Czechoslovak Mathematical Journal

Jiří Vanžura

Tensor-invariants of submanifolds

Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 3, 437–448

Persistent URL: `http://dml.cz/dmlcz/101045`

Terms of use:

© Institute of Mathematics AS CR, 1971

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

TENSOR-INVARIANTS OF SUBMANIFOLDS

JIŘÍ VANŽURA, Praha*)

(Received June 2, 1970)

INTRODUCTION

The present paper is a continuation of my previous paper [2], the knowledge of which is supposed. We use also freely the notation from [2]. The paper is devided into two paragraphs. In the first paragraph sheaves of tensor invariants are introduced and their structure is studied. In the second paragraph we define a process of prologation for tensor invariants and we show that knowing all tensor invariants of order l of differentiability we get by prolongation all tensor invariants of order $l + 1$ of differentiability.

1. SHEAVES OF TENSOR INVARIANTS

If X is a differentiable vector field on a differentiable manifold, let us denote by L_X the Lie derivative with respect to X. Let us denote by $\tilde{\mathcal{T}}^l_{(r,s)}$ *the sheaf of germs of all differentiable* (r, s)*-tensor fields on* $\tilde{J}^l$. Here for $\tilde{\mathcal{T}}^l_{(0,0)}$ we have one more notation from [2], namely $\tilde{\mathcal{D}}^l$. We set $\tilde{\mathcal{T}}^l = \bigoplus_{r,s\geqq 0} \tilde{\mathcal{T}}^l_{(r,s)}$. $\tilde{\mathcal{T}}^l$ has the natural structure of a sheaf of bigraded algebras over the sheaf of rings $\tilde{\mathcal{D}}^l$. For any differentiable vector field X on $\tilde{J}^l$ we have clearly $L_X\tilde{\mathcal{T}}^l_{(r,s)} \subseteq \tilde{\mathcal{T}}^l_{(r,s)}$. Let us define a subsheaf $\tilde{\mathcal{R}}^l_{(r,s)} \subset \tilde{\mathcal{T}}^l_{(r,s)}$ in the following way: let $g_x(T) \in \tilde{\mathcal{T}}^l_{(r,s)}$, where $x \in \tilde{J}^l$ and T is a differentiable (r, s)-tensor field defined on an open neighborhood U_1 of x; $g_x(T) \in \tilde{\mathcal{R}}^l_{(r,s)}$ if and only if for any differentiable vector field X defined on an open neighborhood U_2 of x and lying in $\tilde{D}^l$ there exists a neighborhood $U \subseteq U_1 \cap U_2$ of x on which $L_X T = 0$. Obviously $g_x(T) \in \tilde{\mathcal{R}}^l_{(r,s)}$ if and only if $L_X T = 0$ on a neighborhood of x for all elements X of a set of local generators of the pseudodistribution $\tilde{D}^l$. Again for $\tilde{\mathcal{R}}^l_{(0,0)}$ we have one more notation from [2], namely $\tilde{\mathcal{A}}^l$. Setting $\tilde{\mathcal{R}}^l = \bigoplus_{r,s\geqq 0} \tilde{\mathcal{R}}^l_{(r,s)}$ we can easily see from the "derivative" properties of the Lie derivative, that $\tilde{\mathcal{R}}^l$ has the induced structure of a sheaf of bigraded algebras over the sheaf of rings $\tilde{\mathcal{A}}^l$.

---

*) This paper was written during my scholarship stay at the Scuola Normale Superiore of Pisa.

Definition 1. The sheaves $\tilde{\mathscr{R}}^l_{(r,s)}$ and $\tilde{\mathscr{R}}^l$ will be called *the sheaf of* (r, s)*-tensor invariants and the sheaf of tensor invariants* respectively.

