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GREEN'S RELATIONS ON A PERIODIC SEMIGROUP

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1. INTRODUCTION

A semigroup S is called *periodic* if each element of S has a finite order, where the order of $x \in S$ is the order of the cyclic subsemigroup of S generated by x. This type of semigroup has been treated only occasionally in the literature, essential contributions having been made by S. SCHWARZ in [4] and M. YAMADA in [5].

The fact that for each element x of a periodic semigroup S some power of x is idempotent (ex. 1 p. 120 of [1]) leads to defining a natural (equivalence) relation $\mathscr{K}$ on S by: for $a, b \in S$, $a\mathscr{K}b$ if and only if there exists an idempotent e and integers m, n such that $a^n = b^m = e$. The $\mathscr{K}$-classes of S will be denoted by K^e, e idempotent. For each $\mathscr{K}$-class K^e, the notions of maximal subgroup G^e and maximal semigroups contained in K^e were first studied by Schwarz. As a consequnece of his work we have the following facts about S:

(I) for each idempotent e, $ex = xe$ for each $x \in K^e$, and $eK^e = K^e e = G^e = \{x \in K^e \mid ex = xe = x\}$ is the maximal subgroup of S containing e; and

(II) S is a union of groups if and only if each element of S has index one if and only if for each idempotent e, $K^e = G^e$.

The five Green relations $\mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{J}, \mathscr{H}$ were defined in [2] for an arbitrary semigroup and have become a familiar tool among semigroupers. Their definitions, fundamental properties and an intuitive discussion can be found in Chapter 2 of [1]. If S is a semigroup and $x \in S$, then $L_x[R_x, D_x, J_x, H_x]$ will denote the $\mathscr{L}[\mathscr{R}, \mathscr{D}, \mathscr{J}, \mathscr{H}]$-class of S containing x. Of particular interest here are the results: if S is a periodic semigroup, then

(I) for each idempotent e, $H_e = G^e$ (see ex. 1 p. 61 of [1]); and

(II) $\mathscr{D} = \mathscr{J}$ (Theorem 3 of [2]).

It is the purpose of this paper to determine necessary and sufficient conditions on a periodic semigroup S in order that $\mathscr{K}$ coincide with any one of the Green relations. In section 2, where Theorem 2.1 is actually the key to all results, we characterize S

when $\mathscr{K} = \mathscr{D}$ (Theorem 2.9). In section 3 conditions parallel to those in section 2 are found for the cases $\mathscr{K} = \mathscr{L}$ and $\mathscr{K} = \mathscr{R}$ (Theorems 3.5 and 3.6 respectively). Finally, the easy case $\mathscr{K} = \mathscr{H}$ is mentioned for completeness (Theorem 3.8).

Throughout this paper all unexplained notation and undefined terms are those of [1].

2. THE NATURAL EQUIVALENCE AND $\mathscr{D}$

The first theorem provides a steppingstone to results involving $\mathscr{K}$ and each of the Green's relations.

2.1. Theorem. *For each idempotent e in a periodic semigroup, $K^e \cap D_e = G^e$.*

Proof. Since $G^e \subseteq K^e$ and $G^e = H_e \subseteq D_e$, it is immediate that $G^e \subseteq K^e \cap D_e$.

Conversely, let $x \in K^e \cap D_e$; then $ex = xe \in G^e \subseteq D_e$, so $x\mathscr{D}ex$ and $x = pexq$ for some $p, q \in S^1$. This will allow us to express x as sandwiched between two idempotents. That is, since S is periodic, there exist idempotents f, h and positive integers m, n such that $(pe)^m = f$ and $q^n = h$; therefore

$$x = pexq = (pe)^2\, xq^2 = \ldots = (pe)^{mn}\, xq^{mn} = fxh\,.$$

Noting that $fe = (pe)^m\, e = f$, we have

$$S^1ex \subseteq S^1x = S^1fxh = S^1fexh \subseteq S^1exh\,.$$

But it is easily checked that $S^1exh = S^1ex$, hence $S^1ex = S^1x = S^1exh$. This shows that $x\mathscr{L}ex$.

Dually,

$$xeS^1 \subseteq xS^1 = fxhS^1 \subseteq fxS^1 = fexS^1 = fxeS^1\,.$$

Now since $fx = f$ it follows that $fxeS^1 = xeS^1$, so that $xeS^1 = xS^1 = fxeS^1$, and $x\mathscr{R}ex$. Therefore, $x \in L_{ex} \cap R_{ex} = H_{ex} = H_e = G^e$.

