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# THE EXPONENTIAL STABILITY AND PERIODIC SOLUTIONS OF ITO STOCHASTIC EQUATIONS WITH SMALL STOCHASTIC TERMS

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Ito stochastic equation (1) where $a(t, x, \varepsilon)$ and $B(t, x, \varepsilon)$ are periodic in $t$ is investigated in connection with the ordinary differential equation (2). These two equations are close in the sense of 5), 6). The conditions 5), 6) are modifications of vii), viii) from [1]. It is more difficult to verify conditions 5), 6) than vii), viii) from [1], but, on the other hand, they guarantee the uniform convergence of solutions of Ito equation (1) to solutions of (2) (in the sense of Lemma 1) even if $a(t, x, \varepsilon)$ and $B(t, x, \varepsilon)$ are not bounded. The presented Theorems are applied to the parabolic equation (3') and the existence of stable periodic solutions is obtained.

Notation and basic conditions. Let $E_n$ denote the $n$-dimensional Euclidean space, $x$ an $n$-dimensional vector, $\varepsilon$ a small parameter and $t$ the real variable. Let an $n$-dimensional vector $a(t, x, \varepsilon)$ be defined in $\langle 0, \infty) \times E_n \times \langle 0, \delta)$ and a matrix $B(t, x, \varepsilon)$ of type $n \times n$ be defined in $\langle 0, \infty) \times E_n \times (0, \delta)$, where $\delta$ is a positive number. We define the norm of a vector $a$ and the norm of a matrix $B$ in the ordinary manner, i.e. we set $|a| = \sqrt{\sum_i a_i^2}$, $|B| = \sqrt{\sum_{i,j} B_{ij}^2}$. Let $\Omega$ be an abstract space, $\mathscr{F}$ a $\sigma$-field of subsets of $\Omega$ and $P$ be a probabilistic measure defined on $\mathscr{F}$. The norm of a random vector or of a random matrix $z$ is defined by $\|z\| = \sqrt{E|z|^2}$, where $E$ is the expectation.

In the sequel we shall assume that the following conditions are satisfied

1) $a(t, x, \varepsilon), B(t, x, \varepsilon)$ are continuous in $t, x$ for every $\varepsilon$ and $|a(t, x, \varepsilon) - a(t, y, \varepsilon)| \leqq K|x - y|$, $|B(t, x, \varepsilon) - B(t, y, \varepsilon)| \leqq K|x - y|$.

2) Let $w_\varepsilon(t)$ be $n$-dimensional stochastic processes with independent increments which are defined for $t \geqq 0$, $\varepsilon > 0$ and such that $E(w_\varepsilon(t_2) - w_\varepsilon(t_1)) = 0$, $E|w_\varepsilon(t_2) - w_\varepsilon(t_1)|^2 = F_\varepsilon(t_2) - F_\varepsilon(t_1)$, where $F_\varepsilon(t)$ is a continuous function.

3) A continuous nondecreasing function $F(t)$ exists such that

$$F_\varepsilon(t_2) - F_\varepsilon(t_1) \leqq F(t_2) - F(t_1) .$$

Denote $\mathscr{F}_\varepsilon(t)$ the least $\sigma$-field of subsets of $\Omega$ which is generated by increments $w_\varepsilon(t_2) - w_\varepsilon(t_1)$ for $0 \leqq t_1 < t_2 \leqq t$. We have $\mathscr{F}_\varepsilon(t) \subset \mathscr{F}$.

4) Let an initial value $x_0$ be a vector random value which is independent of all $\mathscr{F}_\varepsilon(t)$, $t \geqq 0$ and $E|x_0|^2 < \infty$. We say that $x_0$ is nonstochastic, if it is equal to a constant vector almost everywhere. We assume that $(\Omega, \mathscr{F}, P)$ enables to construc an initial value $x_0(\omega)$ for every given distribution function.

The assumptions 1), 2) and 4) are sufficient for the existence and the uniqueness of a solution of Ito stochastic equation [1];

$$x_\varepsilon(t) = x_0 + \int_0^t a(\tau, x_\varepsilon(\tau), \varepsilon)\,\mathrm{d}\tau + \int_0^t B(\tau, x_\varepsilon(\tau), \varepsilon)\,\mathrm{d}w_\varepsilon(\tau)\,. \tag{1}$$

For the sake of brevity the index $\varepsilon$ at the solution of (1) will be omitted whenever $\varepsilon > 0$. We shall compare equation (1) with

$$y(t) = x_0 + \int_0^t a(\tau, y(\tau), 0)\,\mathrm{d}\tau\,. \tag{2}$$

The dependence of $a(t, x, \varepsilon)$ and $B(t, x, \varepsilon)$ on $\varepsilon$ is subjected to

5) $$\int_0^t (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau), 0))\,\mathrm{d}\tau \to 0$$

as $\varepsilon \to 0$ uniformly with respect to nonstochastic $x_0$, $t \in \langle 0, L\rangle$ for every $L > 0$, where $y(t)$ is the solution of (2) with the initial condition $y(0) = x_0$.

6) $$\int_0^t |B(\tau, y(\tau), \varepsilon)|^2\,\mathrm{d}F_\varepsilon(\tau) \to 0$$

as $\varepsilon \to 0$ uniformly with respect to nonstochastic $x_0$, $t \in \langle 0, L\rangle$ for every $L > 0$. $y(t)$ has the same meaning as in 5).

7) Let the elements of vector $a(t, x, \varepsilon)$ and matrix $B(t, x, \varepsilon)$, i.e. $a_i$ and $B_{ij}$, have lipschitzian partial derivatives with respect to $x_k$.

**Definition 1.** The solution $\bar{z}(t)$ of

$$z(t) = z_0 + \int_{t_0}^t a(\tau, z(\tau))\,\mathrm{d}\tau + \int_{t_0}^t B(\tau, z(\tau))\,\mathrm{d}w(\tau) \tag{3}$$

is called stable, if to every $\varepsilon > 0$ and $t_0 \geqq 0$ a $\delta(t_0, \varepsilon) > 0$ exists such that $\|\bar{z}(t_0) - z(t_0)\| < \delta(t_0, \varepsilon)$ implies $\|\bar{z}(t) - z(t)\| < \varepsilon$ for $t \geqq t_0$.