Let us denote by $\bar{\mathscr{R}}^l_{(r,s)}$ resp. the $\bar{\mathscr{R}}^l$ restriction of $\tilde{\mathscr{R}}^l_{(r,s)}$ resp. $\tilde{\mathscr{R}}^l$ to $\bar{J}^l$ and by $\mathscr{R}^l_{(r,s)}$ resp. $\mathscr{R}^l$ the restriction of $\tilde{\mathscr{R}}^l_{(r,s)}$ resp. $\tilde{\mathscr{R}}^l$ to J^l. Now we are going to study the structure of these sheaves. We shall see in the sequel that it is almost sufficient to study the sheaf $\bar{\mathscr{R}}^l_{(0,1)}$, i.e. the sheaf of 1-form invariants on $\bar{J}^l$. We have easily

Proposition 1. *Let f be a differentiable function defined on an open set $U \subset \tilde{J}^l$ such that $g_x(f) \in \tilde{\mathscr{A}}^l$ for every $x \in U$. Then $g_x(\mathrm{d}f) \in \tilde{\mathscr{R}}^l_{(0,1)}$ for every $x \in U$.*

The proof follows from the commutativity of the exterior and Lie derivatives.

Let $x \in \bar{J}^l$, $n_l = \dim \bar{J}^l$. We know (see [2], proof of Prop. 10, p. 463) that we can find a coordinate neighborhood U of x in $\bar{J}^l$ with coordinates $(u^1, \ldots, u^{n_l})$ such that $(g_y(u^{k+1}), \ldots, g_y(u^{n_l}))$ is a φ-basis of $\bar{\mathscr{A}}^l_y$ for every $y \in U$. If U is taken sufficiently small, then for some local basis $X_1, \ldots, X_k$ of $\mathscr{F}$ around $p\pi^l_0(x)$ $(p : M \times B \to M$ projection) the vector fields $X^l_1, \ldots, X^l_k$, $\partial/\partial u^{k+1}, \ldots, \partial/\partial u^{n_l}$ form a basis of $T(U)$. Moreover every vector field X_i, $1 \leqq i \leqq k$, is a linear combination of the fields $\partial/\partial u^1, \ldots, \partial/\partial u^k$ only. At this place it is useful to introduce the following convention: *the latin indices $i, j, \ldots$ will run over the range* $1, \ldots, k$; *the greek indices $\alpha, \beta \ldots$ will run over the range* $k + 1, \ldots, n_l$. Let us denote by $\omega^1, \ldots, \omega^{n_l}$ the dual basis to the basis X^l_i, $\partial/\partial u^\alpha$. Obviously there is $\omega^\alpha = \mathrm{d}u^\alpha$ and $g_y(\mathrm{d}u^\alpha) \in \bar{\mathscr{R}}^l_{(0,1)}$ for every $y \in U$.

Now let ω be a differentiable 1-form on U. Under the usual summation convention we write

$$\omega = a_i\omega^i + a_\alpha \, \mathrm{d}u^\alpha$$

where a_i, a_α are differentiable functions. Moreover let us write

$$X_i = \varrho_{ir}\frac{\partial}{\partial u^r}, \quad \frac{\partial}{\partial u^j} = \sigma_{js}X_s$$

where ϱ, σ are differentiable matrix functions, σ is the inverse to ϱ. Finally let c^r_{ij} be the structure coefficients of the Lie algebra $\mathscr{F}^l \mid U$ with respect to the basis X_i, i.e. $[X_i, X_j] = c^r_{ij}X_r$. An easy calculation shows that

$$(L_{X_i}\omega^j)(X_r) = -c^j_{ir}, \quad (L_{X_i}\omega^j)\left(\frac{\partial}{\partial u^\alpha}\right) = \frac{\partial \varrho_{ir}}{\partial u^\alpha}\sigma_{rj}\,.$$

Thus we have

$$L_{X_i}\omega^j = -c^j_{ir}\omega^r + \frac{\partial \varrho_{ir}}{\partial u^\alpha}\sigma_{rj}\,\mathrm{d}u^\alpha$$

and using this we get immediately

$$L_{X_i}\omega = (X_ia_j - c^r_{ij}a_r)\,\omega^j + \left(X_ia_\alpha + \frac{\partial \varrho_{ir}}{\partial u^\alpha}\sigma_{rj}a_j\right)\mathrm{d}u^\alpha\,.$$