2.2. Corollary. *For each idempotent e in a periodic semigroup, $K^e \cap L_e = G^e$ and $K^e \cap R_e = G^e$.*

Proof. Again $G^e \subseteq K^e$ and $G^e = H_e \subseteq L_e$, so it is immediate that $G^e \subseteq K^e \cap L_e$. From the preceding theorem, $K^e \cap L_e \subseteq K^e \cap D_e = G^e$.

Similarly, one shows $K^e \cap R_e = G^e$.

2.3. Corollary. *Periodic semigroup S is a union of groups if and only if $\mathscr{K} \subseteq \mathscr{D}$.*

Proof. If S is a union of groups, then for each idempotent e, $K^e = G^e = H_e \subseteq D_e$.

On the other hand, if $\mathscr{K} \subseteq \mathscr{D}$, then for each idempotent e, $K^e = K^e \cap D_e = G^e$. Hence, S is a union of its maximal subgroups.

2.4 Definition (6.4 of [3]). Semigroup S is *weakly commutative* if for each $a, b \in S$ there exist $x, y \in S$ and an integer k such that $(ab)^k = xa = by$.

2.5 Definition. Semigroup S is a *semilattice of semigroups of type* α if S is a disjoint union of semigroups of type $\alpha\{S_i \mid i \in I, I \text{ index set}\}$, and for each $i, j \in I$ there exists a $k \in I$ such that $S_iS_j \subseteq S_k$ and $S_jS_i \subseteq S_k$.

The minimal semilattice congruence $\mathcal{N}$ of an arbitrary semigroup has been determined modulo prime ideals in [3], in which it is also proved (Theorem 6.7) that a periodic semigroup is weakly commutative if and only if the $\mathcal{N}$ and $\mathcal{K}$ relations coincide. The next proposition generalizes Theorem 14 of [4], which deals with the commutative case.

2.6. Proposition. *If S is a weakly commutative periodic semigroup, then $\mathcal{D} \subseteq \subseteq \mathcal{K} = \mathcal{N}$ and each maximal subgroup is a $\mathcal{D}$-class of S.*

Proof. From the characterization given in [3] of the minimal semilattice congruence $\mathcal{N}$, it follows that $\mathcal{J} \subseteq \mathcal{N}$ for an arbitrary semigroup. In particular, if S is a weakly commutative periodic semigroup, then $\mathcal{D} = \mathcal{J} \subseteq \mathcal{N} = \mathcal{K}$. Hence, each $\mathcal{K}$-class is a subsemigroup of S and a union of $\mathcal{D}$-classes.

Moreover, combining $\mathcal{D} \subseteq \mathcal{K}$ with Theorem 2.1 above yields $G^e = K_e \cap D_e = D_e$.

2.7 Definition (p. 1 of [5]). Semigroup S is said to satisfy *Condition C* if for each $a, b \in S$ and any positive integers m, n there exist positive integers r, s and t such that

$$(ab)^r = (a^m b^n)^s = (b^n a^m)^t .$$

2.8 Lemma. *Periodic semigroup S is weakly commutative if and only if it satisfies Condition C.*

Proof. If S is weakly commutative, then $\mathcal{N} = \mathcal{K}$ and S is a semilattice of unipotent homogroups; namely, the $\mathcal{K}$-classes of S. By Theorem 3 of [5], S satisfies Condition C.

Conversely, let S satisfy Condition C. For $a, b \in S$ there exist integers r, t such that $(ab)^r = (ba)^t$. Hence,

$$(ab)^r = b[a(ba)^{t-1}] = [(ba)^{t-1} b] a .$$

2.9 Theorem. *The following are equivalent on a periodic semigroup S:*

(I) *S is a semilattice of groups*

(II) *S is a union of groups and weakly commutative*

(III) $\mathcal{K} = \mathcal{D}$.

Proof. (I) implies (II). Since S is a semilattice of (periodic) groups, then it is trivially both a union of groups and a semilattice of unipotent homogroups. Due to

Theorem 3 of [5], S satisfies Condition C, and by Lemma 2.8 it is weakly commutative.

(II) implies (III). If S is a union of groups, then for each idempotent e, $K^e = G^e = H_e$; thus $\mathscr{K} = \mathscr{H}$. Combining this with the afore-mentioned results of [3], we get $\mathscr{D} = \mathscr{J} \subseteq \mathscr{N} = \mathscr{K} = \mathscr{H}$. However, it follows from the definitions of the Green's relations that $\mathscr{H} \subseteq \mathscr{L} \subseteq \mathscr{D}$ and $\mathscr{H} \subseteq \mathscr{R} \subseteq \mathscr{D}$. So all of the Green's relations coincide with $\mathscr{K}$.

