Zdeněk Frolík

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A SURVEY OF SEPARABLE DESCRIPTIVE THEORY OF SETS AND SPACES

ZDENĚK FROLÍK, Praha[1])

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This is a survey of the theory of analytic and various Borel-like (Lusinian like) sets and spaces as developed in the last decade.

CONTENT

[1]) This is a revised edition of the author's lectures at the University of Bari in 1968.

1. NOTATION AND TERMINOLOGY

1.1. A relation is a class of ordered pairs; $\mathbf{D}\varrho$ and $\mathbf{E}\varrho$ stand for the domain and the range of ϱ. A family is a single-valued relation where $\mathbf{D}\varrho$ is a set; it is denoted by symbols like $\{X_a \mid a \in A\}$, $\{X_a\}$, or $\{a \to X_a \mid a \in A\}$, $\{a \to X_a\}$. The set of all elements with property P is denoted by $\mathbf{E}\{x \mid P(x)\}$. If f and g are two relations then $f < g$ means that f is a restriction of g.

1.2. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are collections of sets we denote by $[\mathcal{M}_1] \cap [\mathcal{M}_2]$ the set of all $M_1 \cap M_2$ with M_i in $\mathcal{M}_i$ and similarly for $\cup$.

1.3. Let $\mathcal{M}$ be a collection of sets. The set consisting of the intersections (unions) of all countable families in $\mathcal{M}$ is denoted by $\mathcal{M}_\delta$ ($\mathcal{M}_\sigma$, resp.). The set consisting of the unions of all countable disjoint families in $\mathcal{M}$ is denoted by $\mathcal{M}_{\sigma d}$. We denote by $\mathbf{B}(\mathcal{M})$ the smallest collection $\mathcal{N} \supset \mathcal{M}$ with the property $\mathcal{N}_\sigma = \mathcal{N}_\delta = \mathcal{N}$, and we call the elements of $\mathbf{B}(\mathcal{M})$ the Borel-$\mathcal{M}$ sets. The smallest $\mathcal{N} \supset \mathcal{M}$ with $\mathcal{N}_\delta = \mathcal{N}_{\sigma d} = \mathcal{N}$ is denoted by $\mathbf{B}_d(\mathcal{M})$.

The following result is easy, however very useful.

Proposition. *Let $\mathcal{M}$ be a collection of sets. Then*

$$\mathbf{B}(\mathcal{M}) = \bigcup\{\mathbf{B}(\mathcal{N}) \mid \mathcal{N} \subset \mathcal{M}, \mathcal{N} \text{ countable}\}$$
$$\mathbf{B}_d(\mathcal{M}) = \bigcup\{\mathbf{B}_d(\mathcal{N}) \mid \mathcal{N} \subset \mathcal{M}, \mathcal{N} \text{ countable}\}\,.$$

The next result is of crucial importance. For the classical proof based on the Borel classification see Kuratowski [1, p. 259].

Theorem. *Let $\mathcal{M}$ be a collection of subsets of a set P. If* $\operatorname{compl}_P \mathcal{M} \subset \mathbf{B}_d(\mathcal{M})$, *then*

$$\mathbf{B}_d(\mathcal{M}) = \mathbf{B}(\mathcal{M}) = \mathbf{E}\{X \mid X \in \mathbf{B}_d(\mathcal{M}),\ P - X \in \mathbf{B}_d(\mathcal{M})\}$$

(and $\mathbf{B}_d(\mathcal{M})$ is an σ-algebra).

Proof. Put

$$\mathcal{C} = \mathbf{E}\{X \mid X \in \mathbf{B}_d(\mathcal{M}),\ P - X \in \mathbf{B}_d(\mathcal{M})\}$$

Clearly:

$$\mathcal{M} \subset \mathcal{C} \subset \mathbf{B}_d(\mathcal{M}) \subset \mathbf{B}(\mathcal{M})\,,$$

and hence it is enough to show that $\mathbf{B}(\mathcal{C}) \subset \mathcal{C}$. First observe that if $X_1, X_2 \in \mathcal{C}$, then $X_1 \cap X_2 \in \mathbf{B}_d(\mathcal{M})$ and $P - (X_1 \cap X_2) = ((P - X_1) \cap X_2) \cup (X_1 \cap (P - X_2)) \cup$ $\cup ((P - X_1) \cap (P - X_2)) \in \mathbf{B}_d(\mathcal{M})$, and hence $X_1 \cap X_2 \in \mathcal{C}$; thus $\mathcal{C}$ is closed under taking finite intersections. From this fact it follows that $\mathscr{C}$ is closed under finite unions and that the differences of elements of $\mathscr{C}$ belong to $\mathscr{C}$ because $X_1 - X_2 = X_1 \cap (P - X_2)$. Now if $\{C_n\}$ is a sequence in $\mathscr{C}$, then $\{B_n\}$ with $B_n = C_n - \bigcup\{C_k \mid k < n\}$ is a disjoint sequence in $\mathscr{C}$, and hence $\bigcup\{C_n\} = \bigcup\{B_n\} \in \mathbf{B}_d(\mathscr{M})$; since $P - \bigcup\{C_n\} = \bigcap\{P - C_n\} \in \mathbf{B}_d(\mathscr{M})$, we get that $\bigcup\{C_n\} \in \mathscr{C}$. Hence $\mathscr{C}_\sigma \subset \mathscr{C}$. Since compl $(\mathscr{C}) \subset \mathscr{C}$, we get also $\mathscr{C}_\delta \subset \mathscr{C}$. The proof is complete.

1.4. Baire sets. A zero set (another term: an exact closed set) in a space P is a set of the form $\mathbf{Z}(f) = \mathbf{E}\{x \mid fx = 0\}$ where f is a continuous real-valued function on P; the collection of all zero sets in P is denoted by zero (P). The complements of the zero sets are called cozero sets (another term: exact open sets), and the collection of all these sets is denoted by cozero (P). Since clearly zero $(P) \subset (\text{cozero } (P))_\delta$, it follows from Theorem 1.3 that

$$\mathbf{B}(\text{zero } (P)) = \mathbf{B}_d(\text{cozero } (P))\,;$$

the elements of $\mathbf{B}(\text{zero } (P))$ are called the Baire sets in P. It should be remarked that

$$\mathbf{B}_d(\text{zero } (P)) \neq \mathbf{B}(\text{zero } (P))$$

in general, e.g. if P is the closed unit interval of reals. The Baire sets in a space P form a σ-algebra, that means in particular, the complements of Baire sets are Baire sets. The following theorem will be needed:

Theorem. *If $\mathscr{B}$ is a countable collection of Baire sets in a space P, then there exists a continuous mapping f of P into a separable metrizable space M such that $B = f^{-1}[f[B]]$ for each B in $\mathscr{B}$, and each $f[B]$ is a Baire set in $\mathbf{E}f$. (The space M can be taken to be the Hilbert cube.)*

Proof. Choose a countable collection $\mathscr{F}$ of zero sets in P such that $\mathscr{B} \subset \mathbf{B}(\mathscr{F})$. Choose continuous functions f_F $(0 \leqq f_F \leqq 1$ if you want$)$ with $\mathbf{Z}(f_F) = F$ for F in $\mathscr{F}$, and consider the reduced product f of P into $\mathrm{R}^{\mathscr{F}}$ (defined by $fx = \{f_F x \mid F \in \mathscr{F}\}$). For each F in $\mathscr{F}$ let $F' = \mathbf{E}\{y \mid y \in \mathrm{R}^{\mathscr{F}}, f_F y = 0$; consider the set $\mathscr{F}'$ of all F', $F \in \mathscr{F}$, and finally $\mathbf{B}(\mathscr{F}')$.

1.5. Distinguishable sets. A set X in a space P is called distinguishable if there exists a continuous mapping of P into a separable metrizable space such that $f[X] \cap f[P - X] = \emptyset$. It is easy to see that a set $X \subset P$ is distinguishable in P if and only if there exists a countable set $\mathscr{F}$ of bounded continuous functions on P such that for each x in X and y in $P - X$ there exists an f in $\mathscr{F}$ such that $fx \neq fy$.

Theorem. *The set of all distinguishable sets in a space P form a σ-algebra containing the Baire sets in P. If $\mathscr{D}$ is a countable collection of distinguishable sets in P then there exists a continuous mapping into a separable metrizable space* *such that* $f[D] \cap f[P - D] = \emptyset$ *for each* D *in* $\mathscr{D}$. *For the metrizable space one can take the Hilbert cube.*

Proposition. *If a compact set* X *in a space* P *is distinguishable in* P *then* X *is a zero set in* P (*in particular,* X *is closed*).

Proof. Choose a continuous mapping f of P into a metrizable space M such that $X = f^{-1}[Y]$ where $Y = f[X]$. The set Y is compact, hence closed in M. Since M is metrizable, Y is a zero set, and hence X is a zero set.

1.6. Borel-closed and Borel-open sets. The elements of $\mathbf{B}(\text{closed}\,(P))$ are called Borel-closed sets in P, and the Borel-open sets in P are the elements of $\mathbf{B}(\text{open}\,(P))$. In general

$$\mathbf{B}(\text{closed}\,(P)) \neq \mathbf{B}(\text{open}\,(P)),$$

and X is Borel-open if and only if $P - X$ is Borel-closed. If P is perfectly normal (in particular, if P is metrizable), then zero (P) = closed (P), and hence Baire sets, Borel-closed, and Borel-open sets coincide.

1.7. The space Σ of irrationals. Denote by N the set and the discrete space of the natural numbers, by S the set of all finite sequences in N, and by Σ the set of all infinite sequences in N. Thus $\Sigma = \mathsf{N}^{\mathsf{N}}$. Endow Σ with the product topology, i.e. the topology of pointwise convergence. For s in S put

$$\Sigma s = \mathbf{E}\{\sigma \mid \sigma \in \Sigma,\ s < \sigma\}.$$

Clearly $\{\Sigma s\}$ is a base for open sets in Σ, and $s < t$ if and only if $\Sigma s \supset \Sigma t$. The space Σ is known to be homeomorphic with the space of all irrational numbers on the real line. For the further use denote by S_n the set of all elements of S of length n, $n = 1, 2, \ldots$ For σ in Σ we denote by σ_n the only element s in S_n with $s < \sigma$.

Remark. The fact that Σ is homeomorphic with the irrationals follows from the following theorem of Mazurkiewicz:

If P is a separable completely metrizable space such that

(a) *no non-void open set is compact, and*

(b) *the closed-open sets form an open base for P, then P is homeomorphic with Σ.*

The proof is very simple. Take a complete metric for P, and observe that for every $\varepsilon > 0$, each open non-void set can be written as an infinite disjoint union of closed-open sets of diameter less than ε. Using this fact, one can construct disjoint open covers $\{Ps \mid s \in \mathsf{S}_n\}$ by sets of diameter less than $1/n$ such that $Ps < Pt$ if $t < s$. Let $h : \Sigma \to P$ assign to each σ the only point of $\bigcap\{Ps \mid s < \sigma\}$. Clearly $h[\Sigma s] = Ps$, and hence h is a homeomorphism.

1.8. Souslin sets. Let $\mathcal{M}$ be a collection of sets. A Souslin family in $\mathcal{M}$ is a single-valued relation M with $\mathbf{D}M = \mathsf{S}$, and $\mathbf{E}M \subset \mathcal{M}$. The Souslin set of M is the set

$$\mathbf{S}M = \bigcup\{\bigcap\{Ms \mid s < \sigma\} \mid \sigma \in \Sigma\} .$$

The relation associated with M, denoted by $\tilde{M}$, is the set of all $\langle\sigma, y\rangle$ with $\sigma \in \Sigma$, and $y \in Ms$ for all $s < \sigma$; clearly $\mathbf{E}\tilde{M} = \mathbf{S}M$. Denote by $\mathbf{S}(\mathcal{M})$ the collection of all $\mathbf{S}M$ with $M : \mathsf{S} \to \mathcal{M}$; the elements of $\mathbf{S}(\mathcal{M})$ are called the Souslin sets derived from $\mathcal{M}$, or simply Souslin $\mathcal{M}$-sets. It is elementary that

$$\mathbf{B}(\mathbf{S}(\mathcal{M})) \subset \mathbf{S}(\mathcal{M}) .$$

It can be proved (see Section 14)

$$\mathbf{S}(\mathbf{S}(\mathcal{M})) = \mathbf{S}(\mathcal{M}) .$$

Denote by $\mathbf{S}_d(\mathcal{M})$ the set of all $\mathsf{S}M$ with $(\tilde{M})^{-1}$ single-valued. It is easy to show

$$\mathbf{B}_d(\mathbf{S}_d(\mathcal{M})) = \mathbf{S}_d(\mathcal{M}) ,$$

and it can be proved (see Section 14)

$$\mathbf{S}_d(\mathbf{S}_d(\mathcal{M})) = \mathbf{S}_d(\mathcal{M}) .$$

The elements of $\mathbf{S}(\text{closed}\,(P))$ are called the Souslin sets in P, and the elements of $\mathbf{S}_d(\text{closed}\,(P))$ are called the sets with a disjoint Souslin representation. In this case the idempotency of $\mathbf{S}$ and $\mathbf{S}_d$ will be proved in Sections 4 and 6.

1.9. A connection between $\mathbf{B}$ and $\mathbf{S}$. Sometimes it is convenient to describe Borel-$\mathcal{M}$ sets by means of the Souslin operation. The following result will be needed in Section 11.

Theorem (Frolík [9] and [12]). *Let $\mathcal{M}$ be a collection of subsets of a set P, and let* compl $\mathcal{M}$ *consist of complements in P of sets in $\mathcal{M}$. Then $X \subset P$ is Borel-$\mathcal{M}$ if and only if there exist*

$$F : \mathsf{S} \to \mathcal{M} , \quad \textit{and} \quad G : \mathsf{S} \to \text{compl}\,\mathcal{M}$$

such that

$$X = \mathbf{S}F , \quad P - X = \mathbf{S}G ,$$

and for each σ and τ in Σ there exists an n with

(*) $$F_{\sigma_n} \cap G_{\tau_n} = \emptyset .$$

Proof. Let $\mathscr{C}$ be the collection of subsets of P satisfying the property described in Theorem. It is routine to show that

$$\mathscr{C}_\sigma = \mathscr{C}_\delta = \mathscr{C} \supset \mathscr{M} ,$$

and hence it is enough to show that $\mathscr{C} \subset \mathbf{B}(\mathscr{M})$. Assume the contrary and take an X in $\mathscr{C} - \mathbf{B}(\mathscr{M})$. Under the assumption in Theorem, there exist $s_1 \in \mathsf{S}_1$ and $t_1 \in \mathsf{S}_1$ such that $\tilde{F}[\Sigma s_1] \subset B \subset P - \tilde{G}[\Sigma t_1]$ for no B in $\mathbf{B}(\mathscr{M})$, and by induction, $s_{n+1} \in \mathsf{S}_{n+1}$, $t_{n+1} \in \mathsf{S}_{n+1}$ such that $s_{n+1} > s_n$, $t_{n+1} > t_n$, and

$$F[\Sigma s_{n+1}] \subset B \subset P - \tilde{G}[\Sigma t_{n+1}]$$

for no B in $\mathbf{B}(\mathscr{M})$. Pick σ and τ in Σ with $s_n = \sigma_n$ and $t_n = \tau_n$ for each n. Clearly the relation (*) is fulfilled for no n; this contradiction completes the proof.

Remark. If P is an uncountable separable completely metrizable space, then the collection of the Souslin sets is strictly larger than the collection of all Baire sets.

A Souslin family $\{Ms\}$ is said *regular* (or monotonic) if $Ms \subset Mt$ whenever $t < s$. If $\{Ms\}$ is a Souslin family, and if define

$$Ns = \bigcap\{Mt \mid t < s\} ,$$

then N is a regular Souslin family, and

$$\mathbf{S}M = \mathbf{S}N .$$

The family N is called the regularization of M. The following simple observation will be frequently used quite carelessly.

Proposition. *If M is regular, and if all $\{Ms \mid s \in \mathbf{S}_n\}$ are disjoint, then*

$$\mathbf{S}M = \bigcap\{\bigcup\{Ms \mid s \in \in \mathsf{S}_n\} \mid n \in \mathsf{N}\} .$$

1.10. All spaces in this subsection are assumed to be separated (i.e. Hausdorff) and uniformizable, hence completely regular. A compactification of a space P is a compact space K containing P as a dense subspace. Among the all compactifications of P there exists a compactification βP (called a Čech-Stone compactification of P) characterized by each of the following equivalent properties:

(a) If K is a compactification of P then there exists a continuous mapping of βP onto K that is the identity on P.

(b) If f is a continuous mapping of P into a compact space C then a continuous mapping of βX into C extends f;

(c) Condition b with C a compact interval of reals.

There is one useful theorem we shall need.

Theorem. *Let f be a continuous mapping of a compact space K onto a compact space K_1. If the restriction of f to a dense subspace P of K onto $f[P]$ is proper then* $f[K - P] = K_1 - f[P]$.

By a proper mapping we mean a continuous closed mapping such that the preimages of points are compact.

Proof. Let $y \in f[P]$, $K_2 = P \cap f^{-1}[y]$, $x \in K - P$. Choose a closed neighborhood U of x in K with $U \cap K_2 = \emptyset$, and put $F = U \cap P$. The mapping $f : P \to f[P]$ is closed and hence $f[F]$ is closed in $f[P]$; thus $\mathrm{cl}_{K_1} f[F] \cap f[P] = f[F]$. Since $y \notin f[F]$, and $x \in \mathrm{cl}\, F$, hence $fx \in \mathrm{cl}_{K_1} F$, necessarily $fx \neq y$.

2. SEPARATION THEOREMS

2.1. Definition. Given a collection $\mathscr{M}$ of sets, two sets X and Y are said *to be separated by sets in* $\mathscr{M}$, or simply $\mathscr{M}$-separated, if there exist $X_1, Y_1 \in \mathscr{M}$ with $X \subset X_1$, $Y \subset Y_1$, and $X_1 \cap Y_1 = \emptyset$.

For example, if P is a space, and if $\mathscr{M}$ is the collection of all open sets in P, then P is called separated (= Hausdorff) if any two distinct singletons are separated, regular if it is T_1 and if each singleton (x) and any closed F disjoint to (x) are separated, normal if any two disjoint closed sets are separated. Note the following important proposition that has been already used in 1.10.

2.2. If P is a separated space then each compact set $K \subset P$ and any closed set disjoint to K are separated, in particular, every compact subspace of P is closed. If P is regular, then any two disjoint closed sets, one of which is compact, are separated by open sets.

2.3. Separation Lemma. *Let $\mathscr{M}$ be an additive and multiplicative collection of subsets of a set P. Assume that $\mathscr{F}$ is a finite collection of subsets of P such that any two disjoint finite intersections of elements of*

$$[\mathscr{F}] \cap [\mathrm{compl}_P(\mathscr{M})]$$

are $\mathscr{M}$-separated.

Then for each $M \subset \bigcap \mathscr{F}$, $M \in \mathscr{M}$, there exists a family $\mathscr{M}_M = \{M_F \mid F \in \mathscr{F}\}$ ranging in $\mathscr{M}$ such that $M_F \supset F$ and

$$\bigcap \mathscr{M}_M \subset M .$$

In particular, if $\bigcap \mathscr{F} = \emptyset$, then $\bigcap \mathscr{M}_\emptyset = \emptyset$.

Proof. By induction, see [F 14], proof of Lemma 1.

In the next section we shall need the following corollary to Separation Lemma.

2.4. *If $\mathscr{F}$ is a finite collection of compact sets in a separated space P, and if U is a neighborhood of $\bigcap\mathscr{F}$, there exist neighborhoods U_F of F, $F \in \mathscr{F}$, such that $\bigcap\{U_F \mid F \in \mathscr{F}\} \subset U$.*

In Section 5 we shall need the following corollary.