Now, studying the sheaf $\mathscr{R}^l_{(0,1)}$ we want to get some information about solutions of the differential system $L_{X_i}\omega = 0$, $i = 1, \ldots, k$. For this we use up the results of [1]. The differential system

$$X_i a_j - c^r_{ij} a_r = 0 \,, \quad X_i a_\alpha + \frac{\partial \varrho_{ir}}{\partial u^\alpha} \sigma_{rj} a_j = 0$$

on U is equivalent to the system

$$\frac{\partial a_i}{\partial u^j} - \sigma_{jr} c^s_{ri} a_s = 0 \,, \quad \frac{\partial a_\alpha}{\partial u^j} - \frac{\partial \sigma_{jr}}{\partial u^\alpha} a_r = 0 \,.$$

Now let $I^i(U \times \mathbf{R}^{n_l}, U, p_1)$ (briefly I^i) denote the fiber manifold of i-jets of all cross sections of the trivial fiber manifold $(U \times \mathbf{R}^{n_l}, U, p_1)$, where p_1 is the natural projection. If we denote by a_i, a_α the natural coordinates in $\mathbf{R}^{n_l}$ we have on I^1 the associated coordinate system $(u^i, u^\alpha, a_j, a_\beta, a_{j;i}, a_{j;\alpha}, a_{\beta;i}, a_{\beta;\alpha})$. Now let us consider on I^1 the differential system Φ (see [1] Def. 3.1, p. 12) generated by the functions

$$\Phi^j_i = a_{i;j} - \sigma_{jr} c^s_{ri} a_s \quad 1 \leqq i \,, \quad j \leqq k$$

$$\Phi^j_\alpha = a_{\alpha;j} - \frac{\partial \sigma_{jr}}{\partial u^\alpha} a_r \quad k + 1 \leqq \alpha \leqq n_l \,.$$

We denote as in [1] by $p\Phi$ the first prolongation of Φ (see [1], Def. 4.1, p. 16) and by $\mathscr{J}\Phi$ the set of integral jets of Φ (ibid., Def. 3.3., p. 13). We are going to prove that $(u^\alpha, u^i, a_j, a_\beta)$ is a regular chart with respect to Φ at any $x_0 \in \mathscr{J}\Phi$ (ibid., Def. 7.2., p. 41). Setting $\gamma_0 = n_l$, $\gamma_\alpha = n_l$, $\gamma_i = 0$ we can easily see that the first condition of Def. 7.2. from [1] is satisfied. As to the second condition we must prove that the functions Φ^i_j, Φ^i_α, $\partial^\beta_\# \Phi^i_j$, $\partial^\beta_\# \Phi^i_\alpha$, $\partial^j_\# \Phi^i_h$ $j \leqq i$, $\partial^j_\# \Phi^i_\alpha$, $j \leqq i$ form a system of generators of $p\Phi$ on I^2. Here $\partial^i_\#$ denotes the formal derivative by u^i (ibid., p. 15). Clearly it suffices to prove the following two equalities

$$\text{(1)} \qquad \partial^i_\# \Phi^j_h - \partial^j_\# \Phi^i_h = c^s_{rh}(\sigma_{ir} \Phi^j_s - \sigma_{jr} \Phi^i_s)$$

$$\partial^i_\# \Phi^j_\alpha - \partial^j_\# \Phi^i_\alpha = \frac{\partial \sigma_{ir}}{\partial u^\alpha} \Phi^j_r - \frac{\partial \sigma_{jr}}{\partial u^\alpha} \Phi^i_r \,.$$

The simple calculation gives

$$\partial^i_\# \Phi^j_h = a_{h;ji} - \frac{\partial \sigma_{jr}}{\partial u^i} c^s_{rh} a_s - \sigma_{jr} c^s_{rh} a_{s;i} =$$

$$= a_{h;ji} - \frac{\partial \sigma_{jr}}{\partial u^i} c^t_{rh} a_t - \sigma_{jr} c^s_{rh} (\Phi^i_s + \sigma_{iu} c^t_{us} a_t) =$$

$$= a_{h;ji} - \left(\frac{\partial \sigma_{jr}}{\partial u^i} c^t_{rh} + \sigma_{jr} \sigma_{iu} c^s_{rh} c^t_{us} \right) a_t - \sigma_{jr} c^s_{rh} \Phi^i_s \,.$$