The solution $\bar{z}(t)$ is called uniformly stable if $\delta$ is independent of $t_0$. The solutions of (3) are called uniformly exponentially stable if constants $\beta$, $S$, $0 < \beta < 1$, $S > 0$ exist such that $\|z^{(1)}(t) - z^{(2)}(t)\| \leqq \beta\|z^{(1)}(t_0) - z^{(2)}(t_0)\|$ for $t \geqq t_0 + S$, where $z^{(1)}(t)$, $z^{(2)}(t)$ are arbitrary solutions of (3).

We may apply this definition to the ordinary differential equation (2) too. Then the condition for exponential stability reads $|y^{(1)}(t) - y^{(2)}(t)| \leqq \beta|y^{(1)}(t_0) - y^{(2)}(t_0)|$ for $t \geqq t_0 + S$.

**Definition 2.** A process $z(t)$ is called periodic with a period $T$, if it is defined for all $t$ and if for every integer $s$, for all numbers $t_1, t_2, \ldots, t_s$ and for all Borel sets $A_1, A_2, \ldots, A_s$ we have $P(z(t_1) \in A_1,\ z(t_2) \in A_2, \ldots, z(t_s) \in A_s) = P(z(t_1 + T) \in A_1,\ z(t_2 + T) \in A_2, \ldots, z(t_s + T) \in A_s)$.

**Definition 3.** A process $w(t)$ is said to have periodic increments with period $T$, if for every $t$, $h > 0$ and for every Borel set $A$ the condition

$$P(w(t + h) - w(t) \in A) = P(w(t + h + T) - w(t + T) \in A)$$

is fulfilled.

**Theorem 1.** *Let assumptions* 1) *to* 7) *be fulfilled, let* $a(t, x, \varepsilon)$ *and* $B(t, x, \varepsilon)$ *be periodic in* $t$ *with the period* $T$ *and let the processes* $w_\varepsilon(t)$ *have periodic increments with period* $T$. *If the solutions of* (2) *are uniformly exponentially stable, then a positive number* $\varepsilon_0$ *exists such that the solutions of* (1) *are uniformly stable and uniformly exponentially stable for* $0 \leqq \varepsilon \leqq \varepsilon_0$.

The proof of Theorem 1 is based on several lemmas. The first lemma gives us an estimate of the difference of solutions of (1) and (2), which have the same initial value.

**Lemma 1.** *Let assumptions* 1) *to* 6) *be fulfilled; then to every* $\eta > 0$ *and* $L > 0$ *an* $\varepsilon_0 > 0$ *exists such that* $\sup\limits_{t \in \langle 0, L\rangle} E(|x(t) - y(t)|^2 \mid \mathscr{F}^{(0)}) < \eta$ *almost everywhere for* $0 \leqq \varepsilon \leqq \varepsilon_0$ *and for all initial values* $x_0$ *fulfilling* 4), *where* $x(t)$ *is the solution of* (1) *with initial value* $x_0$, $y(t)$ *is the solution of* (2) *with the same initial value* $x_0$, $E(\,|\,)$ *is the conditional expectation and* $\mathscr{F}^{(0)}$ *is an* $\sigma$*-field independent of* $\mathscr{F}_\varepsilon(t)$ *containing* $\mathscr{F}(x_0)$ *with* $\mathscr{F}(x_0)$ *being the least* $\sigma$*-field generated by* $x_0$.

Proof. Suppose that $x_0$ is an nonstochastic initial value; then

$$(1,1)\qquad \|x(t) - y(t)\| \leqq \left\|\int_0^t (a(\tau, x(\tau), \varepsilon) - a(\tau, y(\tau), 0))\,\mathrm{d}\tau\right\| + \left\|\int_0^t B(\tau, x(\tau), \varepsilon)\,\mathrm{d}w_\varepsilon(\tau)\right\|.$$

The second term of this inequality can be estimated by means of 1) and 5),

$$(1,2)\quad \left\|\int_0^t (a(\tau, x(\tau), \varepsilon) - a(\tau, y(\tau), 0))\,\mathrm{d}\tau\right\| \leqq \left\|\int_0^t (a(\tau, x(\tau), \varepsilon) - a(\tau, y(\tau), \varepsilon))\,\mathrm{d}\tau\right\| +$$

$$+ \left\|\int_0^t (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau), 0))\,\mathrm{d}\tau\right\| \leqq K\int_0^t \|x(\tau) - y(\tau)\|\,\mathrm{d}\tau +$$

$$+ \left|\int_0^t (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau), 0))\,\mathrm{d}\tau\right| \leqq K\int_0^t \|x(\tau) - y(\tau)\|\,\mathrm{d}\tau + \varphi_1(\varepsilon)$$

where $\varphi_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We shall estimate the last term in (1,1) by means of (4,7) from [1] and by 6)

$$(1,3)\quad \left\|\int_0^t B(\tau, x(\tau), \varepsilon)\,\mathrm{d}w_\varepsilon(\tau)\right\| \leqq \sqrt{\left(n\int_0^t E|B(\tau, x(\tau), \varepsilon)|^2\,\mathrm{d}F_\varepsilon(\tau)\right)} \leqq$$

$$\leqq K\sqrt{\left(n\int_0^t E|x(\tau) - y(\tau)|^2\,\mathrm{d}F(\tau)\right)} + \sqrt{\left(n\int_0^t |B(\tau, y(\tau), \varepsilon)|^2\,\mathrm{d}F_\varepsilon(\tau)\right)} \leqq$$

$$\leqq K_1\sqrt{\int_0^t E|x(\tau) - y(\tau)|^2\,\mathrm{d}F(\tau)} + \varphi_2(\varepsilon)$$

where $\varphi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$.