2.5. *Let $\mathscr{M}$ be an additive and σ-multiplicative collection of subsets of P (hence $P \in \mathscr{M}$). Assume that $\mathscr{A}$ is a countable collection of sets in P such that any two disjoint finite intersections of elements of $[\mathscr{A}] \cap [\operatorname{compl}(\mathscr{M})]$ are $\mathscr{M}$-separated. Assume that a family*

$$\{M_{\mathscr{F}} \mid \mathscr{F} \subset \mathscr{A},\ \mathscr{F} \text{ finite}\}$$

ranging in $\mathscr{M}$ is given such that $\bigcap\mathscr{F} \subset M_{\mathscr{F}}$ for each $\mathscr{F}$. Then there exists a family

$$\{K_A \mid A \in \mathscr{A}\}$$

ranging in $\mathscr{M}$ such that $K_A \supset A$ for all A in $\mathscr{A}$, and

$$\bigcap\{K_A \mid A \in \mathscr{F}\} \subset M_{\mathscr{F}}$$

for each finite $\mathscr{F} \subset \mathscr{A}$.

3. USCO-COMPACT CORRESPONDENCES

The basic properties of analytic and Souslin sets follow immediately from the results of this section on closed-graph and usco-compact correspondences. For a deeper theory we refer to Frolík [16].

A correspondence f of a space P into a space Q is a triple $\langle \operatorname{gr} f, P, Q\rangle$, usually written $\operatorname{gr} f : P \to Q$, where $\operatorname{gr} f$, the so-called graph of f, is a subset of $P \times Q$. If convenient, we write f instead of $\operatorname{gr} f$, and $\operatorname{gr} f$ instead of f; this convention is commonly used when dealing with mappings. In particular, $\mathbf{D}f$ is used instead of $\mathbf{D} \operatorname{gr} f$, and similarly for $\mathbf{E}f$. If $(\operatorname{gr} f)^{-1}$ is single-valued, then f is called a fibration or a disjoint correspondence.

3.1. Definition. A correspondence $f : P \to Q$ is – *usco* (i.e. upper semi-continuous) if the preimage of each closed set is closed, – *compact* if the values are compact (i.e. the sets $f[(x)]$ are compact), – *usco-compact* if f is usco and compact, – *dusco* if f is disjoint and usco-compact.

3.2. Examples. a) Each continuous mapping is usco-compact. b) The inverse of a proper (= perfect) mapping is dusco-compact. Recall that a mapping $p : P \to Q$ is proper if p is continuous, the preimages of points are compact, and the images of closed sets are closed, see 1.10. In particular, if π is the projection of $K \times P$ into P and K is compact, then π is proper.

3.3. *Let $f : P \to Q$ be usco. Then* $\mathbf{D}f$ *is closed in* P, *and the restriction of* f *to any subspace of* P *is usco, and also the restriction to any closed subspace of* Q *is usco. This is also true for usco-compact etc.*

3.4. *If $f : P \to Q$ is usco-compact and if P is Lindelöf or compact then so is* $\mathbf{E}f$.

Proof.Let $\mathscr{U}$ be a finitely additive open covering of $\mathbf{E}f$. For each U in $\mathscr{U}$ let U' be the set of all $x \in P$ with $f[(x)] \subset U$. Clearly $\{U' \mid U \in \mathscr{U}\}$ is an open cover of P. Now the both statements follow by a straightforward argument.

3.5. *The composite of two usco-compact (dusco-compact) correspondences is usco-compact (dusco-compact, respectively).*

Proof. Use 3.4.

3.6. *If $\{f_a : P_a \to Q_a\}$ are usco-compact then so is f: $\Pi\{P_a\} \to \Pi\{Q_a\}$ defined by $f[(\{x_a\})] = \Pi\{f_a[(x_a)]\}$, and if $P_a = P$ for each a, then so is usco-compact the correspondence $f : P \to \Pi\{Q_a\}$ defined by $f[(x)] = \Pi\{f_a[(x)]\}$.*

Proof is left to the reader, see, e.g., Frolík [16].

3.7. *Let $\{f_a : P_a \to Q\}$ be a family of usco-compact correspondences. Then $f : \Sigma\{P_a\} \to Q$, defined by $\langle\langle a, x\rangle, y\rangle \in f$ if and only if $\langle x, y\rangle \in f_a$, is usco-compact.*

Obvious. Recall that $\Sigma\{P_a\}$ is the set of all $\langle a, x\rangle$, $x \in P_a$, endowed with the finest topology such that all $\{x \to \langle a, x\rangle\} : P_a \to P$ are continuous.

3.8. Theorem. *Let $\{f_a : P_a \to Q\}$ be a family of usco-compact (dusco-compact) correspondences; then $f = \wedge\{f_a\} : \Pi\{P_a\} \to Q$ defined by*

$$f[(\{x_a\})] = \bigcap\{f_a[(x_a)]\}\,,$$

is usco-compact (dusco-compact) provided that Q is separated.

Proof. Let U be an open neighborhood of $f[\{x_a\}]$; clearly there exists a finite set A of indices such that $\bigcap\{f_a[(x_a)] \mid a \in A\} \subset U$. By 2.4 there exist open neighborhoods U_a of $f_a[(x_a)]$, $a \in A$, such that $\bigcap\{U_a \mid a \in A\} \subset U$. Since f_a's are usco, there exist neighborhoods V_a of x_a such that

$$f_a[V_a] \subset U_a\,.$$

Now clearly

$$f[(\{y_a\})] \subset U$$

provided that $y_a \in V_a$ for a in A. This proves that f is usco.

The following result is crucial for the theory of analytic sets.

3.9. Theorem. *Let $\mathscr{B}$ be an open covering of a space P, and for each B in $\mathscr{B}$ let $f_B : P_B \to Q$ be an usco-compact correspondence. Then*

$$f : P \times \Pi\{P_B \mid B \in \mathscr{B}\} \to Q$$

defined by

$$f[(\langle x, \{x_B\}\rangle)] = \bigcap\{f_B[(x_B)] \mid x \in B \in \mathscr{B}\}$$

is usco-compact provided Q is separated.

Proof. Similar to 3.8.

3.10. Definition. A correspondence $f : P \to Q$ is said to be *closed-graph* if the graph of f is a closed set in the product space $P \times Q$.

Proposition. *Each of the following conditions is necessary and sufficient for a correspondence $f : P \to Q$ to be closed-graph:*

a) *If $\langle x, y\rangle \in P \times Q - f$ then $U \times V \cap f = \emptyset$ for some neighborhoods U of x and V of y.*

b) *If $x \in P$, and if $y \in Q - f[(x)]$ then there exists a neighborhood U of x and a neighborhood V of y such that $f[U] \cap V = \emptyset$.*

c) *For any x in P and any local base $\mathscr{U}$ at x we have $\bigcap\{\mathrm{cl}\, f[U] \mid U \in \mathscr{U}\} = f[(x)]$.*

Proof. By definition condition a is necessary and sufficient. Clearly conditions a, b and c are equivalent each to the other one.

As an immediate consequence we get (see 1.8 for the definition of Souslin sets)

Theorem. *A set $X \subset Q$ is a Souslin set in a space Q if and only if $X = \mathrm{E}f$ for some closed-graph correspondence $f : \Sigma \to Q$.*

The relationship between closed-graph and usco-compact correspondences is described in the next theorem.

3.11. Theorem. *An usco-compact correspondence into a separated space is closed graph. A closed graph correspondence into a compact space is usco-compact.*

Proof. Let $f : P \to Q$ be a correspondence. First assume that f is usco-compact and Q is separated, and we are given any x in P and any $y \notin Q - f[(x)]$. Since Q is separated the compact sets $f(x)$ and (y) are separated, i.e. there exist disjoint open sets $W \supset f[(x)]$ and $V \supset (x)$. Since f is usco, there exists a neighborhood U of x in P such that $f[U] \subset W$. Thus

$$f[U] \cap V = \emptyset\,.$$

By Proposition 3.10 the correspondence f is closed-graph.

Assume now that f is closed-graph, and Q is compact. The values are closed, hence compact. To prove that f is usco, choose any x in P and an open neighborhood W of $K = f[(x)]$ in Q. The set $F = Q - W$ is closed, hence compact. For each y in F there exist an open neighborhood V_y of y and a neighborhood U_y of x such that

$$f[U_y] \cap V_y = \emptyset .$$

A finite family $\{V_y \mid y \in A\}$ covers compact F; put

$$U = \bigcap\{U_y \mid y \in A\} , \quad V = \bigcup\{V_y \mid y \in A\} .$$

Clearly

$$f[U] \cap V = \emptyset ,$$

and hence

$$f[U] \subset W .$$

It follows that f is usco.

Remark. The following condition is necessary and sufficient for a correspondence $f : P \to Q$ to be closed-graph:

if C is compact in P, K is compact in Q, and if $f[C] \cap K = \emptyset$, then $f[U] \cap V = \emptyset$ for some neighborhoods U of C, and V of K.

3.12. Theorem. *Replacing usco-compact by closed-graph in* 3.3, 3.7, 3.8, *and* 3.9, *then the resulting assertions hold without any assumption on Q.*

Proof. Routine.

Remark. The assumption that Q is separated in 3.8, 3.9 and 3.11 is essential. On the other hand similar results are true for closed-graph usco-compact correspondences; this enables us to develop the basic properties of analytic sets in general spaces, see [F 16].

4. SOUSLIN AND ANALYTIC SETS

Recall theorem 3.10:

4.1. *A set X in a space P is Souslin if and only if $X = \mathbf{E}f$ for some closed-graph correspondence $f : \Sigma \to P$.*

4.2. Definition. An *analytic set* in a space P is a set of the form $X = \mathbf{E}f$ where $f : \Sigma \to P$ is an usco-compact closed-graph correspondence. The set of all analytic sets in P is denoted by anal (P).

4.3. *Every analytic set is Souslin. In a compact space every Souslin set is analytic,* *and more generally, if X is a Souslin set derived from the closed compact sets in a space P, then X is analytic.*

Proof. The first assertion follows from 4.1, the second is proved by showing that if $X = \mathbf{S}F$ where Fs are closed and compact in a space P then the associated correspondence $\tilde{F} : \Sigma \to P$ is closed-graph and usco-compact.

4.4. *In any space* $\mathbf{S}(\text{Souslin}\,(P)) = \text{Souslin}\,(P)$, *and in any separated space P*

$$\mathbf{S}(\text{anal}\,(P)) = \text{anal}\,(P)\,.$$

Proof. The space Σ and $\Sigma \times \Sigma^S$ are homeomorphic (in particular there exists a continuous mapping of Σ onto $\Sigma \times \Sigma^S$), and 3.9 and 3.12 apply.

Remark. The result is true for any P, see [F 16].

4.5. *Closed sets in Souslin (analytic) sets are Souslin (analytic, respectively).*

Proof. Let C be closed in X, where $X \subset P$. Choose a closed set C_0 in P with $C_0 \cap \cap X = C$. The propositions 3.3 and 3.12 apply.

4.6. *If a space P is analytic in itself then P is a Lindelöf space.*

Proof. Use 3.4.

The concept of the analytic set depends essentially on the surrounding space, because if $X \subset P$ and $f : \Sigma \to X$ is closed-graph then $f : \Sigma \to P$ need not be closed-graph. On the other hand, if $f : \Sigma \to X$ is usco-compact then so is $f : \Sigma \to P$. Now if P is separated then any usco-compact correspondence is closed-graph by 3.11 and hence if $f : \Sigma \to X$ is usco-compact then $f : \Sigma \to P$ is usco-compact and closed-graph. Thus we have proved:

4.7. *If X is analytic in itself, then X is analytic in any separated space $P \supset X$.*

4.8. Definition. An analytic space is *a regular space* which is analytic in itself.

4.9. Theorem. *Every analytic space is paracompact, in particular, normal, and hence uniformizable.*

Proof. Every regular Lindelöf space is paracompact.

The general theorem used in the proof of 4.9 is not elementary (for a proof see Čech [2] or Kelley [1]). On the other hand, the most interesting part of the theory concerns just the uniformizable spaces, and therefore the reader may include uniformizability into the definition without loosing the deep part of the theory. The present author has written almost all his papers on analytic sets in the class of all completely regular spaces, and here we start with a more general setting because several basic theorems can be proved in this setting without any changes. On the other hand, we recall that the basic properties of analytic sets can be derived without any separation axioms, see [F 16]; this paper can be regarded an advance course in compactness. The abstract setting is discussed in section 14, and in [F 17].

4.10. *In the class of all regular spaces the class of all analytic spaces is closed under usco-compact correspondences.*

Proof. Obvious from the definition of analytic spaces, 3.9 and 3.12.

It is proved in [F 16] that the class of all analytic sets in itself is closed under usco-compact closed-graph correspondences. Of course, it follows immediately from the definition and 3.5 that in the class of all separated spaces the subclass of all separated analytic in itself spaces is closed under usco-compact correspondences.

4.11. Theorem. *The following conditions on a separated uniformizable space P are equivalent*:

(a) *P is analytic.*

(b) *P is Souslin in any separated $Q \supset P$.*

(c) *P is Souslin in some compactification of P.*

(d) *P is a Souslin set derived from closed-compact sets in some space.*

Proof is obvious; P has a separated compactification.

4.12. Definition. A *small analytic set* in a space P is a set of the form $\mathbf{E}f$ where f is a closed-graph single-valued usco correspondence of Σ into P. Similarly we define *small Souslin sets* in a space, and *small analytic spaces.*

4.13. Theorem. *All previous results in this section remain true if analytic and Souslin are replaced by small analytic and small Souslin. Any small analytic space in itself is hereditarily Lindelöf, and every metrizable analytic space is small analytic.*

Proof. The first assertion is obvious. To prove the second one, let $f : \Sigma \to P$ be an usco single-valued correspondence onto P, and consider $X \subset P$, $Y = f^{-1}[X]$, and the restriction of f to a correspondence g of X onto Y. Clearly g is usco-compact, Y is Lindelöf, and hence X is Lindelöf by 3.4. To prove the third assertion, assume that a metrizable P is analytic. Since P is analytic, it is Lindelöf, and hence second countable because of the metrizability. Choose a countable open base $\mathscr{B}$ for P, and for each B in $\mathscr{B}$ an usco-compact correspondence $f_B : \Sigma \to P$ onto P such that

$f_B^{-1}[B] \cap f_B^{-1}[P - B] = \emptyset$ (the sets B and $P - B$ are analytic; B is an F_σ and $P - B$ is closed in analytic P). If we take the intersection

$$f : \Sigma^{\mathscr{B}} \to P$$

of $\{f_B\}$, then f is usco-compact onto P by 3.8, and clearly f is single-valued.

Corollary.*) *A metrizable space P is analytic if and only if P is empty or P is the image of Σ under a continuous mapping.*

Proof. Every non-void closed subspace Y of Σ is a retract of Σ, i.e. there exists a continuous mapping r of Σ onto Y such that $r^2 = r$. This is proved as follows: Define $r(x) = x$ for x in Y. To define r on $\Sigma - Y$ consider the sequence $\{S_n'\}$ determined inductively by setting

$$S_n' = \mathbf{E}\{s \mid s \in \mathbf{S}_n,\ \Sigma s \cap Y = \emptyset,\ \Sigma s \cap \bigcup\{\Sigma t \mid t \in S_{n-1}'\} = \emptyset\}\,.$$

Now let r be constant on each Σs, $s \in \bigcup\{S_n'\}$, and let $\mathbf{E} r \subset Y$. Clearly r is continuous and $r^2 = r$.

Remark. $\mathbf{S}$(small anal (P)) = small anal (P) *for any separated space P. In any space P*

$$\mathbf{S}(\text{small Souslin }(P)) = \text{small Souslin }(P)\,.$$

Proof is similar to that for analytic and Souslin sets.

4.14. Remarks. (a) If P is small analytic in itself then P is semi-separated (i.e. T_1). Every compact space is analytic in itself.

(b) I do not know any description of separated spaces that are Souslin in separated spaces, call them here absolute Souslin spaces. Clearly every analytic space is absolute Souslin, and every completely regular absolute Souslin space is analytic. It is almost obvious that absolute Souslin need not be analytic, e.g. an absolutely closed space need not be analytic.

(c) The proof of 4.13 gives the following more general result:

If a sequence of open coverings of an analytic space P is given then there exists an usco-compact correspondence of Σ onto P such that the values refine each of the coverings.

4.15. Choquet's and Sion's definitions. Assume in advance all spaces separated. A space P is *Choquet-analytic* if there exists a continuous mapping of a $F_{\sigma\delta}$ in some compact space onto P. A space is *Sion-analytic* if there exists a continuous mapping

*) E. Michael and A. Stone have proved that every non-void analytic metrizable space is a quotient of Σ.

of a $(\text{compact})_{\sigma\delta}$ onto P. Clearly every Choquet-analytic is Sion-analytic, and by 4.3 every Sion-analytic is analytic in itself.

Theorem. *Assume that X is a Souslin set in a space P, and that there is a countable union Y of compact sets in P such that $X \subset Y$. Then X is Sion-analytic; if P is completely regular then X is Choquet-analytic.*

Proof. There is a closed set C in $\Sigma \times P$ such that X is the projection of C (i.e. $C : \Sigma \to P$ is closed-graph and $\mathbf{E}C = X$). Take a metrizable compactification K of Σ. Then C is a

$$(*) \qquad (\text{compact } (P \times K))_{\sigma\delta}$$

because C is closed in $\Sigma \times Y$, and $\Sigma \times Y$ belongs to (*).

5. SEPARATION OF ANALYTIC SETS

We start with a general lemma.

5.1. *Let $f : \Sigma \to P$ and $g : \Sigma \to P$ be correspondences, and let $\mathbf{E}f \cap \mathbf{E}g = \emptyset$. Let $\mathcal{M}$ be a collection of sets with $\mathbf{B}(\mathcal{M}) \subset \mathcal{M}$. If $\mathbf{E}f \subset X \subset P - \mathbf{E}g$ for no X in $\mathcal{M}$, then there exist σ and τ in Σ such that*

$$f[\Sigma\sigma_n] \subset X \subset P - g[\Sigma\tau_n]$$

for no n in $\mathbb{N}$ and no X in $\mathcal{M}$.

Proof. Since $\mathbf{E}f = \bigcup\{f[\Sigma s] \mid s \in \mathrm{S}_1\}$, $\mathbf{E}g = \bigcup\{g[\Sigma s] \mid s \in \mathrm{S}_1\}$, if $\mathbf{E}f \subset X \subset P - \mathbf{E}g$ for no X in $\mathcal{M}$, then there exist $s_1 \in \mathrm{S}_1$ and $t_1 \in \mathrm{S}_1$ such that $f[\Sigma s_1] \subset X \subset P - g[\Sigma t_1]$ for no X in $\mathcal{M}$. Since $f[\Sigma s_1] = \bigcup\{f[\Sigma s] \mid s \in \mathrm{S}_2, s > s_1\}$, and similarly for $f[\Sigma t_1]$, we can select $s_2 \in \mathrm{S}_2$ and $t_2 \in \mathrm{S}_2$ such that $s_1 < s_2$, $t_1 < t_2$, and $f[\Sigma s_2] \subset X \subset P - g[\Sigma t_2]$ for no X in $\mathcal{M}$. Now the proof goes by induction.