Thus we get

$$\partial^i_{\#}\Phi^j_h - \partial^j_{\#}\Phi^i_h = \left[c^t_{rh}\left(\frac{\partial\sigma_{ir}}{\partial u^j} - \frac{\partial\sigma_{jr}}{\partial u^i}\right) + c^s_{rh}c^t_{us}\sigma_{ir}\sigma_{ju} - c^s_{rh}c^t_{us}\sigma_{jr}\sigma_{iu}\right] a_t +$$

$$+ c^s_{rh}(\sigma_{ir}\Phi^j_s - \sigma_{jr}\Phi^i_s) =$$

$$= \left[c^t_{rh}\left(\frac{\partial\sigma_{ir}}{\partial u^j} - \frac{\partial\sigma_{jr}}{\partial u^i}\right) + c^s_{rh}c^t_{us}\sigma_{ir}\sigma_{ju} - c^s_{uh}c^t_{rs}\sigma_{ir}\sigma_{ju}\right] a_t + c^s_{rh}(\sigma_{ir}\Phi^j_s - \sigma_{jr}\Phi^i_s) =$$

$$= \left[c^t_{rh}\left(\frac{\partial\sigma_{ir}}{\partial u^j} - \frac{\partial\sigma_{jr}}{\partial u^i}\right) + \sigma_{ir}\sigma_{ju}(c^s_{rh}c^t_{us} - c^s_{uh}c^t_{rs})\right] a_t + c^s_{rh}(\sigma_{ir}\Phi^j_s - \sigma_{jr}\Phi^i_s) .$$

The trivial equality $[\partial/\partial u^j, \partial/\partial u^i] = 0$ gives

$$0 = [\sigma_{ja}X_a, \sigma_{ib}X_b] = \sigma_{ja}\sigma_{ib}c^r_{ab}X_r + \sigma_{ja}(X_a\sigma_{ib})X_b - \sigma_{ib}(X_b\sigma_{ja})X_a =$$

$$= \sigma_{ja}\sigma_{ib}c^r_{ab}X_r + \sigma_{ja}\varrho_{av}\frac{\partial\sigma_{ib}}{\partial u_v}X_b - \sigma_{ib}\varrho_{bw}\frac{\partial\sigma_{ja}}{\partial u_w}X_a = \left(\sigma_{ja}\sigma_{ib}c^r_{ab} + \frac{\partial\sigma_{ir}}{\partial u^j} - \frac{\partial\sigma_{jr}}{\partial u^i}\right)X_r$$

and thus we have obtained the equality

$$\frac{\partial\sigma_{ir}}{\partial u^j} - \frac{\partial\sigma_{jr}}{\partial u^i} = -\sigma_{ja}\sigma_{ib}c^r_{ab} \tag{2}$$

which we use immediately in the next. We have

$$\partial^i_{\#}\Phi^j_h - \partial^j_{\#}\Phi^i_h = [-c^t_{rh}c^r_{ab}\sigma_{ja}\sigma_{ib} + \sigma_{ir}\sigma_{ju}(c^s_{rh}c^t_{us} - c^s_{uh}c^t_{rs})]\, a_t +$$

$$+ c^s_{rh}(\sigma_{ir}\Phi^j_s - \sigma_{jr}\Phi^i_s) = -(c^s_{ur}c^t_{sh} + c^s_{rh}c^t_{su} + c^s_{hu}c^t_{sr})\,\sigma_{ir}\sigma_{ju}a_t + c^s_{rh}(\sigma_{ir}\Phi^j_s - \sigma_{jr}\Phi^i_s) .$$

On the other hand we have also

$$0 = [[X_u, X_r], X_h] + [[X_r, X_h], X_u] + [[X_h, X_u], X_r] =$$

$$= (c^s_{ur}c^t_{sh} + c^s_{rh}c^t_{su} + c^s_{hu}c^t_{sr})\,X_t$$

which implies

$$\partial^i_{\#}\Phi^j_h - \partial^j_{\#}\Phi^i_h = c^s_{rh}(\sigma_{ir}\Phi^j_s - \sigma_{jr}\Phi^i_s)$$

and this is the first equality from (1).