According to (1,1), (1,2) and (1,3) we obtain

$$\|x(t) - (t)\| \leqq \varphi(\varepsilon) + K\int_0^t \|x(\tau) - y(\tau)\|\,\mathrm{d}\tau + K_1\sqrt{\left(\int_0^t \|x(\tau) - y(\tau)\|^2\,\mathrm{d}F(\tau)\right)}\,.$$

By Lemma 2 from [1], $\limsup_{\langle 0,L\rangle} \|x(t) - y(t)\| = 0$ as $\varepsilon \to 0$ provided $x_0$ is non-stochastic. Since $x(t) - y(t)$ is a Markov process and since $x_0$ is independent of all increments of $w_\varepsilon(t)$, the statement of Lemma 1 is proved.

In the following lemma we shall estimate the differences of four solutions which, simlarly as in article [2], are used for the proof of stability of equations (1).

**Lemma 2.** *Let assumptions* 1) *to* 7) *be fulfilled; then to every* $\eta > 0$ *and* $L > 0$ *an* $\varepsilon_0 > 0$ *exists such that*

$$\sup_{x_0^{(1)}, x_0^{(2)}, t\in\langle 0,L\rangle} \|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\| \leqq \eta\|x_0^{(1)} - x_0^{(2)}\|$$

$$\textit{for}\quad 0 \leqq \varepsilon \leqq \varepsilon_0\,,$$

*where* $x^{(1)}(t)$, $x^{(2)}(t)$ *is the solution of* (1) *with initial value* $x_0^{(1)}$ *and* $x_0^{(2)}$, *respectively and* $y^{(1)}(t)$, $y^{(2)}(t)$ *solution of* (2) *with initial value* $x_0^{(1)}$ *and* $x_0^{(2)}$, *respectively* ($x_0^{(1)}$ *and* $x_0^{(2)}$ *fulfil* 4) *in the sense that the least* $\sigma$*-field containing both* $\mathscr{F}(x_0^{(1)})$ *and* $\mathscr{F}(x_0^{(2)})$ *is independent of* $\mathscr{F}_\varepsilon(t)$).

Proof. From equations (1), (2) we obtain

$$(2,1)\qquad \|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\| \leqq$$

$$\leqq \left\|\int_0^t (a(\tau, x^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau\right\| +$$

$$+ \left\|\int_0^t (B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon))\, dw_\varepsilon(\tau)\right\| .$$

First, we shall deal with the first expression on the right hand side of (2,1),

$$(2,2)\qquad \left\|\int_0^t (a(\tau, x^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau\right\| \leqq$$

$$\leqq \left\|\int_0^t (a(\tau, x^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), \varepsilon) + a(\tau, y^{(2)}(\tau), \varepsilon))\, d\tau\right\| +$$

$$+ \left\|\int_0^t (a(\tau, y^{(1)}(\tau), \varepsilon) - a(\tau, y^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau\right\| \leqq$$

$$\leqq \left\|\int_0^t (a(\tau, y^{(1)}(\tau), \varepsilon) - a(\tau, y^{(2)}(\tau), \varepsilon) - a(\tau, x^{(1)}(\tau), \varepsilon) + \right.$$

$$\left. + a(\tau, x^{(1)}(\tau) + y^{(2)}(\tau) - y^{(1)}(\tau), \varepsilon))\, d\tau\right\| +$$

$$+ \left\|\int_0^t (a(\tau, x^{(1)}(\tau) + y^{(2)}(\tau) - y^{(1)}(\tau), \varepsilon) - a(\tau, x^{(2)}(\tau), \varepsilon))\, d\tau\right\| +$$

$$+ \left\|\int_0^t (a(\tau, y^{(1)}(\tau), \varepsilon) - a(\tau, y^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau\right\| .$$

Similarly as in [2] we can prove by 7) that

$$(2,3)\qquad |a(t, x, \varepsilon) - a(t, y, \varepsilon) - a(t, u, \varepsilon) + a(t, u + y - x, \varepsilon)| \leqq$$
$$\leqq K|y - x|\, |x - u| ,$$

and

$$(2,4)\qquad \left|\int_0^t (a(\tau, y^{(1)}(\tau), \varepsilon) - a(\tau, y^{(2)}(\tau), \varepsilon) - a(\tau, y^{(1)}(\tau), 0) + a(\tau, y^{(2)}(\tau), 0))\, d\tau\right| \leqq$$
$$\leqq \varphi_3(\varepsilon)\, |y^{(2)}(0) - y^{(1)}(0)| \quad \text{for} \quad t \in \langle 0, L\rangle , \quad \varphi_3(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

From these inequalities we obtain that (2,2) does not exceed

$$K \int_0^t \sqrt{E[|y^{(1)}(\tau) - y^{(2)}(\tau)|^2\, E(|x^{(1)}(\tau) - y^{(1)}(\tau)|^2 \mid \mathscr{F}^{(0)})]}\, d\tau +$$

$$+ K \int_0^t \sqrt{E|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)|^2}\, d\tau + \varphi_3(\varepsilon) \sqrt{E|x_0^{(1)} - x_0^{(2)}|^2} ,$$

where $\mathscr{F}^0$ is the $\sigma$-field generated by $x_0^{(1)}, x_0^{(2)}$ such that the assumptions of Lemma 1 are fulfilled. Using Lemma 1 we obtain an estimate

$$(2,5) \qquad \varphi_4(\varepsilon)\,\|x_0^{(1)} - x_0^{(2)}\| + K\int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|\,\mathrm{d}\tau\,.$$