5.2. Theorem. (Frolík [4], Theorem 5, and [8], Theorem 1.) *Let P be a uniformizable space, $A \subset P$ an analytic set, and C a Souslin set in P with $A \cap C = \emptyset$. Then*

(1) *There exists a Baire set B with $A \subset B \subset P - C$;*

(2) *There exists a continuous mapping φ of P into a separable metrizable space such that $\varphi[A] \cap \varphi[C] = \emptyset$.*

Proof. Choose an usco-compact $f : \Sigma \to P$, and a closed-graph $g : \Sigma \to P$ such that $A = \mathbf{E}f$, $C = \mathbf{E}g$. Assuming the negation of (1), by 5.1 we get σ and τ in Σ such that

$$f[\Sigma\sigma_n] \subset X \subset P - g[\Sigma\tau_n]$$

for no n and no Baire set X. The sets $f\sigma$ and $g\tau$ are disjoint, $\bigcap \operatorname{cl} g[\Sigma\tau_n] = g\tau$, and $f\sigma$ is compact, and therefore

$$f\sigma \cap \operatorname{cl} g[\Sigma\tau_k] = \emptyset$$

for some k. Take a cozero set $N \supset f\sigma$ with $N \subset P - \operatorname{cl} g[\Sigma\tau_k]$. There exists an l such that $f[\Sigma\sigma_n] \subset N$ for $n \geqq l$. Now if $n \geqq l$, and if $n \geqq k$ then

$$f[\Sigma\sigma_n] \subset N \subset P - g[\Sigma\tau_n]$$

which contradicts our assumption and proves (1). The proof of (2) is similar.

5.3. Theorem. *Let A be an analytic set in a space P, and let C be a Souslin set in P disjoint to A. There exists a set U in* $\mathbf{B}$(open (P)) *such that*

$$A \subset U \subset P - C\,.$$

Proof. Let $f : \Sigma \to P$ be a closed-graph usco-compact correspondence with $\mathbf{E}f = A$, and let $g : \Sigma \to P$ be closed-graph with $\mathbf{E}g = C$. Assuming the contrary we reach a contradiction as in 5.2; just put

$$U = P - g[\Sigma\tau_k]\,.$$

5.4. Theorem. (Meyer [1]. *Any two disjoint Souslin sets in a countably compact space P are separated in* $\mathbf{B}$(closed (P)).

Proof. Let $f : \Sigma \to P$, and $g : \Sigma \to P$ be closed-graph correspondences with $\mathbf{E}f \cap \mathbf{E}g = \emptyset$. If $\mathbf{E}f$ and $\mathbf{E}g$ were not separated in $\mathscr{B} = \mathbf{B}(\text{closed}\,(P))$ then there would exist $s_1 \in S_1$, $t_1 \in S_1$ such that $f[\Sigma s_1]$ and $g[\Sigma t_1]$ would not be separated in $\mathscr{B}$, and by induction there would exist σ and τ in Σ such that $f[\Sigma\tau_n]$ and $g[\Sigma\tau_n]$ would not be separated for each n. Since P is countably compact and $\bigcap \operatorname{cl} f[\sigma_n] \cap \bigcap \operatorname{cl} g[\Sigma\tau_n] = \emptyset$, necessarily

$$\operatorname{cl} f\,[\Sigma\sigma_n] \cap \operatorname{cl} g[\Sigma\tau_n] = \emptyset$$

for some n, which contradicts the choice of σ and τ.

5.5. Theorem. *If P is a separated space, $A \subset P$ is analytic, and if $C \subset P - A$ is Souslin in P, then $A \subset B \subset P - C$ for some Borel-open C. If, in addition, C is analytic in P, then there exist Borel-open B_1 and B_2 with $A \subset B_1$, $C \subset B_2$ and $B_1 \cap B_2 = \emptyset$.*

The proof is left to the reader.

5.6. *If $\mathscr{A}$ is a disjoint countable collection of analytic sets in a uniformizable* *space P, then there exists a disjoint family $\{B_A \mid A \in \mathscr{A}\}$ of Baire sets in P with $B_A \supset A$ for each A in $\mathscr{A}$.*

Proof. For each distinct A and A' (hence $A \cap A' = \emptyset$) select Baire sets $B(A, A') \supset$ $\supset A$ and $B(A', A) \supset A'$ such that $B(A, A') \cap B(A', A) = \emptyset$ (this can be done by 5.2), and put $B_A = \bigcap\{B(A, A') \mid A' \in \mathscr{A}, A' \neq A\}$.

5.7. Theorem. *Let $\mathscr{A}$ be a countable collection of analytic sets in a space P, and let*

$$\{B_{\mathscr{F}} \mid \mathscr{F} \subset \mathscr{A}, \mathscr{F} \text{ finite}\}$$

be a family of Baire sets in P such that

$$B_{\mathscr{F}} \supset \bigcap\mathscr{F}$$

for each $\mathscr{F}$. There exists a family $\{B_A \mid A \in \mathscr{A}\}$ of Baire sets such that $B_A \supset A$ for each A, and

$$\bigcap\{B_A \mid A \in \mathscr{F}\} \subset B_{\mathscr{F}}$$

for each finite $\mathscr{F} \subset \mathscr{A}$

Proof. Separation theorem 2.5 applies with $\mathscr{A}$ the given countable collection of analytic sets in P, and $\mathscr{B}$ the collection of all Baire sets in P.

5.8. Thoreem. *A set X in an analytic space P is a Baire set in P if and only if the two spaces X and $P - X$ are analytic.*

Proof. "Only if" is obvious, and "if" follows from 5.2.

5.9. Corollary. *If each closed set in an analytic space P is a Baire set, then each open set is analytic, hence Lindelöf, hence F_σ; thus P is perfectly normal.*

Problem. Is the assumption that P is analytic essential? Is it true with analytic replaced by normal?

5.10. Remark.*) *In an analytic space P the collection of all Baire sets is the smallest collection $\mathscr{M}$ of sets in P such that*

a) $\mathbf{B}(\mathscr{M}) = \mathscr{M}$, *and*

b) *each point of P has arbitrary small neighborhoods that belong to $\mathscr{M}$.*

Proof. See [F 16, Theorem 4.2].

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*) This result has been surprisingly developed in the author's "Stone-Weirstrass theorems for $C(X)$ with sequential topology", Proc. Amer. Math. Soc. 1970.

All spaces in this section are assumed to be separated and uniformizable (i.e. completely regular).

6.1. Theorem and definition. *The following conditions a through g on a space P are equivalent*:

a) *The spaces P and $\beta P - P$ are analytic*;

b) *The space P and $K - P$ are analytic for some compactification K of P*;

c) *The space P and $K - P$ are analytic for any compactification K of P*;

d) *The set P is a Baire set in βP*;

e) *The set P is a Baire set in some compactification K of P*;

f) *The set P is a Baire set in any compactification K of P.*

g) *There exists a proper mapping onto a metrizable space satisfying conditions $a - f$.*

A space satisfying conditions $a - g$ is called *bianalytic*.

Proof. I. If K is a compactification of P then there exists a proper mapping f of βP onto K such that $f[\beta P - P] = K - P$. If $\beta P - P$ is analytic then $K - P$ is analytic (thus condition a implies condition c) as a continuous image of an analytic space, and if $K - P$ is analytic, then $\beta P - P$ is analytic (thus b implies a) as an usco-compact image of an analytic space. Since clearly c implies b, conditions a through c are equivalent.

II. Obviously d implies a, e implies b, and f implies c. The converse implications follow from Theorem 5.8. Thus conditions a through f are equivalent.

III. Assume condition a; there exists a continuous mapping f of βP onto a metrizable space M such that $P = f^{-1}[f[P]]$. Thus $f[P] \cap f[\beta P - P] = \emptyset$, and hence both $f[P]$ and $f[\beta P - P]$ are analytic. The mapping f is proper because $\mathbf{D}f$ is compact; hence the restriction of f to P is proper because $P = f^{-1}[f[P]]$. Thus condition g is fulfilled. To prove the converse implication, assume that f is a proper mapping of P onto Q, and consider the continuous extension of f to a mapping F of βP onto βQ. Since $f[\beta P - P] = \beta Q - Q$ (by 1.10), P satisfies condition a if and only if Q satisfies condition a. This concludes the proof, and in addition we have proved the following theorem.

6.2. Theorem. *The class of all bianalytic spaces is closed under proper mappings (in the class of uniformizable spaces) in both directions. Bianalytic spaces are just the preimages of metrizable bianalytic spaces under proper mappings.*

6.3. The empty space is the only space that is a Baire set in any space; this is obvious. It might be in place to remark that there exists an embedding of the closed unit interval I into a completely regular space P such that each point of I is a G_δ in P but I is not. This example will be needed in Section 8.

6.4. Examples. We shall show that a one-to-one continuous image of a bianalytic (metrizable) space need not be bianalytic, and the union of two bianalytic subspaces need not be bianalytic. The fact needed is the following:

A. If $x \in \beta\mathsf{N} - \mathsf{N}$, then $\mathsf{N} \cup (x)$ is no Baire set in $\beta\mathsf{N}$.

This follows from:

B. If B is a Baire set in $\beta\mathsf{N}$, and $B - \mathsf{N} \neq \emptyset$ then $B - \mathsf{N}$ has a non-void interior in $\beta\mathsf{N} - \mathsf{N}$.

The last proposition is derived immediately from

C. Every zero set in $\beta\mathsf{N} - \mathsf{N}$ has a non-void interior.

See, for example, Gillman-Jerison [1].

Exhibition of the examples: $\mathsf{N} \cup (x)$ is not bianalytic, but $\mathsf{N} \cup (x)$ endowed with the discrete topology is bianalytic. Both N and (x) are bianalytic, but $\mathsf{N} \cup (x)$ is not.

6.5. Examples in 6.4 show that the concept of bianalytic spaces is not any good substitute for a generalization of classical separable absolute Borel set. In the next section one-to-one continuous images of bianalytic spaces will be studied; the external properties of these spaces turn out to be nice, and perhaps the best we can expect. We start with another definition that will be convenient for developing of properties.

Remark. The results are taken from Frolík [4]. For more results in this direction we refer to Frolík [7].

7. BORELIAN SPACES

The spaces studied in this section were introduced in Frolík [4]; they were studied in Rogers [2] in the class of separated spaces under the name of descriptive Borel sets. The definitions and the results in 7.1, 7.2, and 7.3 parallel the pattern in Section 4; the details are left to the reader.

7.1. Definition. A Borelian set in a space P (not necessarily completely regular) is the image of Σ under a closed-graph dusco-compact correspondence into P. A Borelian space is a regular space that is Borelian in itself.

7.2. Theorem. *If P is a separated space then the set Borelian (P) of all Borelian sets in P is invariant under $\mathbf{S}_d$, in particular, under $\mathbf{B}_d$. Every Borelian space is uniformizable, and every Borelian space is a Borelian set in every separated space.*

7.3. Theorem. *Regular dusco-compact images of Borelian spaces are Borelian.*

Proof. 3.5.

Corollary. *Regular one-to-one continuous images of Borelian spaces are Borelian.*

Proof. Any one-to-one continuous mapping is dusco-compact.

7.4. Proposition. *Assume that P is a Borelian subset of a space Q. Then*:

a) *If Q is separated then*

$$P \in ([\text{open}\,(Q)] \cap [\text{closed}\,(Q)]_{\sigma_d \delta}) \,;$$

b) *If Q is regular then*

$$P \in \mathbf{B}(\text{closed}\,(Q)) \,;$$

c) *If Q is uniformizable then*

$$P \in ([\text{closed}\,(Q)] \cap [\text{Baire}\,(Q)])_{\sigma_d \delta} \,.$$

Proof. Let f be a closed graph dusco-compact correspondence of Σ into Q such that $P = \mathbf{E}f$. By Theorem 5.3, and 5.4 there exists a family $\{Fs \mid s \in \mathrm{S}\}$ with $Fs \supset \supset f[\Sigma s]$ such that each family $\{Fs \mid s \in \mathrm{S}_n\}$ is disjoint, and $F(s) \in \mathbf{B}(\text{open}\,(Q))$ if Q is separated, $F(s) \in \mathbf{B}(\text{closed}\,(Q))$ if Q is regular, and Fs are Baire sets if Q is uniformizable.

Clearly $P = \mathbf{S}\{s \to Fs \cap \operatorname{cl} f[\Sigma s]\}$. Since $\{Fs \cap \operatorname{cl} f[\Sigma s] \mid s \in \mathrm{S}_n\}$ are disjoint we have, assuming that the regularization has been done,

$$\mathbf{S}\{s \to Fs \cap \operatorname{cl} f[\Sigma s]\} = \bigcap\{\bigcup\{Fs \cap \operatorname{cl} f[\Sigma s] \mid s \in \mathrm{S}_n\} \mid n \in \mathrm{N} - (0)\}$$

which proves the results.

7.5. Theorem. *Every bianalytic space is Borelian. Borelian spaces are just regular one-to-one continuous images of bianalytic spaces.*

Proof. I. To prove the first statement, in virtue of 7.2 and the fact that Baire $(Q) = \mathbf{B}_d$ (cozero (Q)), it is enough to show that if P is a cozero set in some compactification K of P then P is a Borelian space. Choose a continuous $f : K \to \mathrm{R}$ such that $\mathbf{Z}(f) = K - P$. Then the restriction g of f to a mapping of P onto $f[P] \subset \mathrm{R}$ is perfect and $f[P]$ is a closed subspace of $\mathrm{R} - (0)$. Hence, by 7.3 it is enough to show that $\mathrm{R} - (0)$ is Borelian. But $\mathrm{R} - (0)$ is the disjoint union of the countable space of rationals $\neq 0$, and the space of irrationals, both being Borelian.

II. The proof of the second statement follows immediately from the following factorization theorem for dusco-compact correspondences.

7.6. Proposition. *If $f : P \to Q$ is dusco-compact, then $f = g \circ p^{-1}$, where p is a proper mapping, and g is a one-to-one continuous mapping. If P and Q are separated uniformizable spaces then so is the domain of p.*

Proof. Put $T = \mathbf{E}f$, and let $\mathscr{C}$ be the collection of all $f[(x)]$, $x \in \mathbf{D}f$. Let $\mathscr{F}$ be the set of all $F \subset T$ such that $F \cap C$ is closed in C for each C in $\mathscr{C}$, and $f^{-1}[F]$ is closed in P. It is routine to verify that $\mathscr{F}$ is the collection of all closed sets for some topology on T; let T be endowed with that topology. Clearly $p = f^{-1} : T \to P$ is proper (not necessarily onto!), and the identity mapping g of T into Q is continuous (all directly from the definition of T). Thus $f = g \circ p^{-1}$ is the required factorization of f. If Q is separated then evidently T is separated. Assume that the two spaces P and Q are separated and uniformizable. We have to prove that T is uniformizable. Assume that F is closed in T and $y \in T - F$. If $y \notin \mathrm{cl}_Q F$, then there exists a continuous function r on Q with $ry = 0$ and $r[\mathrm{cl}_Q F] = (1)$, and the continuous function $r' = r \circ g$ on T has the property that $r'y = 0$, $r'[F] = (1)$. It remains the case where $y \in \mathrm{cl}_Q F$; in this case we can write $F = F_1 \cup F_2$ such that $y \notin \mathrm{cl}_Q F_1$, and F_2 is closed in T (the intersection of a closed set in Q with F), and $f[f^{-1}[y]]$ does not meet F_2. Hence $f^{-1}[F_2] = K$ is closed in P, and does not contain $z = f^{-1}y$. Thence there is a continuous function k on P that is 0 at z, and 1 on K. Define r on T by setting

$$rx = k(f^{-1}[x]) .$$

If $X \subset \mathrm{R}$ then

$$r^{-1}[X] = f[k^{-1}[X]] ,$$

and hence r is continuous. Obviously r is 0 at y, and 1 on F_2.

Remark. As an immediate corollary we get that f: $P \to Q$ is usco-compact if and only if $f = g \circ p^{-1}$ where p is proper, and g is continuous; if P and Q are separated and uniformizable, then the middle space can be taken to have the same property. Proposition applies to

$$\{x \to y \mid y \in f[(x)] \times x\} : P \to Q \times P .$$

Now we are ready to prove the external characterizations of Borelian spaces.

7.7. Theorem. *Each of the following conditions a through e is necessary and sufficient for a completely regular space P to be a Borelian space*:

a) *If $P \subset Q$, and if Q is completely regular then*

$$P \in ([\text{closed}\,(Q)] \cap [\text{Baire}\,(Q)])_{\sigma d \delta} .$$

b) *Condition a with $Q = \beta P$.*

c) *Condition a with Q a given compactification of P.*

d) *Condition a with Q any compactification of P.*

e) *Condition a where Q is a Borelian space.*

Proof. Each of the conditions is necessary by 7.4. Each of the conditions is sufficient because if Q is a Borelian space then

$$\text{Borelian}\,(Q) \subset \mathbf{B}_d(\text{closed}\,(Q) \cup \text{Baire}\,(Q))$$

by 7.2 and 7.5.

We note the following result we have just proved:

7.8. *If Q is Borelian then*

$$\text{Borelian}\,(Q) = \mathbf{B}_d(\text{closed}\,(Q) \cup \text{Baire}\,(Q)) = ([\text{closed}\,(Q)] \cap [\text{Baire}\,(Q)])_{\sigma d\delta}\,.$$

7.9. Example. A σ-compact completely regular space need not be Borelian. Take an uncountable compact space K with exactly one cluster point, say x. Take the sum space $\Sigma\{K \mid n \in \mathsf{N}\}$ and identify the points $\langle n, x\rangle$, $n \in \mathsf{N}$. The quotient space Q is clearly a σ-compact space. On the other hand Q is not Borelian. Assuming that f is a dusco-compact correspondence of Σ onto Q, we take the unique σ in Σ with $f\sigma$ containing the point $\mathbf{E}\{\langle n, x\rangle \mid n \in N\}$, and get that

$$f[\Sigma - (\sigma)]$$

is a Lindelöf space. But $Q - f\sigma$ is not Lindelöf. This contradiction shows that Q is not Borelian.

7.10. A small Borelian set in a space P is the image of Σ under a closed-graph dusco-compact single-valued correspondence of Σ into P. Thus the small Borelian sets in P are just the closed-graph one-to-one continuous images of closed subspaces of Σ.

Proposition. *A metrizable Borelian space is small Borelian.*

Proof. Assume that P is metrizable and Borelian. Since P is Borelian, it is second countable. Take a countable base $\{U_n\}$ for P. The sets U_n and their complements are Borelian (because they are Baire sets), and therefore there exists a sequence $\{f_n\}$ of dusco-compact correspondences such that the values of f_n refine the cover $\{U_n, P - U_n\}$ for each n. Take the intersection-correspondence $f = \Lambda\{f_n\}$. By 3.8 f is dusco-compact, and clearly the diameter of each of the values of f is zero. Hence f is single-valued.

Theorem. *The collection of all small Borelian sets in a separated (this is not essential) space is invariant under* $\mathbf{S}_d$, *in particular under* $\mathbf{B}_d$. *In a metrizable space the Borelian sets are invariant under* $\mathbf{B}$.

7.11. Proposition. *The following conditions on a metrizable space are equivalent:*

a) *P is Borelian;*
b) *P is small Borelian;*
c) *P is bianalytic;*
d) *P is separable, and P is a Baire set in every metrizable space $Q \supset P$;*
e) *P is separable, and P is Borel-closed in every metrizable $Q \supset P$; and*
f) *P is separable, and P is Borel-open in every metrizable $Q \supset P$.*

Proof. Recall that every separable metrizable space has a metrizable compactification.

The metrizable spaces satisfying the equivalent conditions in Proposition 7.11 are called Lusinian by Bourbaki, classical Borel sets by Choquet, and metrizable absolute Baire sets or separable absolute Borel metrizable spaces by the author.

8. BB-SETS

In this section all spaces are assumed to be separated and uniformizable (i.e., completely regular).