For the second equality we calculate

$$\partial^i_{\#}\Phi^j_\alpha = a_{\alpha;ji} - \frac{\partial^2\sigma_{jr}}{\partial u^\alpha\,\partial u^i}a_r - \frac{\partial\sigma_{jr}}{\partial u^\alpha}a_{r;i} = a_{\alpha;ji} - \frac{\partial^2\sigma_{jr}}{\partial u^\alpha\,\partial u^i}a_r - \frac{\partial\sigma_{jr}}{\partial u^\alpha}(\Phi^i_r + \sigma_{is}c^t_{sr}a_t) =$$

$$= a_{\alpha;ji} - \left(\frac{\partial^2\sigma_{jt}}{\partial u^\alpha\,\partial u^i} + \frac{\partial\sigma_{jr}}{\partial u^\alpha}\sigma_{is}c^t_{sr}\right)a_t - \frac{\partial\sigma_{jr}}{\partial u^\alpha}\Phi^i_r .$$

Thus we get

$$\partial_{\#}^{i}\Phi_{\alpha}^{j} - \partial_{\#}^{j}\Phi_{\alpha}^{i} =$$

$$= \left[\frac{\partial}{\partial u^{\alpha}}\left(\frac{\partial\sigma_{it}}{\partial u^{j}} - \frac{\partial\sigma_{jt}}{\partial u^{i}}\right) + \frac{\partial\sigma_{ir}}{\partial u^{\alpha}}\sigma_{js}c_{sr}^{t} - \frac{\partial\sigma_{jr}}{\partial u^{\alpha}}\sigma_{is}c_{sr}^{t}\right]a_{t} + \frac{\partial\sigma_{ir}}{\partial u^{\alpha}}\Phi_{r}^{j} - \frac{\partial\sigma_{jr}}{\partial u^{\alpha}}\Phi_{r}^{i} =$$

$$= \left[\frac{\partial}{\partial u^{\alpha}}\left(\frac{\partial\sigma_{it}}{\partial u^{j}} - \frac{\partial\sigma_{jt}}{\partial u^{i}}\right) + \frac{\partial\sigma_{ir}}{\partial u^{i}}\sigma_{js}c_{sr}^{t} - \frac{\partial\sigma_{js}}{\partial u^{\alpha}}\sigma_{ir}c_{rs}^{t}\right]a_{t} + \frac{\partial\sigma_{ir}}{\partial u^{\alpha}}\Phi_{r}^{j} - \frac{\partial\sigma_{jr}}{\partial u^{\alpha}}\Phi_{r}^{i} =$$

$$= \left[\frac{\partial}{\partial u^{\alpha}}\left(\frac{\partial\sigma_{it}}{\partial u_{j}} - \frac{\partial\sigma_{jt}}{\partial u^{i}} + \sigma_{ir}\sigma_{js}c_{sr}^{t}\right)\right]a_{t} + \frac{\partial\sigma_{ir}}{\partial u^{\alpha}}\Phi_{r}^{j} - \frac{\partial\sigma_{jr}}{\partial u^{\alpha}}\Phi_{r}^{i} = \frac{\partial\sigma_{ir}}{\partial u^{\alpha}}\Phi_{r}^{j} - \frac{\partial\sigma_{jr}}{\partial u^{\alpha}}\Phi_{r}^{i}$$

in view of (2), and we have proved the second equality. It is known (see [1], Th. 7.1., p. 42) that for a differential system Φ of order 1 the existence of a regular chart with respect to Φ at $x_0 \in \mathcal{I}\Phi$ is equivalent to the involutiveness (ibid., Def. 7.1., p. 39) of Φ at x_0. Therefore we have proved

Proposition 2. *The differential system Φ is involutive at any point $x_0 \in \mathcal{I}\Phi$.*