Analogously we shall estimate the last term of (2,1). Recalling (4,7) from [1] we get

$$(2,6) \qquad \left\|\int_0^t (B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon))\,\mathrm{d}w_\varepsilon(\tau)\right\|^2 \leqq n\int_0^t \|B(\tau, x^{(1)}(\tau), \varepsilon) -$$
$$- B(\tau, x^{(2)}(\tau), \varepsilon)\|^2\,\mathrm{d}F_\varepsilon(\tau) \leqq 2n\int_0^t \|B(\tau, x^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon) -$$
$$- B(\tau, y^{(1)}(\tau), \varepsilon) + B(\tau, y^{(2)}(\tau), \varepsilon)\|^2\,\mathrm{d}F(\tau) +$$
$$+ 2n\int_0^t \|B(\tau, y^{(1)}(\tau), \varepsilon) - B(\tau, y^{(2)}(\tau), \varepsilon)\|^2\,\mathrm{d}F_\varepsilon(\tau) \leqq$$
$$\leqq 4n\int_0^t \|B(\tau, y^{(1)}(\tau), \varepsilon) - B(\tau, y^{(2)}(\tau), \varepsilon) - B(\tau, x^{(1)}(\tau), \varepsilon) +$$
$$+ B(\tau, x^{(1)}(\tau) + y^{(2)}(\tau) - y^{(1)}(\tau), \varepsilon)\|^2\,\mathrm{d}F(\tau) +$$
$$+ 4n\int_0^t \|B(\tau, x^{(1)}(\tau) + y^{(2)}(\tau) - y^{(1)}(\tau), \varepsilon) - B(\tau, x^{(2)}(\tau), \varepsilon)\|^2\,\mathrm{d}F(\tau) +$$
$$+ 2n\int_0^t \|B(\tau, y^{(1)}(\tau), \varepsilon) - B(\tau, y^{(2)}(\tau), \varepsilon)\|^2\,\mathrm{d}F_\varepsilon(\tau)\,.$$

We shall need the following inequality

$$(2,7) \qquad \int_0^t |B(\tau, y^{(1)}(\tau), \varepsilon) - B(\tau, y^{(2)}(\tau), \varepsilon)|^2\,\mathrm{d}F_\varepsilon(\tau) \leqq \varphi_5^2(\varepsilon)\,|x_0^{(1)} - x_0^{(2)}|^2\,.$$

Without loss of generality this inequality will be proved only for nonstochastic initial values. Choose fixed indices $i, j$ and set $f(t, x, \varepsilon) = B_{ij}(t, z(t), \varepsilon) - B_{ij}(t, y^{(2)}(t), \varepsilon)$, where $y^{(2)}(t)$ is the solution of (2) with the initial value $x_0^{(2)}$ and $z(t)$ the solution of (2) with the initial value $x + x_0^{(2)}$. According to 1) $f(t, x, \varepsilon)$ is continuous in $t, x$ and lipschitzian in $x$. With respect to 7) the partial derivatives $\partial f/\partial x_k$ exist, are lipschitzian in $x$, $f(t, 0, \varepsilon) \equiv 0$, and by 6),

$$(2,8) \qquad \int_0^t |f(\tau, x, \varepsilon)|^2\,\mathrm{d}F_\varepsilon(\tau) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

uniformly with respect to $x_0^{(2)}$, $x$ and $t \in \langle 0, L\rangle$. Let $\mathrm{D}f(t, x, \varepsilon)$ denote the vector consisting of the partial derivatives of $f(t, x, \varepsilon)$ by $x_k$ at the point $[t, x, \varepsilon]$, and let

(,) signify the inner product. First we prove that

$$(2,9)\qquad \int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

uniformly with respect to all vectors $x \neq 0$, $x_0^{(2)}$ and to all $t \in \langle 0, L\rangle$. Actually, we have

$$\int_0^t |f(\tau, x, \varepsilon)|^2 \,\mathrm{d}F_\varepsilon(\tau) = \int_0^t \left|\int_0^1 \left(\mathrm{D}f(\tau, \lambda x, \varepsilon), x\right) \mathrm{d}\lambda\right|^2 \mathrm{d}F_\varepsilon(\tau) =$$

$$= \int_0^t |(\mathrm{D}f(\tau, 0, \varepsilon), x) + (\psi(\tau, x), x)|^2 \,\mathrm{d}F_\varepsilon(\tau)$$

where $\psi(t, x) = \int_0^1 \mathrm{D}f(t, x\lambda, \varepsilon)\,\mathrm{d}\lambda - \mathrm{D}f(t, 0, \varepsilon)$. From this it follows that $|\psi(t, x)| \leqq \leqq K_2|x|/2$. From the previous relation we obtain

$$(2,10)\qquad \int_0^t |f(\tau, x, \varepsilon)|^2 \,\mathrm{d}F_\varepsilon(\tau) = |x|^2 \int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau) +$$

$$+ 2|x|^2 \int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)\left(\psi(\tau, x), \frac{x}{|x|}\right) \mathrm{d}F_\varepsilon(\tau) + |x|^2 \int_0^t \left(\psi(\tau, x), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau)\,.$$

(2,12) and (2,11) is the estimate for the last and the last but one term, respectively,

$$(2,11)\qquad \left|\int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)\left(\psi(\tau, x), \frac{x}{|x|}\right) \mathrm{d}F_\varepsilon(\tau)\right| \leqq$$

$$\leqq \frac{K_2}{2}|x| \sqrt{\left[\int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau)\right]} \sqrt{[F(t) - F(0)]}\,,$$

$$(2,12)\qquad \int_0^t \left(\psi(\tau, x), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau) \leqq \frac{K_2^2}{4}|x|^2 (F(t) - F(0))\,.$$

By (2,10) to (2,12),

$$\int_0^t |f(\tau, x, \varepsilon)|^2 \,\mathrm{d}F_\varepsilon(\tau) \geqq |x|^2 \left\{\int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau) -\right.$$

$$\left. - K_2|x| \sqrt{\left[\int_0^t \left(\mathrm{D}f(\tau, 0, \varepsilon), \frac{x}{|x|}\right)^2 \mathrm{d}F_\varepsilon(\tau)\,(F(t) - F(0))\right]}\right\}.$$