8.1. Definition. A BB-set in a space P is *a Baire set X* in P such that the subspace X of P is Borelian. The set of all BB-sets in P is denoted by $\mathrm{BB}(P)$.

If P is Borelian, then each Baire set in P is Borelian, and hence $\mathrm{BB}(P) =$ *Baire* (P). *If P is Borelian then*

$$\text{(1)} \qquad \text{Borelian}\,(P) = \mathrm{BB}(P) \;\; (= \text{Baire}\,(P))$$

if and only if

$$\text{(2)} \qquad \text{closed}\,(P) \subset \text{Baire}\,(P)\,.$$

Only "if" is evident, and to prove "if", observe that it follows from (1) that $\mathbf{B}(\text{closed}\,(P)) \subset \text{Baire}\,(P)$; since $\text{Borelian}\,(P) \subset \mathbf{B}(\text{closed}\,(P))$, we get (2).

8.2. Theorem. *For any space P the collection* $\mathrm{BB}(P)$ *is a σ-ring.*

Proof. If A and B are two BB-sets, then $A - B$ is a Baire set in P, and hence in A, and that implies that $A - B$ is Borelian. Thus $A - B$ is a BB-set. Clearly BB-sets are closed under countable intersections and countable disjoint unions because Baire sets as well as Borelian sets are. Since $\bigcup A_i = A_0 \cup (A_1 - A_0) \cup (A_2 - A_1 - A_0) \cup \cup \ldots$, we get invariance under arbitrary countable unions.

The BB-sets were introduced in [R 3] under the term descriptive Baire sets. J. D. Knowles and C. A. Rogers proved Theorem 8.2, equivalence of a and b in Theorem 8.5, and Corollary 8.7. All other results, and all proofs are taken from Frolík [8] and [9].

8.3. Lemma. *Let $f : \Sigma \to P$ be a dusco-compact correspondence. There exists a continuous mapping m of P into a separable metrizable space M such that $m \circ f$ is dusco-compact.*

Proof. We need m such that $m \circ f$ is disjoint, because usco-compactness is automatically satisfied. As in the proof of 7.4 there exists a family $\{B_s \mid s \in S\}$ of Baire sets such that $B_s \supset f[\Sigma s]$, and each family $\{B_s \mid s \in S_n\}$ is disjoint. By 1.5 there exists a continuous mapping m into a separable metrizable space M such that

$$m[B_s] \cap m[P - B_s] = \emptyset$$

for each s. Now if σ and τ are two distinct elements of Σ, then $\sigma_n \neq \tau_n$ for some n,

$$(m \circ f)[(\sigma)] \subset m[B_{\sigma_n}],$$
$$(m \circ f)[(\tau)] \subset m[B_{\tau_n}],$$

and $m[B_{\sigma_n}] \cap m[B_{\tau_n}] = \emptyset$ because $\sigma_n \neq \tau_n$.

8.4. Theorem. *In order that a Borelian subspace X of P to be a Baire set in P it is necessary and sufficient that X be distinguishable. In other words,*

$$\mathrm{BB}(P) = \text{Borelian}(P) \cap \text{distinguishable}(P)$$

Proof. Necessity is obvious because each Baire set is distinguishable. Assume that X is distinguishable and Borelian. Since X is distinguishable, there exists a continuous mapping m_1 of P into a separable metrizable space M such that $m_1[X] \cap \cap m_1[P - X] = \emptyset$. Since X is Borelian, there exists a dusco-compact correspondence f of Σ into P such that $X = \mathbf{E}f$. By 8.3 there exists a continuous mapping m_2 of P into a separable metrizable space M_2, such that $m_2 \circ f$ is disjoint. Let m be the reduced product of m_1 and m_2; m is the mapping of P into $M = M_1 \times M_2$ defined by $mx = \langle m_1 x, m_2 x \rangle$. Clearly $m \circ f$ is disjoint (because of m_2), and $m[X] \cap \bigcap m[P - X] = \emptyset$ because of a similar property of m_1. The set $m[X] = \mathbf{E}(m \circ f)$ is Borelian, hence a Baire set in M because M is metrizable. Since $X = m^{-1}[m[X]]$, X is a Baire set in P.

Remarks. We note the following two more general results; (b) will be needed in 8.5:

(a) It is clear that it is enough to assume that X is distinguishable in a more general sense, namely, by continuous mappings into metrizable spaces not necessarily separable.

(b) A set $X \subset P$ is a BB-set in P if and only if it is Borelian, and for each dusco-compact correspondence f of Σ in P with $X = \mathbf{E}f$, there exists a continuous mapping m of P into a separable metrizable space M such that $m \circ f$ is disjoint and $m[X] \cap \cap m[P - X] = \emptyset$.

8.5. Denote by Z_P, or simply Z, the correspondence of $C^*(P)$ into P defined by $Z[(r)] = \mathbf{Z}(r) = \mathbf{E}\{x \mid x \in P,\ rx = 0\}$. Here $C^*(P)$ denotes the Banach space of all bounded continuous functions on P.

Theorem. *The following conditions on a set X in a space P are equivalent:*

(a) *X is a* BB-*set in P;*

(b) *There exists a dusco-compact correspondence f of Σ into P such that $\mathbf{E}f = X$, and $f = Z \circ F$ with F a continuous mapping of Σ into $C^*(P)$;*

(c) *X is Borelian, and each dusco-compact correspondence of Σ into P with $\mathbf{E}f = X$ admits a factorization $f = Z \circ F$ with F a continuous mapping of Σ into $C^*(P)$.*

Proof. Clearly (c) implies (b). Assume (b); the set $F[\Sigma]$ is separable, and so we can take a countable dense $\mathscr{F}$ in $\mathbf{E}F$. It is easy to verify that $\mathscr{F}$ distinguishes the points of X from the points of $P - X$. Hence X is distinguishable, and by 8.2, a Baire set. Thus (b) implies (a). It remains to prove that (a) implies (c). Let f be a dusco-compact correspondence of Σ into P such that $\mathbf{E}f = X$. By Remark (b) in 8.4 there exists a continuous mapping m of P into a separable metrizable space M such that $m[X] \cap m[P - X] = \emptyset$, and $m \circ f$ is disjoint. Choose a metric d for M, and for each σ in Σ put

$$r_\sigma x = \min\left(1, \left(\operatorname{dist}\left(x, (m \circ f)\left[(\sigma)\right]\right)\right)\right),$$
$$r_\sigma \ = \{x \to r_\sigma x \mid x \in M\} : M \to \mathrm{R}\,.$$

Clearly*)

$$F' : \{\sigma \to r_\sigma\} : \Sigma \to C^*(M)$$

is continuous, hence

$$F = m^* \circ F' : \Sigma \to C^*(P)$$

is continuous where $m^* : C^*(M) \to C^*(P)$ is the adjoint of m. Clearly $f = Z \circ F$.

*) This map need not be continuous. One must be more careful at this point.

8.6. Here we derive a characterization of Baire sets in a compact space K by means of continuous mappings of Σ into $C^*(K)$. This is clearly in the lines of the classical result stating that separable absolute Borel sets (= metrizable Borelian spaces, see 7.11) are one-to-one continuous images of closed subspaces of Σ. We need the following.

Lemma. *A space is countably compact (if and) only if $Z_P : C^*(P) \to P$ is usco; a space is compact (if and) only if $Z_P : C^*(P) \to P$ is usco-compact.*

Proof. We shall prove that Z_P is usco-compact if P is compact; this is precisely what is needed for our purpose. The values are compact because they are closed in a compact space. Let $C \subset P$ be closed, let $\mathbf{Z}(f_n) \cap C \neq \emptyset$, and let $\{f_n\}$ converge to f in $C^*(P)$. Choose $\{x_n\}$ in C with $f_n x_n = 0$, and let x be a cluster point of x_n. Then $x \in C$, and clearly $fx = 0$.

Theorem. *A set X in a compact space P is a Baire set in P if and only if there exists a one-to-one continuous mapping F of a closed subspace C of Σ into $C^*(P)$ such that*

$$\{\sigma \to Z(F_\sigma) \mid \sigma \in C\}$$

is a disjoint cover of X.

Proof. 8.5, and 8.6.

8.7. The result in this section gives a sufficient condition for the validity of the inclusion

$$\text{Borelian}\,(P) \subset \text{Baire}\,(P)\,.$$

If P is Borelian then a necessary and sufficient condition is that every open set is a cozero set, see 8.1.

Theorem. *If each open set in P is a Souslin set in P then each Borelian set in P is a Baire set in P.*

Proof. Let f be a dusco-compact correspondence of Σ into P. As in the proof of 8.3 or 7.4 it follows from 5.6 that there exists a family $\{Bs\}$ of Baire sets in P such that $Bs \supset f[\Sigma s]$, and each $\{Bs \mid s \in \mathrm{S}_n\}$ is disjoint. By Separation theorem 5.2, and by our assumption that open sets are Souslin there exists a family $\{As \mid s \in \mathrm{S}\}$ of Baire sets in P such that $f[\Sigma s] \subset As \subset \mathrm{cl}\,[\Sigma s]$. Hence

$$\mathbf{E}f = \bigcup\{\bigcap\{As \cap Bs \mid s < \sigma\} \mid \sigma \in \Sigma\}\,,$$

and the set on the right-hand side is equal to (after regularization)

$$\bigcap\{\bigcup\{As \cap Bs \mid s \in \mathrm{S}_n\} \mid n\}\,.$$

Corollary. *Each Borelian set in P is a Baire set provided that every open set is a Souslin set.*

9. COMPLETE SEQUENCES OF COVERINGS

In this section we describe analytic spaces, Borelian spaces, and one more kind of spaces by existence of complete sequences of coverings with certain properties. All spaces are assumed to be completely regular. The method of complete sequences can be used to develop the properties of analytic and Borelian sets without any use of correspondences. In fact both methods are used in [F 4] in the proofs. The method of complete sequences is intrinsic. The method of complete sequences applies to non-separable theory, and it is not known if the non-separable theory can be based on a general theory of correspondences, see Section 13. All results are taken from Frolík [4], [5], [7], [13] and [14].

9.1. Definition. Let $\mu = \{\mathcal{M}_a \mid a \in A\}$ be a family of coverings of a space P. A *μ-Cauchy filter* is a filter $\mathcal{M}$ on P such that $\mathcal{M} \cap \mathcal{M}_a \neq \emptyset$ for each a in A.

For a discussion of this notion we refer to Frolík [2], [3] and [5].

9.2. Proposition. *Let μ be a complete family of coverings of P. If F is closed subspace of P, then the trace of μ on F is a complete family. If f is a dusco-compact correspondence onto a regular space Q, then the image of μ under f is a complete family on Q.*

Proof. The direct proof of the first statement is obvious and therefore the proof may be left to the reader; on the other hand the first assertion follows immediately from the second one because the inverse of an embedding on a closed subspace is dusco-compact. To prove the second assertion, assume that $\mu = \{\mathcal{M}_a \mid a \in A\}$ is complete, $f : P \to Q$ is dusco-compact onto, and denote by $\nu = \{\mathcal{N}_a \mid a \in A\}$ the image of μ under f; thus $\mathcal{N}_a$ is the set of all $f[M]$, $M \in \mathcal{M}_a$. Since f is disjoint we have that $\mathcal{M}_a$ is the set of all $f^{-1}[N]$, $N \in \mathcal{N}_a$. Assume that $\mathcal{N}$ is a ν-Cauchy, and consider the set $\mathcal{M}$ of all $f^{-1}[N]$, $N \in \mathcal{N}$. Clearly $\mathcal{M}$ is a μ-Cauchy filter on P. Without any loss of generality we may assume that $\mathcal{N}$ is an ultrafilter, then $\mathcal{M}$ is an ultrafilter, and hence $\mathcal{M}$ converges to a point x of P. Consider the compact set $K = f[(x)]$. If $\mathcal{N}$ were convergent to no point of K then one could construct an open neighborhood V of K such that $V \notin \mathcal{N}$; this follows from the compactness of K and the fact that $\mathcal{N}$ is an ultrafilter. Consider the set U of all $z \in P$ such that $f[(z)] \subset V$. Since $V \notin \mathcal{N}$, clearly $U \notin \mathcal{M}$; this contradicts the fact that $\mathcal{M}$ converges to x because U is a neighborhood of x.

9.3. Theorem. *A regular space P is analytic if and only if there exists a complete sequence $\mu = \{\mathcal{M}_n\}$ of countable coverings of P.*

Proof. Assume that $\mu = \{\mathscr{M}_n\}$ is a complete sequence of countable coverings on P. Let $\{M_n^k\}_k$, $n = 1, 2, \ldots$, be a sequence ranging on $\mathscr{M}_n$. For each $\sigma \in \Sigma$ put

$$f\sigma = \bigcap\{\mathrm{cl} \bigcap\{M_n^{\sigma(n)} \mid n \leqq k\}\ k \in \mathsf{N}\}$$

where $\sigma(n)$ is the n-th coordinate of σ. It may be proved that $f : \Sigma \to P$ is usco-compact and surjective.

Conversely, assume that f is an usco-compact correspondence of Σ onto P. Let, as usual, S be the set of all finite sequences of natural numbers, S_n the set of all $s \in \mathsf{S}$ of the length n. Write σ_n for the restriction of $\sigma \in \Sigma$ to the initial segment of N of the length n. Define an order $s < t$ on S_n to mean that $s \neq t$ and the first coordinate of s distinct from that of t is less than that of t. Put

$$\Sigma s = \mathbf{E}\{\sigma \mid \sigma \in \Sigma,\ s < \sigma\}$$

where $s < \sigma$ means that $s = \sigma_n$ for some n, and define

$$Ms = f[\Sigma s] - \bigcup\{f[\Sigma t] \mid t < s\}\,,$$

$$\mathscr{M}_n = \mathbf{E}\{Ms \mid s \in \mathsf{S}_n\}\,.$$

It is easy to see that $Ms \cap Mt = \emptyset$ for $s \neq t$, $s \in \mathsf{S}_n$, $t \in \mathsf{S}_n$ and that $Ms \cap Mt \neq \emptyset$ implies that either s is a section of t or t is a section of s. It follows that if $\mathscr{M}$ is a Cauchy filter, and $M_n \in \mathscr{M} \cap \mathscr{M}_n$, then there exists a σ in Σ such that if $M_n = Ms$ then $s < \sigma$. Since f is usco-compact, that implies that $\bigcap\{\mathrm{cl}\, M \mid M \in \mathscr{M}\}$ has a cluster point in $f\sigma$.

9.4. Definition. A *B-structure on a space* P is a complete sequence $\mu = \{\mathscr{M}_n\}$ of countable coverings of P such that

$$(*) \qquad \bigcap\{M_n\} = \bigcap\{\mathrm{cl} \bigcap\{M_k \mid k \leqq n\} \mid n \in \mathsf{N}\}$$

for any sequence $\{M_n\}$ such that $M_n \in \mathscr{M}_n$. A Borelian structure on P is a B-structure $\mu = \{\mathscr{M}_n\}$ on P such that each $\mathscr{M}_n$ is disjoint.

9.5. (a) *If $\mu = \{\mathscr{M}_n\}$ is a B-structure (Borelian structure), and F is either closed or an element of some $\mathscr{M}_n$, then the trace of μ on F is a B-structure (Borelian structure, respectively) on F. Thus the elements of the coverings of a B-structure are analytic* (9.3).

(b) *The image under a dusco-compact correspondence of a B-structure (Borelian structure) is a B-structure (Borelian structure).*

(c) *Any complete countable sequence of closed coverings is a B-structure.*

Proof. The assertions (a) and (c) are evident. To prove assertion (b), by virtue of 9.2 we only need to show that the image of a B-structure satisfies the structural condition 9.4 $(*)$. Denote by $\nu = \{\mathcal{N}_n\}$ the image of μ under f.

Let $\{N_n\}$ be any sequence with $N_n \in \mathcal{N}_n$, and put $M_n = f^{-1}[N_n]$, $M = \bigcap\{M_n\}$, and $N = \bigcap\{N_n\}$. The set $f[M]$ is compact, and clearly $f[M] = N$. Choose any open neighborhood V of N, and put $U = \mathbf{E}\{x \mid f[(x)] \subset V\}$. There exists an n such that $\bigcap\{M_k \mid k \leqq n\} \subset V$, and hence $\bigcap\{N_k \mid k \leqq n\} \subset U$. Since Q is regular, this implies that condition $(*)$ is fulfilled.

9.6. Theorem. *Each of the following two conditions is necessary and sufficient for a space P to be Borelian*:

(a) *There exists a Borelian structure on P*;

(b) *There exists a complete sequence $\{\mathscr{A}_n\}$ of disjoint countable coverings such that the elements of the coverings are analytic sets.*

Proof. By 9.5 (a) condition (a) implies condition (b). We shall prove that (a) is necessary, and (b) is sufficient.

Assume that P is Borelian; choose a dusco-compact correspondence f of Σ onto P. Let $\mathscr{M}_n$ consist of all $f[\Sigma s]$, $s \in S_{n+1}$. It will be verified that $\mu = \{\mathscr{M}_n\}$ is a Borelian structure on P. Since f is disjoint, the coverings $\mathscr{M}_n$ are disjoint, and $\mathscr{M}_{n+1}$ refines $\mathscr{M}_n$. Further, if $\{M_n\}$ is a monotonic sequence such that $M_n \in \mathscr{M}_n$, then there exists a σ in Σ such that $M_n = f[\Sigma\sigma_{n+1}]$, and $f\sigma = \bigcap\{M_n\}$. It follows now from the fact that f is usco-compact that $f\sigma = \bigcap\{\mathrm{cl}\, M_n\}$. If $M_n \in \mathscr{M}_n$ such that $\{M_n\}$ is not monotonic then $\bigcap\{M_i \mid i \leqq k\} = \emptyset$, and the structural condition is evidently fulfilled. It remains to show that μ is complete. Let $\mathscr{M}$ be a μ-Cauchy filter on P; there exists a σ in Σ such that $f[\Sigma\sigma_n] \in \mathscr{M}$ for each n. If no point of $f\sigma$ were a cluster point of $\mathscr{M}$ then, because of compactness of $f\sigma$, there would exist an open neighborhood U of $f\sigma$ with $U \notin \mathscr{M}$. Since f is usco, necessarily

$$f[\Sigma\sigma_n] \subset U$$

for some n, which contradicts the fact that $f[\Sigma\sigma_n] \in \mathscr{M}$, and concludes the proof of necessity of (a).

Assume condition (b). We shall prove that

$$P \in ([\mathrm{closed}\,(K)] \cap [\mathrm{Baire}\,(K)])_{\sigma d\delta}$$

for a given compactification K of P. By Theorem 7.7 P will be Borelian. By 5.6 there exists a family $\{B_A \mid A \in \mathscr{A}_n\}$ of Baire sets in K with $B_A \supset A$ such that each family $\{B_A \mid A \in \mathscr{A}_n\}$ is disjoint. Since $\{\mathscr{A}_n\}$ is complete, $\mathscr{A}_n$'s are disjoint and as we may and shall assume that $\mathscr{A}_{n+1}$ refines $\mathscr{A}_n$, we have

$$P = \bigcup\{\bigcap\{\mathrm{cl}\, A_n \mid n \in \mathrm{N}\} \mid \{A_n\} \in \beta\}$$

where β is the set of all sequences $\{A_n\}$ with $A_n \in \mathscr{A}_n$ that are filter bases (i.e. have finite intersection property). Hence

$$P = \bigcup\{\bigcap\{B_{A_n} \cap \mathrm{cl}\, A_n \mid n \in \mathrm{N}\} \mid \{A_n\} \in \beta\}\,.$$

Since $\{B_A \mid A \in A_n\}$ are disjoint, one can interchange the signs for the union and the intersection, and get

$$P = \bigcap\{\bigcup\{B_A \cap \mathrm{cl}\, A \mid A \in \mathscr{A}_n\} \mid n \in \mathrm{N}\}\,.$$

This concludes the proof of Theorem 9.6.