In view of the general method of construction of solutions of an involutive differential system of order 1 (see [1], §8, p. 51) we can easily see that for any $x \in \bar{J}^l$ there exist its open neighborhood U_x and n_l vector functions $a^{(r)} = (a_i^{(r)}, a_\alpha^{(r)})$ defined on U_x with values in $\mathbf{R}^{n_l}$ which are on U_x solutions of Φ and moreover $a_i^{(r)} = \delta_i^r$, $a_\alpha^{(r)} = \delta_\alpha^r$. Here $r = 1, \ldots, n_l$. Now we can prove

Proposition 3. *For any $x \in \bar{J}^l$ the fiber $\bar{\mathcal{R}}_{(0,1)}^l(x)$ of $\bar{\mathcal{R}}_{(0,1)}^l$ at x as a module over $\bar{\mathcal{A}}_x^l$ has a basis consisting of n_l elements.*

Proof. In view of the above considerations we can find to any $x \in \bar{J}^l$ its open neighborhood U_x and n_l differentiable 1-forms $\omega^1, \ldots, \omega^{n_l}$ such that

(i) $g_y(\omega^i) \in \bar{\mathcal{R}}_{(0,1)}^l$ for any $y \in U_x$ and any $i = 1, \ldots, n_l$

(ii) $\omega^1, \ldots, \omega^{n_l}$ are linearly independent at any $y \in U_x$.

Obviously $g_x(\omega^1), \ldots, g_x(\omega^{n_l})$ are linearly independent. Let $g_x(\omega) \in \bar{\mathcal{R}}_{(0,1)}^l$, where ω is a differentiable 1-form defined on an open neighborhood of x. On a smaller neighborhood we can write

$$\omega = f_1\omega^1 + \ldots + f_{n_l}\omega^{n_l}$$

where $f_1, \ldots, f_{n_l}$ are uniquely determined differentiable functions. For any differentiable vector field X defined on a neighborhood of x and belonging to $\mathcal{F}^l$ we get applying L_X on the previous equality

$$0 = L_X\omega = (Xf_1)\,\omega^1 + \ldots + (Xf_{n_l})\,\omega^{n_l}\,.$$

This implies $Xf_1 = \ldots = Xf_{n_l} = 0$ on a neighborhood of x and thus $g_x(f_1), \ldots, g_x(f_{n_l}) \in \bar{\mathscr{A}}^l_x$. Therefore $g_x(\omega_1), \ldots, g_x(\omega_{n_l})$ is a basis of $\bar{\mathscr{R}}^l_{(0,1)}(x)$.

Let us keep the notation from the preceding proof. We take on U_x the dual basis $X_1, \ldots, X_{n_l}$ to $\omega^1, \ldots, \omega^{n_l}$. For all $i, j = 1, \ldots, n_l$ we have

$$0 = L_X[\omega^i(X_j)] = (L_X\omega^i)(X_j) + \omega^i(L_X X_j) = \omega^i(L_X X_j)$$

which implies $L_X X_j = 0$ for all $j = 1, \ldots, n_l$ and any $X \in \mathscr{F}^l$. Now in the same way as Proposition 3 we get

Proposition 3*. *For any $x \in \bar{J}^l$ the fiber $\bar{\mathscr{R}}^l_{(1,0)}(x)$ of $\bar{\mathscr{R}}^l_{(1,0)}$ at x as a module over $\bar{\mathscr{A}}^l_x$ has a basis consisting of n_l elements.*

Finally combining Propositions 3 and 3* we have immediately

Proposition 4. *For any integers $r \geqq 0$, $s \geqq 0$ and any $x \in \bar{J}^l$ the fiber $\bar{\mathscr{R}}^l_{(r,s)}(x)$ of $\bar{\mathscr{R}}^l_{(r,s)}$ at x as a module over $\bar{\mathscr{A}}^l_x$ has a finite basis.*

2. PROLONGATION OF COVARIANT-TENSOR INVARIANTS

Throughout the first part of this paragraph we shall consider an open set $U \subset \tilde{J}^{l+1} (l \geqq 0)$ with an associated coordinate system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1,\ldots,i_{l+1}})$.

