1)}_{k+2,2v+1} &= R_4 + \tfrac{1}{2}\alpha_2 \cdot \frac{1}{(n+1)(n-k-1)} \,. \end{aligned}$$

We must calculate $A^{(n+1)}_{k+3,y_i}$. We have (see also Lemma 3)

$$S_3 = \sum_{i=1}^{g_{n+1}} A^{(n)}_{k+3,y_i}(\lambda_{n+1} - 1 + i(n+1))^{n-k-3} + O(c^{n-k-3}) =$$
$$= \frac{c^{n-k-2}}{b_{n+1}(n-k-2)} \sum_{u=1}^{b_n/d_{n+1}} A^{(n)}_{k+3,y_0+ud_{n+1}} + O(c^{n-k-3}) \,,$$

where the last sum depends only on the residue class of c modulo d_{n+1}. Using (27) we get

$$A_{k+3,y}^{(n+1)} = \left(R_3 + \tfrac{1}{2}\alpha_x\right)\left(\frac{1}{2} - \frac{1}{n+1}\right) + R_5 +$$

$$+ \frac{1}{b_{n+1}(n-k-2)} \sum_{u=1}^{b_n/d_{n+1}} A_{k+3,y_0+ud_{n+1}}^{(n)} .$$

Here the last sum depend only on the residue class of c modulo $d_{n+1} = (b_n, n+1)$. But this is also true of the number α_x, since it depends only on the residuum class of c modulo 2, and 2 divides $n + 1$.

Remark. We have

$$A_{4,0}^{(6)} - A_{4,1}^{(6)} = \frac{1}{2 \,.\, 2 \,.\, 4 \,.\, 4 \,.\, 6}$$

and from (26), (28) we get

$$A_{l+1,0}^{(2l)} - A_{l+1,1}^{(2l)} = \frac{1}{[(2l-2)!!]^2 \,.\, 2l},$$

$$A_{l+2,0}^{(2l+1)} - A_{l+2,1}^{(2l+1)} = -\frac{1}{2[(2l-2)!!]^2 \,.\, 2l}.$$

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