If sequences of unit vectors $x_i$, of vectors $x_{0_i}^{(2)}$, of numbers $\varepsilon_i \to 0$, $t_i \in \langle 0, L\rangle$ and a number $q > 0$ existed such that $\int_0^{t_i} (\mathrm{D}f(\tau, 0, \varepsilon_i), x_i)^2 \,\mathrm{d}F_{\varepsilon_i}(\tau) \geqq q$, then by choosing $\Theta = q^{\frac{1}{2}}(F(L) - F(0))^{-\frac{1}{2}}(2K_2))^{-1}$ we would obtain $\int_0^{t_i} |f(\tau, \Theta x_i, \varepsilon_i)|^2 \,\mathrm{d}F_{\varepsilon_i}(\tau) \geqq \Theta^2 q/2$

which contradicts (2,8). Relation (2,9) is proved. From (2,10) to (2,12) we obtain

$$(2,13) \qquad \int_0^t |f(\tau, x, \varepsilon)|^2 \, dF_\varepsilon(\tau) \leqq$$

$$\leqq 2|x|^2 \left[ \int_0^t \left( Df(\tau, 0, \varepsilon), \frac{x}{|x|} \right)^2 dF_\varepsilon(\tau) + \frac{K_2^2}{4} |x|^2 (F(t) - F(0)) \right].$$

By (2,8) there exists a function $\varphi_6(\varepsilon) > 0$, $\varphi_6(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$\int_0^t |f(\tau, x, \varepsilon)|^2 \, dF_\varepsilon(\tau) \leqq \varphi_6(\varepsilon).$$

Put

$$\varphi_7(\varepsilon) = \max \left[ \sqrt[3]{\varphi_6(\varepsilon)}, 2 \int_0^t \left( Df(\tau, 0, \varepsilon), \frac{x}{|x|} \right)^2 dF_\varepsilon(\tau) + \frac{K_2^2}{2} \sqrt[3]{\varphi_6^2(\varepsilon)} \, (F(t) - F(0)) \right].$$

We have $(1/|x|^2) \int_0^t |f(\tau, x, \varepsilon)|^2 \, dF_\varepsilon(\tau) \leqq \sqrt[3]{\varphi_6(\varepsilon)} \leqq \varphi_7(\varepsilon)$ for $|x| \geqq \sqrt[3]{\varphi_6(\varepsilon)}$, and by (2,13) and by definition of $\varphi_7$ we find $(1/|x|^2) \int_0^t |f(\tau, x, \varepsilon)|^2 \, dF_\varepsilon(\tau) \leqq \varphi_7(\varepsilon)$ for $0 < |x| < \sqrt[3]{\varphi_6(\varepsilon)}$. Both these inequalities imply $\int_0^t |f(\tau, x, \varepsilon)|^2 \, dF_\varepsilon(\tau) \leqq \varphi_7(\varepsilon) \, |x|^2$. Since $\varphi_7(\varepsilon) \to 0$ as $\varepsilon \to 0$, inequality (2,7) is proved in the case of nonstochastic initial values. Since $y^{(i)}(t)$ are solutions of the nonstochastic equation (2), inequality (2,7) is true also for stochastic initial values. If we use inequality (2,3) in a similar manner as in the case of estimating (2,2) and if we apply inequality (2,7) and the assumption 1) we obtain that (2,6) does not exceed $4nK_3 \, \varphi_4(\varepsilon) \int_0^t \|y^{(1)}(\tau) - y^{(2)}(\tau)\|^2 \, . \, dF(\tau) + 4K^2 n \int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^2(\tau)\|^2 \, dF(\tau) + 2n^3 \, \varphi_7(\varepsilon) \, \|x_0^{(1)} - x_0^{(2)}\|^2$. From (2,1), (2,5) and the last inequality it follows that

$$\|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\| \leqq \varphi_8(\varepsilon) \, \|x_0^{(1)} - x_0^{(2)}\| + K \int_0^t \|x^{(1)}(\tau) -$$

$$- x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\| \, d\tau +$$

$$+ K_4 \sqrt{\left( \int_0^t \|x^{(1)}(\tau) - x^{(2)}(\tau) - y^{(1)}(\tau) + y^{(2)}(\tau)\|^2 \, dF(\tau) \right)}.$$

By Lemma 2 from [1] we conclude that

$$\|x^{(1)}(t) - x^{(2)}(t) - y^{(1)}(t) + y^{(2)}(t)\| \leqq \varphi_9(\varepsilon) \| \, x_0^{(1)} - x_0^{(2)}\|$$

where $\varphi_9(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now we are going to prove the stability of (1) under more general assumptions than those formulated in Theorem 1.

**Lemma 3.** *Let assumptions* 1) *to* 7) *be fulfilled with* 5) *and* 6) *replaced by*:

$$5') \qquad \int_{t_0}^{t_0 + t} (a(\tau, y(\tau), \varepsilon) - a(\tau, y(\tau), 0)) \, d\tau \to 0$$

*as* $\varepsilon \to 0$ *uniformly with respect to* $t_0$, $t \in \langle t_0, t_0 + L\rangle$ *and to nonstochastic initial values* $x_0$ *for every* $L > 0$.

6') $$\int_{t_0}^{t_0+t} |B(\tau, y(\tau), \varepsilon)|^2 \, dF_\varepsilon(\tau) \to 0 \quad as \quad \varepsilon \to 0$$

*uniformly with respect to* $t_0$, $t \in \langle t_0, t_0 + L\rangle$ *and to nonstochastic initial values* $x_0$ *for every* $L > 0$. *Let* $y(t)$ *have the same meaning as in assumptions* 5), 6). *Let a continuous function* $G(t)$ *exist such that* $F(t_0 + t) - F(t_0) \leqq G(t) - G(0)$. *If the solutions of* (2) *are uniformly exponentially stable, then a positive number* $\varepsilon_0$ *exists such that the solutions of* (1) *are uniformly exponentially stable for* $0 \leqq \varepsilon \leqq \leqq \varepsilon_0$.