Definition 2. Let $r \geqq s \geqq -1$ be integers. *A vector field X defined on an open set $V \subset \tilde{J}^r$ is said to be projectable into $\tilde{J}^s$* if there exists a vector field Y on $\pi^r_s(V)$ such that $Y_{\pi(x)} = (\mathrm{d}\pi^r_s)_x X_x$ for any $x \in V$. If such Y exists it is uniquely determined and we shall denote it by $\mathrm{d}\pi^r_s(X)$. *A function f defined on V is said to be projectable into $\tilde{J}^s$* if there exists a function g on $\pi^r_s(V)$ such that $f = g \circ \pi^r_s$. If such g exists it is uniquely determined and will be mostly denoted again by f.

For example a vector field

$$X = a^i \frac{\partial}{\partial x^i} + \sum_{k=0}^{l+1} a^\eta_{i_1\ldots i_k} \frac{\partial}{\partial y^\eta_{i_1\ldots i_k}}$$

on U is projectable into $\tilde{J}^l$ if and only if the functions a^i, $a^\eta_{i_1\ldots i_k}$ for $0 \leqq k \leqq l$ are projectable into $\tilde{J}^l$.

Now let X be a differentiable vector field on U projectable into $\tilde{J}^l$. For any differentiable function f on $V = \pi^{l+1}_l(U)$ we define differentiable functions $(\delta^{x^i}_{\#} X) f$ on U by

$$(\delta^{x^i}_{\#} X) f = \partial^{x^i}_{\#}((\mathrm{d}\pi^{l+1}_l X) f) - X(\partial^{x^i}_{\#} f).$$

Instead of $\delta^{x^i}_{\#}$, $\delta^{x^i}_{\#}$ we shall often write $\delta^i_{\#}$, $\partial^i_{\#}$. We have

Proposition 5. *Let f, f_1, f_2 be differentiable functions on V, X, X_1, X_2 differentiable vector fields on U projectable into $\tilde{J}^l$. There is*

$$(\delta^i_\# X)(f_1 + f_2) = (\delta^i_\# X) f_1 + (\delta^i_\# X) f_2$$
$$(\delta^i_\# X)(f_1 f_2) = (\delta^i_\# X) f_1 \cdot (f_2 \circ \pi^{l+1}_l) + (f_1 \circ \pi^{l+1}_l) \cdot (\delta^i_\# X) f_2$$
$$(\delta^i_\#(X_1 + X_2)) f = (\delta^i_\# X_1) f + (\delta^i_\# X_2) f .$$

Proof. The only non obvious equality is the second one

$$\begin{aligned}(\delta^i_\# X)(f_1 f_2) &= \partial^i_\#[(\mathrm{d}\pi X)(f_1 f_2)] - X[\partial^i_\#(f_1 f_2)] = \\ &= \partial^i_\#[(\mathrm{d}\pi X) f_1 \cdot f_2 + f_1 \cdot (\mathrm{d}\pi X) f_2] - X[\partial^i_\# f_1 \cdot f_2 + f_1 \cdot \partial^i_\# f_2] = \\ &= \partial^i_\#((\mathrm{d}\pi X) f_1) \cdot f_2 + (\mathrm{d}\pi X) f_1 \cdot \partial^i_\# f_2 + \partial^i_\# f_1 \cdot (\mathrm{d}\pi X) f_2 + f_1 \cdot \partial^i_\#((\mathrm{d}\pi X) f_2) - \\ &- X(\partial^i_\# f_1) \cdot f_2 - \partial^i_\# f_1 \cdot (\mathrm{d}\pi X) f_2 - (\mathrm{d}\pi X) f_1 \cdot \partial^i_\# f_2 - f_1 \cdot X(\partial^i_\# f_2) = \\ &= (\delta^i_\# X) f_1 \cdot f_2 + f_1 \cdot (\delta^i_\# X) f_2 .\end{aligned}$$

We denote by $\mathscr{P}^1 T^{l+1}_x$ *the vector space of* 1*-jets of all projectable into* $\tilde{J}^l$ *differentiable vector fields at* $x \in \tilde{J}^{l+1}$ and by T^l_y the tangent vector space of $\tilde{J}^l$ at $y \in \tilde{J}^l$. For $1 \leqq i \leqq n$, $x \in U$, $y = \pi^{l+1}_l(x)$ we define maps $\chi^i_x : \mathscr{P}^1 T^{l+1}_x \to T^l_y$ by

$$\chi^i_x(j^1_x(X)) f = ((\delta^i_\# X) f)(x) .$$

It can be easily seen from Prop. 5 that $\chi^i_x(j^1_x(X))$ is really a vector.