9.7. *A uniformizable space P is an $F_{\sigma\delta}$ in βP if and only if there exists a complete sequence of closed countable coverings of P.*

Proof. Assume that P is an $F_{\sigma\delta}$ in βP. Hence

$$P = \bigcap\{\bigcup\{F_{nk} \mid k \in \mathrm{N}\} \mid n \in \mathrm{N}\}\,.$$

Let $\mathscr{F}_n$ be the set of all $F_{nk} \cap P$, $k \in \mathrm{N}$. It is easy to verify that $\{\mathscr{F}_n\}$ is complete.

Conversely assume that $\{\mathscr{F}_n\}$ be a complete sequence of countable closed coverings of P. The space P is analytic by 9.3, and hence paracompact by 4.9.

One can prove (see Frolík [5], Theorem 7) that

$$P = \bigcap\{\bigcup\{\mathrm{cl}_{\beta P}\, F \mid F \in \mathscr{F}_n\} \mid n \in \mathrm{N}\}\,.$$

The point is that $\bigcap\{\mathrm{cl}\, F_k \mid k \leqq n\} = \emptyset$ whenever $\bigcap\{F_k \mid k \leqq n\} = \emptyset$ for closed sets F_k in P, and this follows from the normality of P.

Remark. A σ-compact space need not be Borelian by 7.6. A space P is called a B-space if there exists a B-structure on P. By 9.6 every Borelian space is a B-space, and by 9.7 every σ-compact space is a B-space. The class of all B-spaces will be studied in Section 10.

10. B-SPACES AND K-BOREL SETS

All spaces are assumed to be completely regular.

10.1. Definition. A space P is called a *B-space* if P is regular, and if there exists a B-structure on P.

Every Borelian space is B-space, and the converse is not true by 9.7, Remark. No external characterization of B-spaces is known, however, in one particular case we can prove that the external behaviour of B-spaces is nice; this works, in particular, for metrizable spaces. We need the following concept:

10.2. Definition. A *quasi-classical space* is an usco-compact image of a separable metrizable space. A space P is quasi-classical at infinity if $\beta P - P$ is quasi-classical.

The class of all quasi-classical spaces is closed under usco-compact correspondences, and hence, if $K - P$ is quasi-classical for some compactification K of P, then $\beta P - P$ is quasi-classical and hence, $K - P$ is quasi-classical for each compactification of P. For example, every separable metrizable space is both quasi-classical, and quasi-classical at infinity.

10.3. Lemma. *Assume that $\{\mathscr{A}_n\}$ is a complete sequence of countable coverings of a space P such that each element of $\mathscr{A} = \bigcap\{\mathscr{A}_n\}$ is analytic. Let $Q \supset P$ be a space such that $Q - P$ is quasi-classical. Then*

$$P \in ([\text{closed}\,(Q)] \cap [\text{Baire}\,(Q)])_{\sigma\delta}\,.$$

Proof. Let $\mathscr{A}$ be the union of all $\mathscr{A}_n$, $n \in N$. By our assumption there exists an usco-compact correspondence k of a separable metrizable space R onto $Q - P$. Choose a countable base $\mathscr{B}$ of R. For each finite $\mathscr{F} \subset \mathscr{A}$ and each B in $\mathscr{B}$ let $Z(\mathscr{F}, B)$ be a Baire set in Q such that $\bigcap\mathscr{F} \subset Z(\mathscr{F}, B)$, and if there exists a Baire set Z with $\bigcap\mathscr{F} \subset Z \subset Q - k[B]$, then $Z(\mathscr{F}, B) \subset Q - k[B]$. Let for each finite $\mathscr{F} \subset \mathscr{A}$, $Z_{\mathscr{F}}$ be the intersection of all $Z(\mathscr{F}, B)$, $B \in \mathscr{B}$.

Apply Theorem 5.7 to Q, the collection $\mathscr{A}$ and the family $\{Z_{\mathscr{F}}\}$; we get a family $\{Z_A \mid A \in \mathscr{A}\}$ of Baire sets such that $A \subset Z_A$, and $\bigcap\{Z_A \mid A \in \mathscr{F}\} \subset Z_{\mathscr{F}}$ for each finite $\mathscr{F} \subset \mathscr{A}$. We shall prove that

$$P = \bigcap\{\bigcup\{Z_A \cap \text{cl}_Q\, A \mid A \in \mathscr{A}_n\} \mid n \in \mathrm{N}\}\,.$$

The inclusion $\subset$ being evident, assume that a point $x \in Q - P$ is in the set on the right-hand side. Thus there exist A_n in $\mathscr{A}_n$ such that

$$(**) \qquad x \in \bigcap\{Z_{A_n} \cap \text{cl}_Q\, A_n\}\,.$$

Consider the set

$$K = \bigcap\{\text{cl}_P \bigcap\{A_k \mid k \leqq n\} \mid n \in \mathrm{N}\}\,.$$

Since $\mathscr{A}$ is complete, the set K is compact (it may be empty!), and for each neighborhood U of K there exists an $n_U \in \mathrm{N}$ such that

$$\text{cl}_P \bigcap\{A_k \mid k \leqq n_U\} \subset U\,.$$

Choose a y in R with $x \in ky$. The set ky is compact, and if V is a neighborhood of ky in Q then there exists a B_V in $\mathscr{B}$ such that

$$k[B_V] \subset V\,.$$

The sets K and ky are disjoint, because $K \subset P$ and $ky \cap P = \emptyset$. The space Q is uniformizable and therefore there exists a neighbourhood of K, which is a zero set in Q such that

$$Z \cap ky = \emptyset .$$

Put $\mathscr{F} = \mathbf{E}\{A_k \mid k \leqq n_Z\}$, and consider the set $k[B_{Q-Z}]$ $(\subset Q - Z)$. Therefore

$$Z_{\mathscr{F}} \subset Q - k[B_Z] .$$

Hence

$$\bigcap\{Z_{A_n} \mid n \leqq n_Z\} \cap ky = \emptyset ,$$

thus

$$x \notin \bigcap\{Z_{A_n} \mid n \leqq n_Z\} .$$

This contradicts our assumption (**) above, and establishes the converse inclusion.

Remark. In the case of a Borelian structure the proof is much more simpler without any assumption on $Q - P$. By the first separation theorem we can choose Baire sets Z_A, $A \in \mathscr{A}$, in Q such that $Z_A \supset A$ for each A in $\mathscr{A}$, and the collections

$$\{Z_A \mid A \in \mathscr{A}_n\}$$

are disjoint. Then, after the regularization, we may interchange $\bigcup$ and $\bigcap$ in

$$\bigcap\{\bigcup\{Z_A \cap \mathrm{cl}_Q\, A \mid A \in \mathscr{A}_n\} \mid n \in \mathrm{N}\} ,$$

and the resulting set is easy to prove to be P (the proof is the same as in the proof of Theorem 7.4). For details see [F 3], Theorem 11 or [F 6], Theorem 7. This method can be used to prove that if there exists a B-structure on a space P such the elements of the coverings are Baire sets in P, then P is Borelian.

10.4. Theorem. *If P is a B-space that is quasi-classical at infinity then*

$$(*) \qquad P \in ([\text{closed}\,(Q)] \cap [\text{Baire}\,(Q)])_{\sigma\delta}$$

for each space $Q \supset P$. In particular, this is true if P is a metrizable B-space.

Proof. If $\{\mathscr{A}_n\}$ is a B-structure on P, then each element of $\mathscr{A} = \bigcup\{\mathscr{A}_n\}$ is analytic by 9.5. Hence, by 10.3, formula (*) is true for each compactification Q of P. It follows that (*) is true for any $Q \supset P$.

10.5. Theorem. *Each of the following conditions is necessary and sufficient for a metrizable space P to be Borelian (= separable absolute Borel set):*

(1) *There exists a B-structure on P;*

(2) *There exists a Borelian structure on P;*

(3) *There exists a complete sequence* $\{\mathscr{A}_n\}$ *of countable coverings of* P *such that the elements of* $\bigcup\{\mathscr{A}_n\}$ *are analytic;*

(4) *Condition* (3) *with* $\mathscr{A}_n$*'s disjoint.*

Proof. Use 10.3, 10.4, and the fact that every separable metrizable space has a metrizable compactification.

10.6. Problems. (a) Is it true that every B-space is of the form $(*)$ in 10.4 in any space Q?

(b) Assume that $P \in \mathbf{B}(\text{closed}\,(\beta P))$; is P a B-space?

(c) Assume that $P \in \mathbf{B}(\text{closed}\,(\beta P))$; is $P \in \mathbf{B}(\text{closed}\,(K))$ for any compactification of P?

(d) Problem (c) with the additional assumption that P is metrizable.

These problems are restated in 10.7, and 10.8.

10.7. Call a space P an *absolute Borel-closed space* if $P \in \mathbf{B}(\text{closed}\,(Q))$ for each $Q \supset P$. Clearly each Borelian space is absolute Borel-closed space, and I do not know if every B-space is an absolute Borel-closed space.

10.8. A space P is called *K-Borel* if $P \in \mathbf{B}(\text{closed}\,(\beta P))$, or equivalently, if $P \in \mathbf{B}(\text{closed}\,(K))$ for some compactification K of P or equivalently, if $P \in \mathbf{B}(\text{compact}\,(Q))$ for some $Q \supset P$. I do not know whether any K-Borel space is absolute Borel-closed space, even if P is metrizable. See also Section 13.1.

11. RESPECTABILITY OF COMPOSITES

No assumption on spaces is made in this section. For an application in Section 12 we shall need the corollary 11.2 (a) to Theorem 11.1. For the convenience we shall give a direct proof of 11.2 (a) for the case of uniformizable separated spaces. For the detailed proofs see Frolík [15].

A correspondence $f : P \to Q$ is called *graph-Souslin* or *graph-analytic* if the graph of f is, respectively, Souslin or analytic in $P \times Q$.

11.1. Theorem. *Let* $f_2 : P \to Q$ *and* $f_2 : Q \to R$ *be correspondences for topological spaces, and let* $f : P \to R$ *be the composite* $f_2 \circ f_1$. *Then*

(a) *If* f_1 *and* f_2 *are graph-Souslin, and if one of them is graph-analytic, then* f *is graph-Souslin.*

(b) *If* f_1 *and* f_2 *are graph-analytic, then so is* f.

(c) *If* f_1 *and* f_2 *are closed-graph, and if one of them is graph-compact, then* f *is closed-graph.*

(d) *If* f_1 *and* f_2 *are graph-compact, then so is* f.

11.2. Corollaries to 11.1. Let π be the projection of $P \times Q$ into P. *Then*:

(a) *If S is a Souslin set in $P \times Q$ then $\pi[S]$ is a Souslin set in P whenever Q is analytic.* (ROGERS [4].)

(b) *If S is closed in $P \times Q$ then $\pi[S]$ is closed in P provided that Q is compact.*

Proof. Use the obvious relation

$$\pi[S] \times Q = Q \times Q \circ S .$$

11.3. Proof of 11.2 (a) for the case when P and Q are uniformizable and separated. Choose a compactification K of P. There exists a Souslin set R in $K \times Q$ such that

$$S = R \cap (P \times Q) .$$

The set R is analytic (as a Souslin set in an analytic space), and hence the projection A of R into K is analytic, hence Souslin in K. Clearly $\pi[S] = P \cap A$. Thus $\pi[S]$ is Souslin in P.

Remark. If P and Q are arbitrary spaces, and if S is analytic in $P \times Q$ (or closed and compact in $P \times Q$) then so is $\pi[S]$ in P. This follows from the following proposition ([F 16, Theorem 2]): if $f : R \to P \times Q$ is usco-compact and closed-graph then $\pi \circ f$ is closed-graph (and usco-compact). See also Remark 14.3.

11.4. The proof of 11.1 (c), (d) is easy, and therefore left to the reader. One more result is needed for (b):

Lemma. *Assume that $K_1 \subset P \times Q$ and $K_2 \subset Q \times R$ are compact, and U is a neighborhood of $K = K_2 \circ K_1$ in $P \times R$. There exist a neighborhood U_1 of K_1 in $P \times Q$, and a neighborhood U_2 of K_2 in $Q \times K$ such that $U_1 \circ U_2 \subset U$.*

Proof. Assume the contrary. Hence there exists a neighborhood U of K such that $U_2 \circ U_1 - U \neq \emptyset$ for each neighborhood U_i of K_i, $i = 1, 2$. It follows there exist nets $\alpha_1 = \{\langle x_a, y_a \rangle\}$ and $\alpha_2 = \{\langle y_a, z_a \rangle\}$ such that α_i is eventually in each neighborhood of K_i, $i = 1, 2$, and $\langle x_a, z_a \rangle \notin U$ for all a. Now we use the following useful result which has been used twice for filters:

if a net $\{t_a\}$ is frequently in each neighborhood of a compact set K in a space T, then some subnet of $\{t_a\}$ converges to a point of K.

By this result we may and shall assume that α_1 converges to a point $\langle x, y \rangle$ of K_1, and α_2 converges to a point $\langle y, z \rangle$ of K_2. Then $\{\langle x_a, z_a \rangle\}$ converges to $\langle x, z \rangle$, which contradicts our assumption that $\langle x_a, z_a \rangle \notin U$ for all a.

Finally to prove the results on nets $\{t_a\}$ clustering around a compact set K in T, assume that no point of K is a cluster point of $\{t_a\}$. For each t in K there exists an open neighborhood U_t of t such that $\{t_a\}$ is eventually in $T - U_t$. Thus $\{t_a\}$ is eventually in the complement of each finite union of U_t's, in particular, in the complement of some neighborhood of K.

11.5. To prove 11.1 (a), (b) we need*) the notion of a certain composite-product of parametrizations of graphs.

Definition. If $k_1 : T_1 \to P \times Q$, and $k_2 : T_2 \to Q \times R$ are correspondences, we define a correspondence

$$k = k_2 \square k_1 : T_1 \times T_2 \to P \times R$$

by setting

$$k[(\langle t_1, t_2 \rangle)] = k_2[(t_2)] \circ k_1[(t_1)] .$$

Theorem. *If k_i are closed-graph, and one of them is usco-compact, then $k = k_2 \square k_1$ is closed-graph: If both k_1 and k_2 are usco-compact then so is k.*

Proof. 11.1 (c), (d), and 11.4.

11.6. Proof of 11.1 (a), (b): Apply Theorem 11.5.

12. THE SOUSLIN GRAPH THEOREM

Here we just state the main results from FROLÍK [10]. It should be remarked that the original Banach's proof of the classical Banach theorem got a nice setting in MARC WILDE's paper [1], see also KELLEY [2], Closed-Graph Theorem. The result in 12.1 is a generalization of SCHWARTZ [1] as well as MARTINEAU [1] results.

12.1. Souslin-graph theorem. *Assume that E is a* T.L.S. *which is inductively generated by homomorphisms from non-meager* T.L.S., *and that F is a locally convex* T.L.S. *which is analytic. Then*

if f is a homomorphism of E into F such that the graph of f is a Souslin set in $E \times F$ then f is continuous.

The proof follows immediately from

12.2. Theorem. *Let $g : E_1 \to E$, and $h : F \to F_1$ be continuous homomorphisms, where E_1, E, and F are* T.L.S., *and F_1 is a locally convex* T.L.S. *Assume that E is non-meager, and F is analytic. Then*

if the graph of a homomorphism $f : E \to F$ is a Souslin set in $E \times F$ then $k = h \circ f \circ g$ is continuous.

---

*) A much simpler proof goes as follows (for (*a*)): let $C = E\{\langle x, y, z \rangle \mid f_1 x = y, f_2 y = z\}$; C is analytic, and the graph of f is the projection of C.

The proof follows from Propositions 11.3, 12.3 and 12.4. In fact the following more general result is established by that:

Proposition. *Assume that E_1, E, F, and F_1 are inductively continuous groups, g and h are continuous homomorphisms, and F is analytic. Then for each symmetric closed set K in F_1 such that $\bigcup n \circ K = F_1$ there exists an n such that $k^{-1}[nK]$ is a neighborhood of the zero in E_1.*

12.3. Any Souslin set in any space has the property of Baire. More generally, the collection of all sets with the property of Baire in any space is invariant under the Souslin operation.

Recall that a set X has the *property of Baire* in a space P (is *almost open* in another terminology) if $(F - X) \cup (X - F)$ is meager for some closed (or equivalently: open) set F in P. For a proof see Kuratowski [1].

12.4. Assume that X is non-meager, and has the property of Baire in an inductively continuous group G. Then $X - X$ $(= \mathbf{E}\{x - y \mid x \in X,\ y \in X\})$ is a neighborhood of zero in G.

For a proof we refer to Kuratowski [1] or Kelley [2], or Čech [2].

12.5. It would be useful to know more about analytic locally convex linear spaces. We refer to Treves [1], Appendix, for examples.

13. REMARKS TO NON-SEPARABLE THEORY*)

Classical separable theory deals with respectable sets in separable metrizable spaces. In the preceding sections an extension of the separable theory to completely regular spaces (basic properties of analytic sets to general spaces) was described. Restricting respectable spaces to metrizable we get exactly concepts of classical separable theory. The analytic spaces are just completely regular absolute Souslin sets, bianalytic spaces are just absolute Baire sets (in tight extensions!), Borelian spaces have also a description as absolute something between Baire and Borel-closed. Thus Baire sets and Borel-closed sets led to separable theory.

Classical non-separable theory deals with respectable sets in the class of all metrizable spaces. For non-separable classical theory we refer to A. H. Stone papers. In metrizable spaces Baire set, Borel-closed sets, and Borel-open sets coincide. It turns out that absolute Borel-open spaces give a non-separable theory. After developing the basic properties of these spaces, a connection with separable theory will be de-

*) For every good recent results (1969, 1970) of two A. H. Stone students see the Stone's paper in Proc. Top. Conf. in Pullman (April 1970), Washington State University.

scribed. For details we refer to [F 11]. All spaces are assumed to be completely regular.

13.1. Definition. A *tight extension* of a space P is a space $Q \supset P$ such that P is dense in Q. An *absolute Borel-open* (*absolute Souslin-open, absolute* G_δ) space is a space P that has the respective property in each of its tight extensions.

For more results on G_δ-spaces we refer to Frolík [1], [2], [3] and recent papers by Wicke and Worrell, where one can find further references. It seems to be clear why the absoluteness is restricted to tight extensions.

Theorem. *Each of the following conditions is necessary and sufficient for a space P to be an absolute Borel-open space (absolute Souslin-open space, absolute G_δ-space):*

a) *P is Borel-open (Souslin-open, G_δ, respectively) in βP.*

b) *P is Borel-open (Souslin-open, G_δ, respectively) in some compactification of P.*

c) *P is Borel-open (Souslin-open, G_δ, respectively) in each compactification of P.*

Proof. It is routine that the set of the stated conditions is necessary and sufficient, and c implies b, and b implies a. Therefore we have to prove that a implies c. Let K be a compactification of P, and let $f : \beta P \to K$ be the continuous extension of the identity of P, and let φ : open $(\beta P) \to$ open (K) be defined by

$$\varphi U = \mathbf{E}\{y \mid f^{-1} y \subset U\}\,.$$

It is easy to verify that if $P = \mathbf{S}U$ with $U : \mathsf{S} \to$ open (βP), then $P = \mathbf{S}\varphi \circ U$ when $(\varphi \circ U)\, s = \varphi U s$, for each s in S (this proves the part concerning the Souslin-open set), and if $P = \bigcap\{U_n \mid n \in \mathsf{N}\}$ with U_n open in βP, then $P = \bigcap\{\varphi U_n\}$ with φU_n open in K, and this proves the part concerning G_δ's. Finally assume that P is Borel-open in βP; then by Theorem 1.9

$$P = \mathbf{S}U\,, \quad \beta P - P = \mathbf{S}F$$

where $U : \mathsf{S} \to$ open (βP), $F : \mathsf{S} \to$ closed (βP), and for each σ and τ in Σ there exists an n such that

$$U\sigma_n \cap F\tau_n = \emptyset\,.$$

Put $U' = \varphi \circ U$, and define F' by setting $F's = f[Fs]$ for s in S then clearly

$$P = \mathbf{S}U'\,, \quad K - P = \mathbf{S}F'\,,$$

and also

$$U'\sigma_n \cap F'\tau_n = \emptyset$$

whenever

$$U\sigma_n \cap F\tau_n = \emptyset\,;$$

this proves that P is Borel-open in K by Theorem 1.9.