Proof. Constants $0 < \beta < 1$ and $S > 0$ exist such that $\|y^{(1)}(t) - y^{(2)}(t)\| \leqq \leqq \beta\|y^{(1)}(t_0) - y^{(2)}(t_0)\|$ holds for $t \geqq t_0 + S$ and all solutions of (2). By Lemma 2 we can choose $\varepsilon_1 > 0$ for the given $(1 - \beta)/2$ and $S$ such that $\|x^{(1)}(t) - x^{(2)}(t) - - y^{(1)}(t) + y^{(2)}(t)\| \leqq (1 - \beta)2^{-1} \|x^{(1)}(t_0) - x^{(2)}(t_0)\|$ for $t \in \langle t_0, t_0 + 2S\rangle$, $0 \leqq \varepsilon \leqq \leqq \varepsilon_1$. According to 5'), 6') and to the existence of the function $G(t)$ we can choose $\varepsilon_1$ independently of $t_0$. The initial values were chosen so that $y^{(1)}(t_0) = x^{(1)}(t_0)$, $y^{(2)}(t_0) = x^{(2)}(t_0)$. From these inequalities we obtain easily

$$\|x^{(1)}(t) - x^{(2)}(t)\| \leqq \frac{1 + \beta}{2} \|x^{(1)}(t_0) - x^{(2)}(t_0)\| \quad \text{for} \quad t \in \langle t_0 + S, t_0 + 2S\rangle .$$

Since the solutions of Itô equations are unique and continuable, we have

(3,1) $$\|x^{(1)}(t) - x^{(2)}(t)\| \leqq \frac{1 + \beta}{2} \|x^{(1)}(t_0) - x^{(2)}(t_0)\| \quad \text{for} \quad t \geqq t_0 + S .$$

It remains to prove that every solution $x(t)$ of (1) is stable. Choose a fixed solution of (1) and denote it by $\bar{x}(t)$. Let $\bar{y}(t)$ be the solution of (2) with the initial value $\bar{y}(t_0) = = \bar{x}(t_0)$. The exponential stability of $\bar{y}(t)$ and the continuous dependence on initial values imply that to every $\eta > 0$ and $t_0 \geqq 0$ a $\delta > 0$ exists such that $\|y(t) - \bar{y}(t)\| \leqq \leqq \eta/3$ for $t \geqq t_0$ whenever $\|y(t_0) - \bar{y}(t_0)\| < \delta$. By Lemma 1, to $\eta/3$ an $\varepsilon_2 > 0$ exists such that $\|x(t) - y(t)\| < \eta/3$ for $t \in \langle t_0, t_0 + S\rangle$, $0 \leqq \varepsilon \leqq \varepsilon_2$, where $x(t)$ is the solution of (1) with the initial value $x(t_0) = y(t_0)$. From this we obtain

$$\|x(t) - \bar{x}(t)\| \leqq \|x(t) - y(t)\| + \|\bar{x}(t) - \bar{y}(t)\| + \|\bar{y}(t) - y(t)\| \leqq \eta$$

for $0 \leqq \varepsilon \leqq \varepsilon_0$ and $t \in \langle t_0, t_0 + S\rangle$, where $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. Recalling (3,1) we conclude that $\bar{x}(t)$ is stable. If $a(t, x, 0)$ is periodic then $\delta$ need not depend on $t_0$ and it means that $\bar{x}(t)$ is uniformly stable.

In the next part of the article the construction of a periodic solution will be carried out. There are many methods which are suitable for the proof of the existence of a periodic solution and some of them can be used under less restrictive assumptions; however, we utilise the uniform exponential stability of solutions in our construction and therefore, the convergence of the method is rather rapid and the estimate (4,3) holds.

**Theorem 2.** *Let the assumptions of Theorem* 1 *be fulfilled; then equations* (1) *have periodic solutions* $\bar{x}_\varepsilon(t)$ *with period* $T$ *for* $0 < \varepsilon \leqq \varepsilon_0$, *equation* (2) *has periodic solution* $\bar{y}(t)$ *and* $\lim_{\varepsilon \to 0} \sup_t E|\bar{x}_\varepsilon(t) - \bar{y}(t)|^2 = 0$ *where* $\varepsilon_0$ *has the meaning given in Theorem* 1.

Remark. Since the solutions of (1) are uniformly exponentially stable, the periodic solutions are determined uniquely in the sense that their distribution functions are determined uniquely.

The proof of Theorem 1 is based on Lemma 4. In order to simplify the wording of this lemma and its proof we introduce a new notation. Let $X(t, \xi, \omega)$ be a solution of (1) with nonstochastic initial value $X(t_0, \xi, \omega) = \xi$. We can write $x(t, x_0) = X(t, x_0(\omega), \omega)$ for general initial values fulfilling 4). Since every solution of (1) is a Markov process, we have

$$F(t, \lambda_1, \ldots, \lambda_n) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \Phi(t, \lambda_1, \ldots, \lambda_n;\, t_0, \xi_1, \ldots, \xi_n) \, \mathrm{d}F(t_0, \xi_1, \ldots, \xi_n)$$

for the distribution functions of $x(t, x_0)$, where $\Phi(t, \lambda_1, \ldots, \lambda_n;\, t_0, \xi_1, \ldots, \xi_n)$ is the distribution function of $X(t, \xi, \omega)$. This dependence can be written in the form $F(t, \lambda) = \mathcal{O}_{t,t_0}\{F(t_0, \lambda)\}$. We say that $F_m(\lambda_1, \ldots, \lambda_n)$ converge to $F(\lambda_1, \ldots, \lambda_n)$, if this convergence has its ordinary meaning in all points of continuity of $F(\lambda_1, \ldots, \lambda_n)$. Obviously the expression

$$E(\gamma(X(t, \xi, \omega))) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \gamma(\lambda_1, \ldots, \lambda_n) \, \mathrm{d}\Phi(t, \lambda_1, \ldots, \lambda_n;\, t_0, \xi_1, \ldots, \xi_n)$$

is a continuous and bounded function of $\xi$, where $\gamma$ is any continuous and bounded function. By Alexandrov's theorem we obtain from this that the operator $\mathcal{O}_{t,t_0}$ is continuous.

**Lemma 4.** *Let the assumptions of Theorem* 1 *be satisfied, then to every distribution function* $F(0, \lambda_1, \ldots, \lambda_n)$, $\int |\lambda|^2 \, \mathrm{d}F(0, \lambda_1, \ldots, \lambda_n) < \infty$ *a distribution function* $F^*(\lambda_1, \ldots, \lambda_n) = \lim_{m \to \infty} \mathcal{O}_{mT,0}\{F(0, \lambda_1, \ldots, \lambda_n)\}$ *exists and* $\int |\lambda|^2 \, \mathrm{d}F^*(\lambda_1, \ldots, \lambda_n) < \infty$.