(III) implies (I). From Corollary 2.3, $\mathscr{K} \subseteq \mathscr{D}$ implies that S is a union of groups; that is, each $\mathscr{K}$-class is a maximal group. By Theorem 4.6 of [1], henceforth referred to as Clifford's Theorem, S is a semilattice of $\mathscr{D} = \mathscr{K}$-classes.

3. THE NATURAL EQUIVALENCE AND $\mathscr{L}, \mathscr{R}, \mathscr{H}$

We now proceed by a series of lemmas and definitions towards necessary and sufficient conditions that $\mathscr{K} = \mathscr{L}$. These conditions will be seen to parallel those above for the case $\mathscr{K} = \mathscr{D}$.

3.1 Lemma. *For an arbitrary semigroup S, any $\mathscr{R}$-class R which is a union of groups is a right group.*

Proof. It is well-known that the set of idempotents of any $\mathscr{R}$-class forms a right zero semigroup, and for any $\mathscr{D}$-related idempotents e, f, groups H_e and H_f are isomorphic. Thus, if E is the set of idempotents of R, and H is a fixed $\mathscr{H}$-class contained in R, it is easily established that $R \cong ExH$. In light of Theorem 1.27 of [1], R is a right group.

3.2 Lemma. *If an arbitrary semigroup S is a semilattice of right groups, then $\mathscr{N} = \mathscr{J}$ and each of these right groups is a $\mathscr{J}$-class of S.*

Proof. Let $\mathscr{M}$ be a semilattice congruence on S such that each $\mathscr{M}$-class is a right group. Now a right group is completely simple, thus S is a union of completely simple semigroups. By Clifford's Theorem, S is a semilattice of $\mathscr{J}$-classes; hence, $\mathscr{N} \subseteq \mathscr{J}$. However, we have already remarked that $\mathscr{J} \subseteq \mathscr{N}$ in any semigroup; thus, $\mathscr{N} = \mathscr{J}$.

Also, $\mathscr{N} \subseteq \mathscr{M}$, where each $\mathscr{M}$-class is a right group. Recalling the characterization of a right group given in the preceding lemma, let $M \cong ExG$ be an arbitrary $\mathscr{M}$-class of S. For any $(e, a), (f, b) \in M$ it is easily verified that $S^1(e, a) S^1 = S^1(f, b) S^1$. That is, $x\mathscr{M}y$ implies $x\mathscr{J}y$; or $\mathscr{M} \subseteq \mathscr{J} = \mathscr{N}$. So, $\mathscr{M} = \mathscr{N} = \mathscr{J}$ and each right group is a $\mathscr{J}$-class of S.

3.3 Definition. Semigroup S is *left* [*right*] *weakly commutative* if for each $a, b \in S$ there exist $x \in S$ and an integer k such that

$$(ab)^k = bx \qquad [(ab)^k = xa] .$$

3.4 Lemma. *Let a periodic semigroup S be a union of groups. Then S is left weakly commutative if and only if $\mathscr{R} = \mathscr{D}$.*

Proof. Suppose S is left weakly commutative. Let $a, b \in S$ and $a\mathscr{L}b$. Then $S^1a = S^1b$ and there exist $x, y \in S^1$ such that $a = xb$, $b = ya$. Now the hypothesis that S is left weakly commutative implies there exist $z \in S$ and an integer k such that $(xb)^k = bz$. Also, the index of a is one, for a is in some maximal group, so choose integer $n > k$ such that $a^n = a$. Then

$$a = a^n = (xb)^n = (xb)^k (xb)^{n-k} = bz(xb)^{n-k} \in bS^1 .$$

Similarly, one shows $b \in aS^1$. Hence, $a\mathscr{R}b$ and $\mathscr{L} \subseteq \mathscr{R}$. Therefore, $\mathscr{D} = \mathscr{L} \circ \mathscr{R} \subseteq \mathscr{R} \circ \mathscr{R} = \mathscr{R}$, and $\mathscr{R} = \mathscr{D}$.

Conversely, suppose $\mathscr{D} = \mathscr{R}$. By Clifford's Theorem S is a semilattice of $\mathscr{R}$-classes. So for any $a, b \in S$, $ab\mathscr{R}ba$. Hence, $abS^1 = baS^1$ and there exists an $x \in S^1$ such that $ab = b(ax)$.

3.5 Theorem. *The following are equivalent on a periodic semigroup S:*

(I) *S is a semilattice of right groups*

(II) *S is a union of groups and left weakly commutative*

(III) $\mathscr{K} = \mathscr{L}$.