Remark. A similar result is true for each Borel-open class, see WILLARD [1].

13.2. Definition. A *classical absolute Borel set* (*Souslin set*) is a metrizable space that is a Baire set (Souslin set) in every metrizable space in which it is embedded.

Theorem. *Each of the following conditions is necessary and sufficient for a metrizable space P to be a classical absolute Borel* (*Souslin*) *set*:

a) *P is Borel-open* (*Souslin-open*) *set in a completion Q of P.*

b) *P is an absolute Borel-open* (*Souslin-open*) *space.*

c) *P is of the form $A \cap G$ in βP when A is Borelian* (*analytic*) *in βP, and G is a G_δ.*

Proof. By definition condition a is necessary and it is obvious that b is sufficient. To prove that a implies b, assume that P is Borel-open (Souslin-open) in a completion Q of P. By a well-known result of E. Čech [1] Q is an absolute G_δ. Thus P is a Borel-open (Souslin-open) set in each compactification of Q, and hence, by Theorem 13.1, condition b is fulfilled. Thus each of the conditions a and b is necessary and sufficient. The proof is completed by showing that a is equivalent to c. We start with the easier implication. Assume that P is a Borel-open (Souslin-open) set in a completion Q of P. Because of metrizability of Q, P is a Baire set (Souslin set) in Q. Hence $P = Q \cap B$ with B a Baire set (Souslin set) in βQ. Consider the canonical mapping f of βP onto βQ, and put $G = f^{-1}[Q]$, $A = f^{-1}[B]$. It is easy to verify that condition c is fulfilled. Finally assume condition c. Take a dusco-compact (usco-compact) correspondence g of Σ into βP with $\mathbf{E}f = A$. Take a metric d for P such that, if Q is a completion of $\langle P, d\rangle$, and if f is the canonical mapping of βP onto βQ, then $f^{-1}[Q] \subset G$. This follows from the fact that P is paracompact; if $G = \bigcap\{U_n\}$ with U_n open, then d should be chosen such that the closures in βP of sets of diameter less than $1/n + 1$ are contained in U_n. In the case when f is dusco-compact we may and shall assume that d is chosen such that the correspondence

$$h = f \circ g$$

is dusco-compact (see 8.3). If $B = \mathbf{E}h$, then

$$Q \cap B = P$$

in βQ (compare with the concluding part of the proof of Theorem 13.1).

Remark. Condition c in Theorem 13.2 is a realization of a very general approach to non-separable theory. Assume that we are given a class $\mathscr{K}$ of respectable spaces in a separable theory. We define the corresponding class in the non-separable theory to be the class of open P that are of the form $G \cap K$ in βP with K in $\mathscr{K}$, and G a G_δ in βP. For more details we refer to FROLÍK [9] and [11].

In conclusion we shall state without outlines of proofs several characterizations from [11].

13.3. Intrinsic characterizations. Recall that all spaces in this section are assumed to be completely regular.

13.3.1. *A space P is absolutely Souslin-open if and only if there exists a Souslin family $U : \mathsf{S} \to \text{open}\,(P)$ such that $\mathbf{S}(U) = P$, and a family $\{\mathscr{U}(s) \mid s \in \mathsf{S}\}$, each $\mathscr{U}(s)$ being an open cover of $U(s)$, such that the following condition is fulfilled:*

If $\mathscr{M}$ is a filter on P and if there exists a σ in Σ with $\mathscr{M} \cap \mathscr{U}(s) \neq \emptyset$ for all $s < \sigma$, then $\mathscr{M}$ has a cluster point. (This is Theorem 16A in [F 11].)

13.3.2. If M is a Souslin family we define the *co-Souslin set* associated with M to be the set

$$\mathbf{coS}\,(M) = \bigcap\{\bigcup\{M(s) \mid s < \sigma\} \mid \sigma \in \Sigma\}\,.$$

The following is easy to check:

If P is a set, and if $M : \mathsf{S} \to \exp P$, and $N : \mathsf{S} \to \exp P$ are such that $Ms = P - Ns$ for all s then

$$\mathbf{S}(P) = P - \mathbf{coS}\,(N)\,.$$

13.3.3. Theorem. *The following condition is necessary and sufficient for a space P to be absolutely Borel-open:*

There exist U and $\{\mathscr{U}(s)\}$ with the properties in Theorem 13.3.1, and an order-preserving mapping $V : \mathsf{S} \to \text{open}\,(P)$ such that $\{\{V(s) \mid s < \sigma\} \mid \sigma \in \Sigma\}$ is a complete family of coverings, and for each $\sigma, \tau \in \Sigma$ there exists an n with $U\sigma_n \subset V\tau_n$. (This is Theorem 16B in [F 11].)

The proof uses important Theorem 1.9.

13.3.4. Theorem. *The following condition is necessary and sufficient for a space P to be Borel-closed in βP:*

There exist $F, H : \mathsf{S} \to \text{closed}\,(P)$ such that: (a) *$\{\{Hs \mid s < \sigma\} \mid \sigma \in \Sigma\}$ is a complete family of coverings,* (b) *if $\mathscr{M}$ is a filter on P and $Ms \in \mathscr{M}$ for all $s < \sigma$ and some $\sigma \in \Sigma$, then $\mathscr{M}$ has a cluster point, and* (c) *for each $\sigma, \tau \in \Sigma$ there exists an n with $F\sigma_n \subset H\tau_n$.* (This is Theorem 16C in [F 11].)

The proof again depends on Theorem 1.9.

13.4. Further intrinsic characterizations. Assume that P is a space and $\alpha = \langle \alpha_1, \alpha_2 \rangle$ is a pair such that α_i-Cauchy filters are defined. For example α_i may be a metric on P, a uniformity on P, or a family of coverings of P. Another example: if M is a Souslin family of subsets of P then an M-Cauchy filter is a filter $\mathscr{M}$ such that $Ms \in \mathscr{M}$ for all $s < \sigma$ for some σ in Σ. An α-Cauchy filter is a filter that is α_i-Cauchy for $i = 1, 2$. We say that α is complete (on P) if every α-Cauchy filter has a cluster point in P.

One can prove that a space P is analytic if and only if there exists a complete Souslin family M of subsets of P with $\mathbf{S}M = P$. Indeed if f is an usco-compact correspondence of Σ onto P then $\{f[\Sigma s] \mid s \in \mathsf{S}\}$ is a complete Souslin family, and if M is complete Souslin family on P then $\{\sigma \to \tilde{M}\sigma\} : \Sigma \to P$ is usco-compact.

13.4.1. Theorem. *Each of the following two conditions is necessary an sufficient for metrizable space P to be absolutely Souslin space*:

a) *There exist a sequence $\{\mathscr{U}_n\}$ of open coverings of P, and a Souslin family M* of subsets of P with $\mathbf{S}M = P$ such that $\langle\{\mathscr{U}_n\}, M\rangle$ *is complete on P.*

b) *There exist a continuous pseudometric d on P and an M as in a) such that $\langle d, M\rangle$ is complete.* (See [F 11], Section 1.)

13.4.2. By a *Borel structure* on a space P we mean a Souslin family M such that each $\{Ms \mid s \in \mathsf{S}_n\}$ is a disjoint cover of P, $Ms \subset Mt$ if $t < s$, and what is more important, if $\sigma, \tau \in \Sigma$ then the sets $M\sigma_n$ and $M\tau_n$ are functionally separated for large enough n. If P is a proximity space (or if a proximity is clearly given, e.g. if P is a metric space), then by a Borel structure on P we mean a Borel structure on the induced topological space such that the last condition is strenthened by replacing "functionally separated" by "distant". A Borel structure M is said to be complete if the sequence $\{\{Ms \mid s \in \mathsf{S}_n\}\}$ is complete. Up to the indexing a complete Borel structure on P is nothing else than a Borelian structure on P.

13.4.3. Theorem. *Each of the following conditions is necessary and sufficient for a metrizable space P to be absolutely Borel*:

a) *There exist a Borel structure M on P, and a sequence $\{\mathscr{U}_n\}$ of open coverings of P such that $\langle M, \{\mathscr{U}_n\}\rangle$ is complete.*

b) *There exist a metric d for P and a Borel structure M on $\langle P, d\rangle$ such that $\langle M, d\rangle$ is complete.*

Proof. See section 2 in [F 11] where further similar characterizations are given.

Remark. Define a Borel structure on a subset X of a space P to be a Borel structure M on X with the last condition understood in the whole space; we call M relatively complete if every M-Cauchy filter on X has a cluster point in X whenever it has a cluster point in P. One can prove that a subset of a metrizable space P is a Baire set in P if and only if there exists a relatively complete Borel structure on X in P.

13.5. Further characterizations.

13.5.1. Theorem. *A space P is a classical absolutely Souslin (Borel) space if and only if P is homeomorphic to a closed subspace of a product space $G \times B$ where G*

is a completely metrizable space, and B is a separable classical absolutely Souslin (Borel) space.

Proof. [F 11, Theorem 12.]

13.5.2. Theorem. *A metrizable space P is classical absolutely Souslin (Borel) space if and only if for each completion R of P there exists a continuous mapping f of P into a separable classical absolute Souslin (Borel) space, that extends to no point of $R - P$.*

14. ABSTRACT THEORY IN PAVED SPACES

It is an interesting and useful fact that the fundamental concepts of analytic set and Borelian set can be defined and developed in general setting, starting with any collection of sets that is subject in several basic results to one of the two natural conditions, finite multiplicativity or finite additivity. For the proofs and more details we refer to Frolík [17]. In addition we shall define all the concepts over a given space Q, not necessarily the space Σ of all irrationals, and show what properties of Q are needed for the validity of the fundamental theorems. For example to get the stability of the Souslin operation on Q (i.e , $\mathbf{S} \circ \mathbf{S} = \mathbf{S}$) it is enough to assume that Q maps continuously onto $Q^{\aleph_0}$. This is the reason for recommending this section also to the readers interested just in the descriptive theory in topological spaces. For technical reasons, and to point out the topological standpoint, we shall work with pairs $\langle S, \mathscr{M} \rangle$ where S is a set and $\mathscr{M}$ is a collection of subsets of S.

14.1. A *paved space* is a pair $P = \langle S, \mathscr{M} \rangle$ where S is set and $\mathscr{M}$ is a collection of subsets of S; S is denoted by $|P|$, and $\mathscr{M}$ is denoted by st P and called the pavement of P. The elements of the pavements are called the stones of P. If we assume that the empty set is a stone then we get precisely the concept of paved space as introduced by P. Meyer [1]. Every topological space is regarded to be a paved space; the stones are just the closed sets. For a paved space P we denote by top P the topological space whose underlying set is $|P|$ and the pavement of top P is the smallest topology (to mean the closed sets) containing the pavement of P.

A correspondence $f : Q \to P$ of paved spaces is called *usco* if the preimages of stones are the stones. Clearly the identity mapping of top P into P is usco. It is evident that if a mapping f of a topological space Q into P is usco, then $f : Q \to$ top P is usco. This is not true for correspondences in general. Nevertheless this is true for usco-compact correspondences whenever the paved space is finitely multiplicative. To introduce usco-compact correspondences it is enough to define compact sets in paved spaces. Define a set X in P to be compact if X is compact in top P. The argument establishing the Alexander lemma shows that a set X in P is compact if and only if for each collection $\mathscr{F}$ of stones such that $(X) \cup \mathscr{F}$ is a filter subbase (i.e. it has the finite intersection property) the intersection of $\mathscr{F}$ meets X.

Proposition. *The composite of two usco or usco-compact correspondences has the respective property. If f is an usco-compact correspondence of a topological space Q into a finitely multiplicative paved space P then $f : Q \to \operatorname{top} P$ is usco-compact.*

In the case of topological spaces the Souslin sets are defined as the images of Σ under the closed-graph correspondences. It may seem to be natural to define a correspondence $f : Q \to P$ to be closed-graph if $f : Q \to \operatorname{top} P$ is closed graph. With this definition we would not get the Souslin sets in P as closed-graph images of Σ; we would get the Souslin sets in top P. We are going to introduce the so-called S-correspondences to get that the Souslin sets in a paved space are just the images of Σ under S-correspondences. Proposition 14.2 says something about the relationship between S-correspondences and closed-graph correspondences. More information is given in 14.6.

14.2. By a *Souslin family over $\mathscr{B}$ in $\mathscr{M}$* we mean a single-valued relation M with $\mathbf{D}M$ being a family of sets $\mathscr{B}$ and $\mathbf{E}M \subset \mathscr{M}$. The *Souslin set of M*, designated by $\mathbf{S}M$, is the set

$$\bigcup\{\bigcap\{MB_a \mid x \in B_a \in \mathscr{B}\} \mid x \in \bigcup\mathscr{B}\}\,.$$

Thus if $\mathscr{B} = \{\Sigma s \mid s \in \mathrm{S}\}$ then we get just the Souslin $\mathscr{M}$-sets. The collection of all Souslin sets in $\mathscr{M}$ over $\mathscr{B}$ is denoted by $\mathbf{S}_{\mathscr{B}}(\mathscr{M})$.

The relation associated with M is the set $\tilde{M}$ of all pairs $\langle x, y\rangle$ such that $x \in \bigcup\mathscr{B}$ and $y \in MB_a$ for each B_a containing x. Clearly

$$\mathbf{E}\tilde{M} = \mathbf{S}M\,.$$

Proposition. *Let f be a correspondence of a topological space Q into a paved space P. If f is associated with a Souslin family over an open cover of Q in* st P *then the graph of f is a closed set in $Q \times$* top P. *If $f : Q \to$* top P *is closed-graph then f is associated with a Souslin family in* st (top P) *over any open base for Q.*

Definition. An *S-family over a topological space Q* in a paved space P is a Souslin family M in st P over an open countable cover of Q. The correspondence

$$\tilde{M} : Q \to P$$

is said to be associated with M. Finally, an *S-correspondence* is a correspondence associated with an S-family.

The following two theorems are fundamental for developing of Souslin, analytic and Borelian sets.

Theorem 1. *Let $f : R \to Q$ be an usco-compact correspondence for topological spaces. If $g : Q \to P$ is an S-correspondence then so is the composite $h = g \circ f$ provided that the pavement of P is finitely additive.*

Remark. Theorem 1 is formulated for a correspondence associated with a Souslin family over a countable cover. The proof is the same for any cardinal of the cover and therefore we get the following

Corollary. (Theorem in [F 16].) *If $f : R \to Q$, and $g : Q \to P$ are correspondences for topological spaces, if f is usco-compact and if g is closed-graph then h is closed-graph.*

Theorem 2. *Assume that P is a paved space, $\mathscr{B}$ is an open cover of a topological space Q. For each B_a in $\mathscr{B}$ let $f_a : Q_{B_a} \to P$ be a correspondence of a topological space Q_{B_a} into P. Define a correspondence f of* (see Theorem 3.9)

$$Q' = Q \times \Pi\{Q_{B_a} \mid B_a \in \mathscr{B}\}$$

into P by setting

$$\langle\langle x, \{x_a \mid B_a \in \mathscr{B}\}\rangle, y\rangle \in \operatorname{gr} f$$

if and only if

$$y \in \bigcap\{f_a[(x_a)] \mid x \in B_a \in \mathscr{B}\} .$$

If $\mathscr{B}$ is countable, and if all f_a are S-correspondences then so is f. If, in addition, there is a subcover $\mathscr{C}$ of $\mathscr{B}$ such that all f_a with B_a in $\mathscr{C}$ are usco-compact then f is usco-compact provided that the pavement of P is finitely multiplicative.

Corollary. *The intersection of a countable number of S-correspondences is an S-correspondence. If in addition one of the correspondences is usco-compact then so is the intersection provided that the pavement is finitely multiplicative.*

Remark. As a consequence of the general setting of Theorem 2 we get that the Souslin product of closed-graph correspondences is closed-graph, and the Souslin product of usco-compact closed-graph correspondences is usco-compact and closed-graph.

14.3. Definition. Let Q be a topological space, and let P be a paved space. A *Souslin set in P over Q* is the image of Q under an S-correspondence of Q into P. An *analytic set in P over Q* is the image of Q under an usco-compact S-correspondence of Q into P. *Souslin sets with disjoint representation* (called *d-Souslin sets*) and *Borelian sets* are defined in an obvious way. We use $\mathbf{S}_Q(P)$, $\mathbf{A}_Q(P)$, $\mathbf{S}^d_Q(P)$ and $\mathbf{A}^d_Q(P)$ to denote the collection of all, respectively, Souslin, analytic, d-Souslin or Borelian sets in P over Q. If $\mathscr{M}$ is a collection of sets then $\mathbf{S}_Q(\mathscr{M})$ etc. have the obvious meaning. If $Q = \Sigma$, then the subscript Q is omitted and we speak just about Souslin etc. sets in P or in $\mathscr{M}$. It follows from the definitions that:

Proposition 1. *If X is a stone in a finitely multiplicative P, and if Y is Souslin or analytic or with a disjoint Souslin representation or Borelian in P over Q, then so is $X \cap Y$.*

Proposition 2. *Let P be a paved space, and let Q be a topological space. Every Souslin (d-Souslin) set in P over Q is Souslin (d-Souslin) in* top *P over Q. If P is finitely multiplicative then every analytic (Borelian) set in P over Q is analytic (Borelian) in* top *P over Q.*

Proof. The first statement follows from Proposition 14.2, the second from the fact that if $f : Q \to P$ is usco-compact then $f : Q \to \text{top } P$ is usco-compact provided that Q is a topological space and P is finitely multiplicative.

Thus in the case of finitely multiplicative paved spaces if X is respectable in P then X is respectable in top P. There are several theorems saying that if X is respectable in P provided that X is respectable in P in a weaker sense, e.g. distinguishable. For results of this sort see 14.7.

The following results don't need any comment.

Theorem 1. *Assume that f is an S-correspondence of a topological space Q into a finitely additive paved space P, and let $A \subset Q$ be the image of a topological space R under an usco-compact correspondence of R into Q (this assumption is fulfilled if A is analytic in Q over R). Then $f[A]$ is Souslin in P over R, and if in addition f is usco-compact then $f[A]$ is analytic in P over R.*

Proof. Apply Theorem 1 in 14.2.

Theorem 2. *Let P be a paved space, and let Q be a topological space that continuously maps onto $Q^{\aleph_0}$. Then*

$$\mathbf{S}_Q(\mathbf{S}_Q(P)) = \mathbf{S}_Q(P),$$

and if P is finitely multiplicative then

$$\mathbf{S}_Q(\mathbf{A}_Q(P)) = \mathbf{A}_Q(P).$$

If, in addition, Q bijectively continuously maps onto $Q^{\aleph_0}$, then the formulas are true with the supper script d attached.