Proof. Let $x_0(\omega)$ be a random value fulfilling 4) with distribution function $F(0, \lambda)$ and $x_1(\omega)$ be a random value fulfilling 4) with distribution function $F(sT + qT, \lambda) = \mathcal{O}_{sT+qT,0}\{F(0, \lambda)\}$, where $\lambda = [\lambda_1, \ldots, \lambda_n]$, $s$ is the least integer with $sT > S$ ($S$ is from Definition 1) and $q$ is an arbitrary number from $0, 1, \ldots, s - 1$. According to Lemma 3,

$$(4,1) \qquad E|x(sTl, x_1(\omega)) - x(sTl, x_0(\omega))|^2 \leqq$$
$$\leqq \beta_1 E|x(sT(l-1), x_1(\omega)) - x(sT(l-1), x_0(\omega))|^2 \leqq$$
$$\leqq \beta_1^l E|x_1(\omega) - x_0(\omega)|^2 , \quad 0 < \beta_1 < 1 .$$

This inequality yields for distribution functions,

$$F(sT(l+1) + qT, \lambda) \leqq F(sTl, \lambda + \Theta^l \gamma e) + \Theta^l \gamma ,$$
$$F(sTl, \lambda) \leqq F(sT(l+1) + qT, \lambda + \Theta^l \gamma e) + \Theta^l \gamma$$

where $\Theta = \sqrt[3]{\beta_1}$, $\gamma = \max_q \sqrt[3]{E|x_1(\omega) - x_0(\omega)|^2}$,

$$F(t, \lambda) = \mathcal{O}_{t,0}\{F(0, \lambda)\} , \quad e = [1, 1 \dots 1] .$$

Both previous inequalities imply that

$$(4,2) \qquad F(sTl, \lambda - \Theta^l \gamma_1 e) - \Theta^l \gamma_1 \leqq F(sT(l+k) + qT, \lambda) \leqq$$
$$\leqq F(sTl, \lambda + \Theta^l \gamma_1 e) + \Theta^l \gamma_1 , \quad \gamma_1 = \frac{\gamma}{1 - \Theta} ,$$

i.e.

$$\limsup_{k \to \infty} F(sTk + qT, \lambda) - \liminf_{k \to \infty} F(sTk + qT, \lambda) \leqq$$
$$\leqq F(sTl, \lambda + \Theta^l \gamma_1 e) - F(sTl, \lambda - \Theta^l \gamma_1 e) + 2\Theta^l \gamma_1$$

for arbitrary $l$. For proving that the distribution functions converge, put $h(l, u, v) = F(sTl, ue + v + \Theta^l \gamma_1 e) - F(sTl, ue + v - \Theta^l \gamma_1 e)$, where $u$ is a real variable and $v$ is a vector orthogonal to $e$. First we shall prove that there exists only a countable set of values of $u$ such that $\liminf_{l \to \infty} h(l, u, v) = 2p(u) > 0$ for a fixed $v$. Consider a finite number of values of $u$ for which $\liminf_{l \to \infty} h(l, u_i, v) = 2p(u_i) > 0$ for a fixed $v$. There exists an $l_0$ with $h(l, u_i, v) \geqq p(u_i)$, $2\Theta^l \gamma_1 \leqq \min_{i \neq j} |u_i - u_j|$ for $l \geqq l_0$. Using the properties of distribution functions we obtain $\sum p(u_i) \leqq 1$. Since the values of $p(u)$ are nonnegative, there can exist only countably many positive values of $p(u)$. This means that the distribution functions $F(sT(l+k) + qT, \lambda)$ converge to a distribution function $F^*(\lambda)$ almost everywhere. By (4,2) we obtain that $F^*(\lambda)$ is independent of $q$, $F(lT, \lambda)$ converge to $F^*(\lambda)$ and

$$(4,3) \qquad F(sTl, \lambda - \Theta^l \gamma_1 e) - \Theta^l \gamma_1 \leqq F^*(\lambda) \leqq F(sTl, \lambda + \Theta^l \gamma_1 e) + \Theta^l \gamma_1 .$$

Obviously, $F^*(\lambda)$ fulfils the first condition of Lemma 4. From (4,1) we have $\int |\lambda|^2 \, dF(lT, \lambda) \leqq C$ where $C$ is a constant independent of $l$ and, by Helly theorem, $\int |\lambda|^2 \, dF^*(\lambda) < \infty$.

Next, turn to the proof of Theorem 2. Let $x_0(\omega)$ be a random value fulfilling 4). $F^*(\lambda)$ denote the distribution function which has been constructed in Lemma 4. Let $\bar{x}_0^{(\varepsilon)}(\omega)$ be a random value fulfilling 4) with the distribution function $F^*(\lambda)$, Let $\bar{x}_\varepsilon(t)$ be a solution of (1) with the initial value $\bar{x}_0^{(\varepsilon)}(\omega)$. Since the operator $\mathcal{O}_{t,t_0}$ is continuous, we have

$$\mathcal{O}_{T,0}\{F^*(\lambda)\} = \mathcal{O}_{T,0}\{\lim_{m\to\infty} F(mT, \lambda)\} = \lim_{m\to\infty} \mathcal{O}_{T,0}\{F(mT, \lambda)\} =$$
$$= \lim_{m\to\infty} \mathcal{O}_{(m+1)T,0}\{F(0, \lambda)\} = F^*(\lambda),$$

where $F(0, \lambda)$ is a distribution function of $x_0(\omega)$. We have proved that the distribution function of $\bar{x}_\varepsilon(t)$ is periodic in $t$ with the period $T$. Since $a$, $B$ are periodic in $t$ and $w_\varepsilon(t)$ have periodic increments, (Definition 3) the solution $\bar{x}_\varepsilon(t)$ is also periodic according to Definition 2. By Theorem 1 the solutions are uniformly exponentially stable, and consequently, $F^*(\lambda)$ is determined uniquely.