Proof. (I) implies (II). From Lemma 3.2, $\mathscr{N} = \mathscr{J} = \mathscr{D}$ and each of these right groups is a $\mathscr{D}$-class of S. Again, a right group is completely simple, so each $\mathscr{D}$-class (and likewise S) is a union of groups.

To complete the proof it suffices, due to Lemma 3.4, to show that $\mathscr{R} = \mathscr{D}$. To see this let L_e be an $\mathscr{L}$-class of S. We know that L_e is a union of $\mathscr{H}$-classes, each a maximal subgroup of S. Let H_f be any $\mathscr{H}$-class contained in L_e. The idempotents of any $\mathscr{L}$-class form a left-zero semigroup, so $ef = e$. However, the idempotents of right group D_e form a right-zero semigroup, so $ef = f$. That is, $e = ef = f$; hence, $L_e = H_e$. But L_e was arbitrary, so $\mathscr{L} = \mathscr{H}$. So, $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{H} \circ \mathscr{R} \subseteq \mathscr{R}$, and $\mathscr{R} = \mathscr{D}$.

(II) implies (III). Since S is a union of groups, then each $\mathscr{K}$-class is a maximal group and $\mathscr{K} = \mathscr{H}$. Also, by Lemma 3.4 $\mathscr{R} = \mathscr{D}$, thus $\mathscr{K} = \mathscr{H} = \mathscr{L} \cap \mathscr{R} = \mathscr{L} \cap \mathscr{D} = \mathscr{L}$.

(III) implies (I). Using Corollary 2.2, for any idempotent e, $K^e = L_e = K^e \cap L_e = G^e = H_e$. This implies that S is a union of groups and $\mathscr{L} = \mathscr{H}$; therefore, $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{H} \circ \mathscr{R} \subseteq \mathscr{R}$ and $\mathscr{R} = \mathscr{D}$.

Once more from Clifford's Theorem we infer that S is a semilattice of $\mathscr{R}$-classes. In this case each $\mathscr{R}$-class is a union of groups, so the result follows from Lemma 3.1.

Lemmas dual to those in 3.1, 3.2 and 3.4 yield the analog:

3.6 Theorem. *The following are equivalent on a periodic semigroup S:*

(I) *S is a semilattice of left groups*

(II) *S is a union of groups and right weakly commutative*

(III) $\mathscr{K} = \mathscr{R}$.

3.7 Proposition. *Let periodic semigroup S be a union of groups. Then all Green's relations on S coincide if and only if S is weakly commutative.*

Proof. If $\mathscr{H} = \mathscr{L} = \mathscr{R} = \mathscr{D}$, then by Lemma 3.4 and its dual, S is both right and left weakly commutative. To see that S is weakly commutative, let $a, b \in S$. Then there exist $x, y \in S$ and integers r, s such that $(ab)^r = bx, (ab)^s = ya$. Hence, $(ab)^{rs} = [(ab)^r]^s = (bx)^s = b[x(bx)^{s-1}]$ and $(ab)^{rs} = [(ab)^s]^r = (ya)^r = [(ya)^{r-1} y] a$.

On the other hand, a weakly commutative semigroup is trivially both right and left weakly commutative; so again by Lemma 3.4 and its dual, $\mathscr{R} = \mathscr{D} = \mathscr{L}$. By definition, $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$, and we know $\mathscr{D} = \mathscr{J}$ in the periodic case.

For the sake of completeness in our discussion of $\mathscr{K}$ and the Green's relations, we conclude with necessary and sufficient eonditions that $\mathscr{K} = \mathscr{H}$. Their proof is immediate from the introductory comments and Clifford's Theorem.

3.8 Theorem. *The following are equivalent on a periodic semigroup S:*

(I) *S is a semilattice of completely simple semigroups*

(II) *S is a union of groups*

(III) $\mathscr{K} = \mathscr{H}$.

References

[1] *Clifford, A. H.*, and *Preston, G. B.:* "The Algebraic Theory of Semigroups", Vol. *1*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

[2] *Green, J. A.:* "On the structure of semigroups", Annals of Math., *54* (1951), 163—172.

[3] *Petrich, M.:* "The maximal semilattice decomposition of a semigroup," Math. Zeitschr., *85* (1964), 68—82.

[4] *Schwarz, Š.:* "Contribution to the theory of periodic semigroups", Czech. Math. J., *3* (1953), 7—21, (Russian, English summary).

[5] *Yamada, M.:* "A remark on periodic semigroups", Sci. Rep. Shimane Univ. *9* (1959), 1—5.

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