Proof. Apply Theorem 2 in 14.2 and the following obvious

Lemma. *If there exists a continuous mapping of Q_1 onto Q then*

$$\mathbf{S}_{Q_1}(P) \supset \mathbf{S}_Q(P), \quad \textit{and} \quad \mathbf{A}_{Q_1}(P) \supset \mathbf{A}_Q(P)$$

for any paved space P. If there exists a one-to-one continuous mapping of Q_1 onto Q then the formulas are true with the supper script d attached.

Corollary. *Assume that P is a finitely multiplicative paved space, and Q is a topological space that maps continuously (bijectively) on $Q^{\aleph_0}$. Then $X \subset |P|$ is analytic (Borelian) in P over Q if and only if X is Souslin (d-Souslin) in P over Q, and X is contained in an analytic (Borelian) set in P over Q.*

Proposition 3. *Assume that Q is a topological space and P is a paved space. Then*

$$\operatorname{st} \mathbf{S}_Q(P) \supset (\operatorname{st} P)_\delta .$$

If Q is regular and contains an infinite discrete set then

$$\operatorname{st} \mathbf{S}_Q(P) \supset (\operatorname{st} P)_\sigma .$$

Corollary a. *For any collection of sets we have*

$$\begin{aligned} \mathbf{B}(\mathbf{S}(\mathscr{M})) &= \mathbf{S}(\mathscr{M}) \supset \mathbf{B}(\mathscr{M}) \supset \mathscr{M} , \\ \mathbf{B}(\mathbf{A}(\mathscr{M})) &= \mathbf{A}(\mathscr{M}) \supset \mathbf{B}(\mathscr{M}) \supset \mathscr{M} . \end{aligned}$$

A similar theorem for $\mathbf{S}^d$, $\mathbf{A}^d$ shows that

Corollary b. *For any collection $\mathscr{M}$ we have*

$$\begin{aligned} \mathbf{B}_d(\mathbf{S}^d(\mathscr{M})) &= \mathbf{S}^d(\mathscr{M}) \supset \mathbf{B}_d(\mathscr{M}) \supset \mathscr{M} , \\ \mathbf{B}_d(\mathbf{A}^d(\mathscr{M})) &= \mathbf{A}^d(\mathscr{M}) \supset \mathbf{B}_d(\mathscr{M}) \supset \mathscr{M} . \end{aligned}$$

Example. If Q is a countably infinite discrete space then $\mathbf{S}_Q(\mathscr{M}) = \mathscr{M}_{\delta\sigma}$. If we want to repeat this operation infinitely times, we must consider $Q^{\aleph_0} = \Sigma$, but then $\mathbf{S}(\mathscr{M})$ may be strictly larger than $\mathbf{B}(\mathscr{M})$, e.g. if $\mathscr{M}$ is the set of all closed sets on the real line.

Remark. In the case of topological spaces the projection of an analytic set is always analytic. This is an immediate consequence of Theorem 2 in Frolík [16] that says that if π is a projection of a product space $R \times P$ onto P, and if $f : Q \to R \times P$ is a closed-graph usco-compact correspondence, then $\pi \circ f$ is closed-graph (and, of course, usco-compact). In the present general setting we haven't defined the product of paved spaces, and therefore no generalization of this important theorem is given, and as a result, the projection technique for developing of analytic sets is not discussed here. For a development of Souslin sets by the projection technique we refer to P. MEYER [1] and 14.6. In the case of general topological spaces the projection technique is developed in Frolík [16] for Souslin and also analytic sets.

14.4. In this subsection we consider the respectable sets over the space Σ of irrational numbers. The first result says that the open base $\{\Sigma s \mid s \in \mathrm{S}\}$ for Σ suffices; this shows that the theory presented in this section extends the classical theory. The further results are related to separation of analytic sets.

Proposition 1. *If f is an S-correspondence of Σ into a finitely multiplicative paved space P then there exists a homeomorphism k of Σ onto itself such that $f \circ k$ is associated with a Souslin family in P over $\{\Sigma s\}$.*

We refer to Frolík [12] for the proof, and also for further related results, e.g., in the case of the Souslin sets, for a replacement of finite multiplicativity by the assumption that the empty set is a stone.

The first separation principle has the following setting.

Proposition 2. *Assume that P is a finitely multiplicative paved space, and let $\mathscr{C}$ be a collection of subsets of $|P|$ such that $\mathbf{B}(\mathscr{C}) = \mathscr{C}$.*

a) *Assume that if X is a compact set in $(\mathrm{st}\, P)_\delta$, and if Y is a stone disjoint to X, then there exists a neighborhood $U \in \mathscr{C}$ of X in* top P *with $U \cap Y = \emptyset$. Then if X is analytic in P, and if X is a Souslin set in P disjoint to X, then $X \subset C \subset |P| - Y$ for some C in $\mathscr{C}$.*

b) *Assume that any two disjoint compact sets in $(\mathrm{st}\, P)_\delta$ are separated by neighborhoods belonging to $\mathscr{C}$. Then any two disjoint analytic sets in P are separated in $\mathscr{C}$.*

Remark. The separation assumption in Proposition 2 are satisfied if top P is a separated space that locally belongs to $\mathscr{C}$. This is needed in the next result that should be compared with 5.10. A set X in P is called bianalytic if X and $P - X$ are analytic. We denote by Bianal (P) the collections of all bianalytic sets in P.

Theorem. *Assume that P is finitely multiplicative paved space such that* top P *is separated and locally belongs to a collection of sets $\mathscr{Z}$. Let $\mathscr{X}$ be the complementary part of $\mathbf{B}(\mathscr{Z})$. Then*

$$\mathscr{X} \supset \mathrm{Bianal}\,(\mathrm{top}\, P) \supset \mathrm{Bianal}\,(P)\,,$$

and if top P *is analytic and* $\mathscr{Z} = \mathrm{st}\, P$ *then*

$$\mathscr{X} = \mathrm{Bianal}\,(\mathrm{top}\, P) = \mathrm{Bianal}\,(P)$$

and top P *locally belongs to $\mathscr{X}$.*

Consider the particular case where top P is a separated completely regular space and st P is the set of all zero-sets in top P. It is clear that Proposition 2 is a generalization of the author's theorems 5.8 and 5.10.

14.5. Here we want to give four further characterizations of Borelian spaces, see Theorem 7.7.

Theorem 1. *Each of the following four conditions is necessary and sufficient for a completely regular space P to be a Borelian space:*

1. *X is an absolute d-Souslin set in the class of all completely regular spaces.*
2. *X is a d-Souslin set in some Borelian space.*
3. *X is a d-Souslin set in some compact separated space.*
4. *X is a d-Souslin set derived from compact sets in a completely regular space.*

Proof. Necessity is obvious, and to prove sufficiency it is enough to observe that a correspondence associated with a Souslin family ranging in compact sets is usco-compact, and that the intersection of a closed-graph correspondence with an usco-compact correspondence is usco-compact.

Lemma. *Let X be a Borelian set in a finitely multiplicative paved space, and let $\mathscr{C}$ be a collection of sets such that any two analytic sets are separated in $\mathscr{C}$. Then*

$$X \in ([\mathscr{C}] \cap [\mathrm{st}\, P])_{\sigma_d \delta} .$$

Proof. See 7.4

Theorem 2. *Assume that P is finitely multiplicative paved space, and let $\mathscr{C}$ with* $\mathbf{B}(\mathscr{C}) = \mathscr{C}$ *be a collection of Borelian sets such that*

$$(*) \qquad \mathscr{C} \text{ separates analytic sets} .$$

Then

$$\mathbf{A}_d(P) = ([\mathscr{C}] \cap [\mathrm{st}\, P])_{\sigma_d \delta} .$$

Condition $(*)$ is satisfied if top P *is separated and locally belongs to $\mathscr{C}$.*

14.6. Graph characterization of S-correspondences. We know that any S-correspondence is closed-graph, and the converse is not true in general. To characterize S-correspondences by a property of graphs it is convenient to introduce several concepts.

Definition. The complements of the stones in a paved space are called *costones.* Thus in a topological space the costones are just the open sets. If Q and P are paved spaces we say that $X \subset |Q| \times |P|$ is a *σ-set* if X is the union of a countable collection of sets of the form $U \times V$ where U is a costone in Q and V is a costone in P. The complements of *σ-sets* are called *δ-sets.* It should be remarked that the product $Q \times P$ has not been defined.

Theorem. *In order that a correspondence f of a topological space Q into a paved space P such that $|P|$ is a stone to be an S-correspondence it is necessary and sufficient that the graph of f to be a δ-set.*

Proof. If f is associated with a Souslin family M in P over a countable cover $\{B_a\}$ then the complement of the graph is the union of all the sets $B_a \times (|P| - MB_a)$, and hence it is a *σ-set*. Conversely, if the complement of the graph of f is the union of a countable family $\{B_a \times V_a\}$ where B_a and V_a are costones in the respective paved spaces, then put

$$MB_a = |P| - V_a .$$

We may and shall assume that one of the sets $B_a \times V_a$ is $Q \times \emptyset$ $(= \emptyset)$. Then f is associated with M.

14.7. D-sets. We know that every analytic or Borelian set in a multiplicative paved space P has the respective property in top P. In this section we are concerned with the question under what sufficient conditions an analytic or Borelian set in top P has the respective property in P. The most important example is given in 14.8. The reader is invited to read simultaneously that section.

Definition 1. A *D-set* in a paved space P is a set X such that there exists a countable collection $\mathscr{C}$ of stones with the property that if $x \in X$ and $y \in P - X$ then $x \in C \subset P - (y)$ for some C in $\mathscr{C}$.

Clearly the collection of D-sets is invariant under countable intersections and countable unions, and each stone in P is a D-*set* in P.

Definition 2. A paved space P is called *first countable* if each stone X in P is the intersection of a sequence $\{C_n\}$ of stones such that $X \subset \operatorname{int} C_n$ for each n (the interior is taken in top P). It should be remarked that this definition has nothing to do with first countable topological spaces.

The main result reads.

Theorem 1. *Assume that P is a first countable paved space such that any two disjoint analytic sets in* top P *are separated by D-sets in P. Then a Borelian set X in* top P *is Borelian in P if and only if X is a D-set.*

We shall see that this is a generalization of Theorem 8.4 on BB-sets. In the case of analytic set the situation is more complicated, and the two main results are given below. In Theorem 1 "only if" is obvious, and "if" follows from the following simple result.

Proposition 1. *Assume that P is a paved space, such that f is an usco correspondence* *of a topological space Q into* top P. *Let $\mathscr{C}$ be a countable collection of stones such that for each y in Q and each x in $P - fy$ there exists a C in $\mathscr{C}$ with*

$$fy \subset \operatorname{int} C \subset C \subset P - (x)\,.$$

Then $f : Q \to P$ is an S-correspondence.

Proof. For each C in $\mathscr{C}$ let U_C be the set of all $y \in Q$ with $fy \subset \operatorname{int} C$. Clearly f is associated with $\{U_C \to C \mid C \in \mathscr{C}\}$.

Proof of Theorem 1. Assume that X is a Borelian set in top P and a D-set in P. Choose a dusco-compact S-correspondence f of Σ into top P with $X = \mathsf{E}f$, and a countable collection $\mathscr{D}$ of stones in P such that X is "distinguished" by $\mathscr{D}$, and for each $n \in \mathsf{N}$, and distinct $s, t \in \mathsf{S}_n$ there exist D-sets X_s, Y_t "distinguished" by $\mathscr{D}$ such that $f[\Sigma s] \subset X_s$, $f[\Sigma t] \subset X_t$. Choose a finitely additive countable collection $\mathscr{C} \supset \mathscr{D}$ of stones such that each element of D of $\mathscr{D}$ is the intersection of a sequence $\{C_n\}$ in $\mathscr{C}$ such that $D \subset \operatorname{int} C_n$ for each n. One can verify that the assumptions of Proposition 1 are fulfilled. By Proposition 1 $f : \Sigma \to P$ is dusco-compact.

As concerns analytic sets we get:

Theorem 2. *Assume that P is a first countable finitely additive paved space, and let $f : \Sigma \to$ top P be an usco-compact correspondence such that there exists a countable collection $\mathscr{D}$ of sets with the following property: if $y \in \Sigma$, and if $x \in P - fy$ then there exists a D in $\mathscr{D}$ with $fy \in D \subset P - (x)$. Then $\mathsf{E}f$ is analytic in P.*

Proof is based on a similar idea as the proof of Theorem 1.

Very important is the following result.

Theorem 3. *Assume that P is a paved space, $f : \Sigma \to$ top P is usco-compact, and there exists an usco mapping g of P into a topological space R such that $g^{-1}[g[\mathsf{E}f]] = \mathsf{E}f$, and g :* top *$P \to R$ is closed-graph. Then $\mathsf{E}f$ is Souslin in P.*

Proof. The composite $h = g \circ f$ is closed-graph by Theorem 14.2.1, and hence $\mathsf{E}h$ is Souslin in R. Since $g : P \to R$ is usco, $\mathsf{E}f = g^{-1}[\mathsf{E}h]$ is Souslin in P.

Remark. If X is Souslin in P, and if X is analytic in top P then X need not be analytic in P even if P has all properties in this subsection. See Example in 14.8.

14.8. Exact sets. For a topological space P denote by exact (P) the set $|P|$ endowed with the collection of all zero sets (called exact closed sets in P, see Čech [2]), and we let Baire (P) to denote $|P|$ endowed with the smallest σ-algebra containing the exact closed sets in P. If P is a paved space we define exact (P) or Baire (P) to be, respectively, exact (top P) or Baire (top P).

Clearly the paved space exact (P) is first countable, finitely additive, and countably multiplicative.

Theorem. *If P is completely regular than the Borelian sets in* exact (P) *are just the BB-sets in P.*

Example. Let K be a compact space with just one cluster point, say x, and let the cardinal of K be at least $\aleph_2$. Let P be the sum of $\{K \mid n \in \mathsf{N}\}$ with the points $\langle x, n\rangle$, $n \in \mathsf{N}$, identified. Evidently K is Souslin in exact (K), and K is analytic, but one can show that K is not analytic in exact (K).

14.9. P. Meyer's proof of stability of the Souslin operation. A paved space is said to be countably compact if each countable filter subbase of stones has non-void intersection. Obviously every compact paved space is countably compact. Observe that Σ endowed with $\emptyset$ and the collection of all sets Σs, $s \in \mathsf{S}$, is countably compact; this space will be denoted by Σ. In this section we assume that the empty set is a stone in every paved space.

Theorem. *Let P be a paved space. If Q is countably compact paved space and if C is the countable intersection of contable unions of sets of the form $X \times Y$ with $X \in$ st P, $Y \in$ st Q, then the projection of C into P is a Souslin set in P. If A is Souslin in P and if $Q = \Sigma$, then there exists an C with the above properties such that A is the projection of C into P.*

Proof is left to the reader.

Definition. If P and Q are paved spaces, we denote by $P \otimes Q$ the set $P \times Q$ endowed with the collection of all sets $X \times Y$ with X or Y a stone in P or Q, respectively.

Lemma. *Under the assumptions on P and Q in Theorem, if C is a Souslin set in $P \otimes Q$ then the projection X of C into P is a Souslin set.*

Proof. Take a countably compact Q' and a countable intersection C' of countable unions of stones in $(P \otimes Q) \otimes Q'$ such that C is the projection of C'. Clearly X is the projection of C' into P: observe that $(P \otimes Q) \otimes Q'$ is isomorphic to $P \otimes (Q \otimes Q')$, and $Q \otimes Q'$ is countably compact.

Proof of the stability of the Souslin operation: apply Lemma and Theorem.

Remark. Notice that this proof is based on the classical projection technique; P. Meyer observed that countably compact may replace analycity. The projection technique was also used by Bourbaki to prove capacitability of Souslin sets. For a development of projection technique in general spaces we refer to [F 16]. It should be remarked that [F 17] can be completed to include the projection technique in general setting. Finally observe that several separation theorems can be stated for Souslin sets in countably compact spaces.

15. THEORY OF SIEVES AND REMARKS

A *sieve* is a pair $<C, <>$ when C is a relation and $<$ is a linear order for the range $\mathbf{E}C$ of C. The *sifted set* of C, designated by $\mathbf{SF}(C)$, is the set of all points x in $\mathbf{D}C$ such that the subset $C[x]$ of $\mathbf{E}C$ is not well ordered (that means: there exists a strictly decreasing sequence in $C[x]$). Now if Q is a linearly ordered set, and if P is a set (possibly endowed with some structures) then by a *sieve in* $P \times Q$ we mean a sieve $<C, <>$ such that $C \subset P \times Q$, and $<\mathbf{E}C, <>$ is an ordered subset of Q. We shall only consider the sieves in $P \times Q$ when P is a paved space, and Q is the ordered space Q or rationals, or the ordered space R of reals. Theory of sieves is a classical powerful tool in studying the Souslin operation and Borel sets. It seems that the first sieve was considered by H. LEBESGUES; on the other hand it was N. LUSIN who defined something very close to Lusin sieve as defined below, and discovered the ideas of everything that could be done. For an excellent introduction to the theory of sieves in the classical setting we refer to Kuratowski [1]. The Polish school has supplied a tremendous volume of material. On the other hand we would like to encourage the interested reader to go through the Lusin's paper [2], and to consult the difficulties in Kuratowski [1]. For many ingenious tricks one should go to W. SIERPINSKI's papers. For contemporary point of view we refer to Rogers [5], where further references could be found; this paper is also recommended as an introduction. It is my opinion that the theory of sieves is open to a fruitful study from the contemporary point of view.

This section is concluded by hitting some problems concerning stadard Borel spaces, standard analytic spaces (as introduced by C. W. MACKEY [1]), Blackwell spaces, and by several remarks to the general theory of compact generated algebras and related notions. The present author intends to publish more developed theory elsewhere; I don't want to set the basic definitions here because I am not sure of the details that could change the whole setting of the theory.

We add just one paragraph to report on Rogers [6]; this is a contribution to the problem how to generalize the property of metrizable spaces that every open set is respectable with respect to closed sets. The subject seems to be open to further investigations.

15.1. Relationship to the Souslin operation. A *Lusin sieve on a paved space* P is sieve C in $P \times \mathsf{Q}$, that can be written as

$$C = \bigcup\{X_n \times (q_n) \mid n \in \mathsf{N}\}$$

where the X_n's are stones in P, and $\{q_n\}$ is a one-to-one sequence in Q.

15.1.1. Lemma. *If a paved space is finitely multiplicative, then every Souslin set in P is the sifted set of a Lusin sieve on P.*

Proof. Assume that $X = \mathbf{S}M$ where $M : \mathsf{S} \to \operatorname{st} P$; we may and shall assume that $Ms \subset Mt$ for $t < s$. Define $r : \mathsf{S} \to \mathsf{Q}$ by setting

$$rs = 1 - 2^{-i_0-1} - \ldots - 2^{-i_0-i_1-\ldots-i_n-n-1}$$

for $s = \{i_k \mid k \leqq n\} \in \mathsf{S}$. Put

$$C = \bigcup\{Ms \times (rs) \mid s \in \mathsf{S}\} .$$

One can check that X is the sifted set of C, and it is obvious that C is a Lusin sieve on P.

On the other hand the sifted set of a Lusin sieve is always a Souslin set. We shall prove more:

15.1.2. Lemma. *Assume that P is a paved space, and that C is a subset of $P \times \mathsf{R}$ such that the projection of $C \cap (P \times I)$ into P is a Souslin set (analytic set) in P for each half open interval I in R. Then the sifted set of C is Souslin (analytic) in P.*

Proof. We may and shall assume that $C \subset P \times]0, 1[$. Arrange all the rationals in $]0, 1[$ in a one-to-one sequence $\{r(n)\}$, and for each $s = \{i_k \mid k \leqq n\} \in \mathsf{S}$ put

$$Rs = [r(i_0) \cdot \ldots \cdot r(i_n),\ r(i_0) \cdot \ldots \cdot r(i_{n-1})[$$

where we set $r(i_0) \cdot \ldots \cdot r(i_{n-1}) = 1$ if $n = 0$. Finally put

$$Ms = \mathbf{D}(C \cap P \times Rs) .$$

Since all Ms, $s \in \mathsf{S}$, are Souslin (analytic) in P, $\mathbf{S}M$ is also Souslin (analytic) in P. One can easily verify that $\mathbf{S}(M)$ is the sifted set of C.