By applying Lemma 4 to (2) we conclude that equation (2) has also a periodic solution $\bar{y}(t)$. Let $y^*(t)$ be a solution of (2) with the initial condition $y^*(t_0) = \bar{x}_\varepsilon(t_0)$ for $0 \leqq t_0 \leqq Ts$. By Lemma 1, $\|\bar{x}_\varepsilon(t) - y^*(t)\| \leqq \varphi(\varepsilon)$ for $t \in \langle t_0, t_0 + Ts\rangle$ and $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$. Since the solutions of (2) are uniformly exponentially stable we obtain

$$\|y^*(t_0 + Ts) - \bar{y}(t_0 + Ts)\| \leqq \beta\|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\|$$

and

$$\|\bar{x}_\varepsilon(t_0 + Ts) - \bar{y}(t_0 + Ts)\| \leqq \varphi(\varepsilon) + \beta\|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\|.$$

Since $\bar{x}_\varepsilon(t)$, $\bar{y}(t)$ are periodic, $\|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\| \leqq \varphi(\varepsilon) + \beta\|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\|$, i.e.

$$\|\bar{x}_\varepsilon(t_0) - \bar{y}(t_0)\| \leqq \frac{\varphi(\varepsilon)}{1 - \beta}.$$

Since the theory of Ito stochastic equations and the theory of parabolic equations are closely related, it is possible to formulate a certain statement about periodic solutions of parabolic differential equations.

**Corollary.** *Let $a(t, x, \varepsilon)$ and $B(t, q, \varepsilon)$ fulfil the assumptions from Theorem 1 and let some assumptions be satisfied such that the parabolic equations*

$$(3') \qquad \frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j} \frac{\partial^2\left[\sum_k B_{ik}(t, y, \varepsilon)\, B_{jk}(t, y, \varepsilon) u\right]}{\partial y_i\, \partial y_j} - \sum_n \frac{\partial[a_i(t,y,\varepsilon)u]}{\partial y_i}$$

*have fundamental solutions (see for example [3], [4], [5]), then for a sufficiently small $\varepsilon$ the parabolic equations (3') have periodic solutions, initial values of which are $\alpha f_0(y)$ where $\alpha$ are real numbers and $f_0(y)$ is a function (which has continuous* *second derivatives and depends on* $\varepsilon$). *These solutions are relatively asymptotically stable in the sense that for every solution* $u(t, y; f_1(y))$ *of* (3′) *with initial value* $f_1(y)$ *and* $\int |y|^2 |f_1(y)| \, dy < \infty$, $\int f_1(y) \, dy = \alpha$ *we have*

$$\lim_{t \to \infty} \int_{\lambda_1}^{\mu_1} \dots \int_{\lambda_n}^{\mu_n} (u(t, y; f_1(y)) - u(t, y; f_0(y))) \, dy_1 \dots dy_n = 0$$

*uniformly with respect to* $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n$.

The equation (3′) has a fundamental solution $p(s, x; t, y)$ and with respect of Remark 5,25 from [6] $\int_G p(s, x; t, y) \, dy$ is a transition function of a Markov process having the differential operator

$$\tfrac{1}{2} \sum_{i,k,j} B_{ik}(t, x, \varepsilon) \, B_{jk}(t, x, \varepsilon) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i a_i(t, x, \varepsilon) \frac{\partial u}{\partial x_i},$$

and which is the solution of stochastic equation

$$x(t) = x_0 + \int_0^t a(\tau, x(\tau), \varepsilon) \, d\tau + \int_0^t B(\tau, x(\tau), \varepsilon) \, dw(\tau), \tag{4}$$

where $w(t)$ is the Wiener process [6].

Let $x(t, x_0(\omega))$ be the solution of (4) with the initial value $x_0(\omega)$, where $F_0(\lambda)$ is the distribution function of $x_0(\omega)$. We can express the distribution function of $x(t, x_0(\omega))$ by

$$P(x(t, x_0(\omega)) \leqq \lambda) = \int_{-\infty}^{\lambda} \int_{-\infty}^{\infty} p(0, x; t, y) \, dF_0(x) \, dy.$$

By Theorem 2 there exists a periodic solution $\bar{x}(t)$ with the distribution function $F^*(t, \lambda)$. We have

$$\int_{-\infty}^{\lambda} \int_{-\infty}^{\infty} p(0, x; T, y) \, dF^*(0, x) \, dy = F^*(0, \lambda).$$

Obviously, $F^*(0, \lambda)$ has a continuous density which we denote by $f^*(\lambda)$ and for which $\int_{-\infty}^{\infty} p(0, x; T, y) f^*(x) \, dx = f^*(y)$. This means that $\int_{-\infty}^{\infty} p(0, x; t, y) f^*(x) \, dx$ is a periodic solution of (3′).

Let $f_1(y)$ be nonnegative and $\int |y|^2 f_1(y) \, dy < \infty$, $\int f_1(y) \, dy = 1$. Let $x_1(\omega)$ be a random value with density $f_1(y)$ and let $x_0(\omega)$ be a random value with density $f^*(y)$. If we use the random values just constructed in (4,1), we obtain in a similar manner as by (4,2),

$$F^*(\lambda - \Theta' \gamma e) - \Theta' \gamma \leqq F(lT, \lambda) \leqq F^*(\lambda + \Theta' \gamma e) + \Theta' \gamma,$$

where $e = [1, 1, \dots, 1]$, from this we can easily prove the statement of the Corollary for $\alpha = 1$. Other $f_1(y)$, for which $\int |y|^2 |f_1(y)| \, dy < \infty$ are linear combinations of nonnegative initial functions and since equation (3') is linear, the statement holds for such $f_1(y)$, too. The periodic solution $u(t, y; f^*(y))$ depends on $\varepsilon$ and if $\bar{y}(t)$ is the periodic solution of (2), then the last statement of Theorem 2 yields $\int |y - \bar{y}(t)|^2 \,.$ $. \, u(t, y; f^*(y)) \, dy \to 0$ as $\varepsilon \to 0$.

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