As a corollary we get the following

Theorem. *If P is a topological space then the sifted sets of Souslin (analytic) sets in $P \times \mathsf{R}$ are Souslin (analytic) sets in P; each Souslin set in P is the sifted set of a Lusin sieve.*

The reader is invited to get more from the Lemmas.

15.2. Lusin sieves. For every well ordered set X we use $\tau(X)$ to denote the unique ordinal such that X is order isomorphic to the ordered set of all ordinals less than $\tau(X)$. If an ordered set X is not well ordered we put $\tau(X) = \omega_1$ (for convenience).

If X is a well ordered subset of R then $\tau(X) < \omega_1$ (i.e. $\tau(X)$ is a countable ordinal). Thence

$$\mathbf{SF}(C) = \mathbf{E}\{x \mid \tau(C[x]) = \omega_1\} .$$

We say that a *sieve C is bounded* over a set X if

$$\sup \{\tau(C[x]) \mid x \in X\} < \omega_1 .$$

The following result is simple, but very important (essentially Lusin [2], p. 72).

15.2.1. Lemma. *Let C be a Lusin sieve on a paved space P. For each countable ordinal α the set*

$$C_\alpha = \mathbf{E}\{x \mid \tau(C[x]) \geqq \alpha\}$$

is a Borel set in P (and $\bigcap\{C_\alpha \mid \alpha < \omega_1\} = \mathbf{SF}(C)$).

Proof. Assume that

$$C = \bigcup\{X_n \times (q_n) \mid n \in \mathsf{N}\}$$

where X_n are stones, and $\{q_n\} : \mathsf{N} \to \mathsf{Q}$ is one-to-one. Define X_n^α, $n \in \mathsf{N}$, $\alpha < \omega_1$ by setting

$$X_n^0 = X_n , \quad \text{and for} \quad \alpha > 1$$
$$X_n^\alpha = X_n \cap \bigcap\{X_n^\beta \mid \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit cardinal} ,$$

and

$$X_n^\alpha = X_n \cap \bigcup\{X_i^{\alpha-1} \mid q_i < q_n\} \quad \text{otherwise} .$$

One can verify that

$$C_\alpha = \bigcup\{X_n^\alpha \mid n \in \mathsf{N}\} .$$

Thence the sets C_α are Borel in P, and by definition

$$\mathbf{SF}(C) = \bigcap\{C_\alpha \mid \alpha < \omega_1\} .$$

Corollary. *If a Lusin sieve C on P is bounded over a set Y then there exists a Borel set B in P with*

$$\mathbf{SF}(C) \subset B \subset P - Y .$$

In particular, $\mathbf{SF}(C)$ *is a Borel set whenever C is bounded on P –* $\mathbf{SF}(C)$.

Call a *sieve C bounded* if C is bounded on the complement of $\mathbf{SF}(C)$ (to be more precise, say on the complement of $\mathbf{SF}(C)$ in $\mathbf{D}C$). By Corollary 15.2.1, if a Lusin sieve is bounded then the sifted set is a Borel set. One of the best results of Lusin [2] is the converse in the case when the paved space is R (or R^n, as he remarks). We state here the Rogers formulation (Rogers [8, Lemma 2]).

15.2.2. Lemma. *Assume that P is finitely multiplicative paved space, A is an analytic set in* top P (*or more generally, an usco-compact image of Σ in* top P), *and C is a Souslin set in* top $P \times Q$. *If $A \cap \mathbf{SF}(C) = \emptyset$, then C is bounded over A.*

Remark. The assumption that A is an usco-compact image of Σ is essential. For example, take an analytic set X in R, that is not Borel, and express X as the sifted set of Lusin sieve C on R. By 15.2.1 C is not bounded. Consider the subspace $P = \mathsf{R} - X$ of R, and the Lusin sieve $C' = C \cap (P \times Q)$ on P. Clearly C' is not bounded, and the sifted set of C' is empty, thence Borel.

Theorem. *Assume that P is a finitely multiplicative paved space, and that A is analytic in* top P (*it is enough to assume that A is an usco-compact image of Σ in* top P). *If X is a Souslin set in P disjoint from A, then*

$$X \subset B \subset P - A$$

for some Borel set B in P; particularly, if $X = P - A$ then X is Borel in P.

Corollary. *If $\mathcal{M} = \mathbf{B}(\mathrm{st}\ P)$ then*

$$\exp(P - A) \cap \mathrm{st}\ \mathbf{S}(P) = \mathbf{S}(\mathcal{M} \cap \exp(P - A))\,.$$

It should be remarked that Theorem is a refinement of the first separation principle as formulated before.

15.3. The second principle. The first theorem is the Rogers generalization of Kunugui's generalization of the Lusin's second separation principle (Rogers [5], Theorem 14).

15.3.1. Theorem. *Let A and B be Souslin sets derived from the bi-Souslin sets in a space P. Then there are sets C, D that are complements of Souslin sets in P and that satisfy*

$$C \cap D = \emptyset\,, \quad A - B \subset C\,, \quad B - A \subset D\,.$$

The next result is the Rogers generalization of a result of Kôndo (Rogers [5], Theorem 16).

15.3.2. Theorem. *For any space P*

$$\mathbf{S}^d\,(\text{bi-Souslin}\,(P)) = \text{bi-Souslin}\,(P)\,.$$

This result follows by the method of the proof of 7.4 from the following (Rogers [5], Theorem 15)

15.3.3. *If A and B are disjoint Souslin sets derived from bi-Souslin sets in a space P, then there are disjoint bi-Souslin A_1 and B_1 with $A \subset A_1$, $B \subset B_1$.*

The proof of the results of this section depends on the following fundamental result of Lusin (see Rogers [5], Theorem 11).

15.3.4. *Let P be a space, and let A and B be subsets of $P \times Q$ such that the sets*

$$P \times Q - A \quad \text{and} \quad B$$

are Souslin in $P \times Q$. The set of all $x \in P$ with $A[x]$ similar to a subset of $B[x]$ is a Souslin set in P.

15.4. Uniformization. Assume that C is a 'respectable' set in the product space $P \times Q$. Is there a single-valued relation $f \subset C$ with $\mathbf{D}C = \mathbf{D}f$, $\mathbf{E}C = \mathbf{E}f$ such that f is as good as C in $P \times Q$? If $P, Q = \mathsf{R}$ and C is co-analytic then f can be chosen co-analytic, and this is not true for Borel or analytic. One can consider a similar problem: is there 'respectable' compact-valued section? This problem is studied in Rogers [5].

15.5. Measurable spaces. A *measurable space*, shortly an *M-space* is a paved space P such that the pavement is a σ-algebra. A *measurable mapping*, or shortly an *M-mapping*, is an usco mapping of M-spaces. An *M-quotient mapping* is an M-mapping $f : P \to Q$ such that X is a stone in Q whenever $f^{-1}[X]$ is a stone in P. For example, if P is a topological space then the Baire space Baire (P) of P is an M-space, and if $f : P \to Q$ is continuous then $f :$ Baire $(P) \to$ Baire (Q) is an M-mapping, but the converse is not true in general.

A collection $\mathscr{M}$ of sets in a paved space P is said to be *distinguishing* if for each two distinct points x, y in P there exists an M in $\mathscr{M}$ such that either $x \in M$, $y \notin M$ or $y \in M$, $x \notin M$. An M-space is said to be *separated* if the pavement is distinguishing.

An M-base or a generating collection for an M-space P is a subset $\mathscr{M}$ of the pavement such that the pavement is the smallest σ-algebra on P containing $\mathscr{M}$. An M-space P is separable if it has a countable M-base.

For example for any paved space P the *M-modification of P* is the M-space $\mathbf{M}P$ having the pavement of P for an M-base. It should be remarked that if P is a topological space then $\mathbf{M}P$ is often called the Borel space of or induced by P, and the stones are called Borel sets in P; in our terminology Borel is used in the situations when the smallest collection closed under $\mathbf{B}$ is considered, e.g. Borel-closed sets, Borel-open sets. We have stressed many times that

$$\mathbf{M}P = \text{Baire } P = \text{Borel } P$$

if P is metrizable. In the particular case of metrizable spaces we shall also use classica terminology, e.g. classical separable absolute Borel space.

15.5.1. Embeddings into standard spaces. Denote by 2 the discrete space consisting from two points, 0 and 1. If A is a set we denote by Cantor (A) the topological space 2^A. Thus the points are the families $x = \{x_a \mid a \in A\}$ with x_a zero or one, and the topology is the topology of coordinatewise convergence. Denote by B_a^1, $a \in A$, the set of all x in Cantor (A) with $x_a = 1$. The complement of B_a^1 is denoted by B_a^0; clearly B_a^0 is the set of all x with $x_a = 0$. Note that $\{B_a^i\}$ is an open subbase for Cantor (A), B_a^i is closed and open, and the space is compact. Having in mind that Cantor (A) and Cantor (B) are homeomorphic whenever A and B are of the same cardinal, we use the symbol Cantor (m), with m an cardinal, without specifying the meaning. Obviously Cantor (m) is metrizable if and only if m is countable.

An M-embedding is a one-to-one M-mapping $f : P \to Q$ such that each stone in P is the preimage of a stone in Q.

Embedding Lemma. *Assume that $\mathscr{C}$ is an M-base for a separated M-space P. If X is a subset of P denote by γ_X the characteristic function of X; thus $\gamma_X x = 1$ if $x \in X$, and $\gamma_X x = 0$ otherwise. Define*

$$f : P \to \text{Baire}\,(\text{Cantor}\,(\mathscr{C}))$$

by setting

$$fx = \{\gamma_C x \mid C \in \mathscr{C}\}.$$

Then:

A. *f is an M-embedding.*

B. *f is onto if and only if for each non-void $\mathscr{C}' \subset \mathscr{C}$ the set $\bigcap \mathscr{C}' - \bigcup(\mathscr{C} - \mathscr{C}')$ is a singleton.*

C. *The range of f is a closed subspace of* Cantor $(\mathscr{C})$ *if and only if the collection of all C and $P - C$ with C in $\mathscr{C}$ is a compact pavement of P.*

Proof. Write Q for Cantor $(\mathscr{C})$. Evidently f is one-to-one, and to prove that f is an embedding it is enough to observe that

$$f^{-1}[B_C^1] = C$$

for each C in $\mathscr{C}$. To prove B, take a point y in Q, and observe that

$$f^{-1}(y) = \bigcap \mathscr{C}' - \bigcup(\mathscr{C} - \mathscr{C}')$$

where $\mathscr{C}'$ is the set of all C in $\mathscr{C}$ with $y_C = 1$. Finally, notice that 'only if' in C is very easy to verify, and to prove 'if', take any point y in the closure $f[P]$ in Q, and observe that the collection $\{D_C \mid C \in \mathscr{C}\}$, where $D_C = C$ if $y_C = 1$, and $D_C = P - C$ otherwise, has the finite intersection property (is a filter subbase), and $\bigcap\{D_C\} \neq \emptyset$ if and only if $y \in f[P]$.

Theorem. *Assume that P is a separated M-space. Then P embeds in* Baire (Cantor (m)) *whenever there exists an M-base of cardinal at most m; particularly, if P is separable then P embeds into* Baire (Cantor $(\aleph_0)$), *and thence P embeds into the Baire space of a compact, metrizable totally disconnected space. Further, P is isomorphic with the Baire space of a Cantor space if and only if there exists an M-Base with the 'coordinating' property in Lemma, statement B. Finally, P is isomorphic with the Baire space of a compact totally disconnected space if and only if it has an M-base that is closed under complementation, and that is a compact pavement.*

15.5.2. Analytic M-spaces. An M-space P is said to be *analytic* if P is isomorphic with the Baire space of an analytic space.

Lemma. *Let f be a continuous mapping of an analytic space P onto a space Q. Then*

$$f : \text{Baire}\,(P) \to \text{Baire}\,(Q)$$

is an M-quotient mapping, thence an isomorphism if injective.

Proof. If $X \subset Q$, and if $f^{-1}[X]$ is a Baire set in P, then $f^{-1}[X]$ and $P - f^{-1}[X]$ are analytic, thence their images X and $Q - X$ are analytic in Q, and hence X is a Baire set in Q by Separation Theorem.

Corollary. *If P is isomorphic with the Baire space of a Borelian space then P is isomorphic with the Baire space of a bi-analytic space.* (Use the fact that every Borelian space is a one-to-one continuous image of a bi-analytic space.)

Remark. It follows from 15.2.2 that if $\mathcal{M}$ is a collection of stones in an analytic M-space P with $\mathbf{B}(\mathcal{M}) = \mathcal{M}$, then

$$\mathbf{S}(\mathcal{M}) \cap \text{st}\, P = \mathcal{M}\,.$$

We shall not need this fact.

Now we are going to prove a very powerful result, that is classical Souslin theorem if the space P is metrizable. By a *Borelian M-space* we mean an M-space that is isomorphic with the Baire space of a Borelian topological space.

Theorem.*) *Let f be an M-mapping of an analytic M-space P' into a separated separable M-space Q'. Then $f : P' \to f[P'] \subset Q'$ is M-quotient, and the subspace*

*) An improved version one can find in the author's "Measurable map with analytic domain and Metrizable range is quotient", Bull. Amer. Math. Soc. 1970.

$f[P']$ of Q' is analytic. If f is one-to-one, and if P' is Borelian then $f[P']$ is metrizable Borelian.

Proof. Let P be an analytic topological space such that P' is the Baire space of P. By Theorem 15.5.1 we can choose a compact metrizable space R such that Q' is a subspace of the Baire space R' of R. Take a countable open base $\mathscr{C}$ for R (obviously the elements of $\mathscr{C}$ are Baire sets in R), and for each C in $\mathscr{C}$ put

$$MC = f^{-1}[C] \times C .$$

One can check immediately that

$$\mathbf{S}M = \operatorname{gr} f$$

and the associated relation $\tilde{M}$ is disjoint. For the definition and properties of $\mathbf{S}M$ for Souslin families over a topological space consult 14.2.

Since all MC are analytic (as Baire sets in an analytic space $P \times R$), the graph of f is analytic. If P is Borelian then all the sets MC are Borelian (because R does be Borelian), and hence the graph of f is Borelian. Thence, the projection $f[P]$ of $\operatorname{gr} f$ into R is analytic, and if, in addition, P is Borelian and f is one-to-one, then $f[P]$ is Borelian. It remains to show that the Baire space of $f[P]$ is a subspace of R', and this is evident because of the metrizability of R (and in general it would follow from the fact that $f[P]$ is analytic), and that the M-mapping

$$f : P' \to \operatorname{Baire} f[P]$$

is an M-quotient. Assume that X is a set in $f[P]$ such that the set $f^{-1}[X]$ is a Baire set in P. Hence the two sets

$$f^{-1}[X] \quad \text{and} \quad P - f^{-1}[X]$$

are analytic, thence the two sets

$$\operatorname{gr} f \cap (f^{-1}[X] \times R) \quad \text{and} \quad \operatorname{gr} f \cap ((P - f^{-1}[X]) \times R)$$

are analytic, thence their projections

$$X \quad \text{and} \quad f[P] - X$$

are analytic, and via Separation Theorem the set X is a Baire set in $f[P]$.

Remark. One can avoid the use of 'Souslin sets over a topological space' by replacing R with the space Σ of irrationals.

Corollary. *If P is an analytic M-space, and if a countable collection $\mathscr{C}$ of stones distinguishes the points of P, then $\mathscr{C}$ is an M-base for P, and P is isomorphic with* *the Baire space of a metrizable analytic space (= classical analytic space). In particular, an analytic space is separable if and only if it is isomorphic with the Baire space of a metrizable analytic space. If P is a Borelian M-space, and if a countable collection $\mathscr{C}$ of stones distinguishes the points of P, then $\mathscr{C}$ is an M-base, and P is isomorphic with the Baire space of a metrizable Borelian space (= classical separable absolute Borel set).*

Definition. A *Blackwell space* is a separable separated M-space P such that every countable collection of stones that distinguishes the points is an M-base. By a *pseudo-Blackwell space* we shall mean a separated M-space such that every M-mapping onto a separable separated M-space is M-quotient. Similarly, by a *pseudo-analytic space* we shall mean a separated M-space such that every M-mapping onto a separable separated M-space Q is M-quotient, and Q is analytic.

Proposition. *Every analytic space is pseudo-analytic, every Blackwell space is pseudo-Blackwell (in fact, a pseudo-Blackwell space is Blackwell if and only if it is separable), every separable analytic space is a Blackwell space. Every quotient of a pseudo-Blackwell space is pseudo-Blackwell.*

Proof. Routine.

The classical Blackwell problem whether or not every Blackwell space is analytic (that is still open) can be extended to pseudo-analytic and pseudo-Blackwell spaces.

15.5.3. Standard spaces. A classical result says that any two uncountable separable absolute Borel sets (= metrizable Borelian spaces) are isomorphic M-spaces (Kuratowski [1]). It follows that if P is a separable separated Borelian M-space then either P is countable, and then the pavement consists of all subsets of P, or P is isomorphic with Baire (Cantor ($\aleph_0$)).

Definition. A *standard M-space* is an M-space P that is isomorphic either to Baire (Cantor (m)) for some m, or to a singleton, or to a countable sum of such spaced (defined in a natural way).

Theorem. *Each stone in a standard M-space is standard. A standard space is separable if and only if each standard subspace is a stone.*

For this theorem, and for further development the following simple result is needed.

Lemma. *If f is a continuous mapping of a product space $P = \Pi\{P_a \mid a \in A\}$ into a separable metrizable space, then f factorizes through a projection of P onto a countable subproduct.*

The concept of a separable standard space was introduced by C. W. Mackey [1] (under the name Standard Borel space), who also gave non-topological characterization of these spaces as in Theorem 15.5.1.

The results of this subsection will be published elsewhere.

15.6. Spaces with respectable open sets. Assumptions on the space that every space is 'respectable' appear in many useful theorems. For example recall that each Borelian set is a Baire set whenever each open set is Souslin. Unfortunately no deep analysis of such conditions has been done. C. A. Rogers [6] proved the following interesting theorem. I don't see the conceptual background of the theorem, and therefore the reader is referred to the original Rogers proof.

Theorem. *Suppose that every open set in a separated topological space P has a Souslin representation $\mathbf{S}(F)$ with all Fs closed, and all $\tilde{F}[\Sigma s]$ being M-sets. Then every open set has a similar representation with $\tilde{F}$ disjoint, and if $f : \Sigma \to P$ is usco-compact such that the sets $f[\Sigma s]$ are M-sets, then $f[\Sigma]$ is Borelian.*

Added in proof:

15.7. Measure-theoretic properties of analytic sets. G. Choquet [2] is responsible for applications in the theory of capacites, M. Sion [3], [5] contributed to applications in measure theory. The present author noticet in "Projection limits of measure spaces", Sixth Berkeley Symposium on Statistics and Probability Theory, that every Baire σ-measure on an analytic space extends (uniquely) to a reg uar Borel measure, and applied this result (with Lemma 15.5.2 and Theorem 15.5.2) to projective limits of measure spaces. The specific properties of Baire measures on analytic spaces are developed in abstract setting in "Capacity-compact measures" (to appear), in paticular one gets a generalization of Bochner-Choksi-Metivier theorem.

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Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).