Proposition 6. *Let X^{l+1} be a differentiable vector field defined on an open neighborhood of $x \in U$ which is the $(l+1)$-th prolongation of a differentiable vector field X defined on an open neighborhood of $\xi = q\pi^{l+1}_0(x)$. X^{l+1} is projectable into $\tilde{J}^l$ and $\chi^i_x(j^1_x(X^{l+1})) = 0$.*

Proof. The projectability of X^{l+1} can be seen for example from its coordinate expression (see [2], p. 456). Moreover $\mathrm{d}\pi^{l+1}_l(X^{l+1}) = X^l$. We denote by h_t the local 1-parameter group generating X and by h^r_t its r-th prolongation. For $x = j^{l+1}_{a_0}(\sigma)$ we have

$$\begin{aligned}X^{l+1}_x(\partial^i_\# f) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} [(\partial^i_\# f)(h^{l+1}_t(j^{l+1}_{a_0}(\sigma)))] = \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} [(\partial^i_\# f)(j^{l+1}_{a_0}(h^0_t \sigma))] = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} \left[\left(\frac{\partial}{\partial x^i}\right)_{x=a_0} f(j^l_x(h^0_t \sigma))\right] = \\ &= \left(\frac{\partial}{\partial x^i}\right)_{x=a_0} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{t=0} f(h^l_t(j^l_x(\sigma)))\right] = \left(\frac{\partial}{\partial x^i}\right)_{x=a_0} [(X^l f)(j^l_x(\sigma))] = \\ &= [\partial^i_\#(X^l f)](j^{l+1}_{a_0}(\sigma)) = [\partial^i_\#(X^l f)](x)\end{aligned}$$

from which our assertion immediately follows.

Now we change slightly our notation. Let $V \subset \tilde{J}^l$ ($l \geqq 0$) be an open set with an associated coordinate system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$, and let ω be a differentiable r-linear form on V. We are going to define on $U = (\pi_l^{l+1})^{-1} V$ differentiable r-linear forms $\partial^i_\# \omega$. Let $x \in U$ and $Y_1, \ldots, Y_r \in T^{l+1}_x$. Let $X_1, \ldots, X_r$ be differentiable vector fields defined on an open neighborhood of x projectable into $\tilde{J}^l$ and such that $X_i(x) = Y_i$. We set

$$(\partial^i_\# \omega)_x (Y_1, \ldots, Y_r) = \{\partial^i_\# [\omega(\mathrm{d}\pi_l^{l+1} X_1, \ldots, \mathrm{d}\pi_l^{l+1} X_r)]\} (x) - \\ - \sum_{u=1}^{r} \omega_y(\mathrm{d}\pi_l^{l+1} X_1, \ldots, \chi^i_x(X_u), \ldots, \mathrm{d}\pi_l^{l+1} X_r)$$

where $y = \pi_l^{l+1}(x)$. Of course we must prove now that $(\partial^i_\# \omega)_x (Y_1, \ldots, Y_r)$ does not depend on the choice of $X_1, \ldots, X_r$. As usual we shall prove that $Y_s = 0$ for some $1 \leqq s \leqq r$ implies $(\partial^i_\# \omega)_x (Y_1, \ldots, Y_r) = 0$. For simplicity let $Y_1 = 0$. We can write

$$X_1 = a^j \frac{\partial}{\partial x^j} + \sum_{k=0}^{l+1} a^\eta_{j_1 \ldots j_k} \frac{\partial}{\partial y^\eta_{j_1 \ldots j_k}}$$

where $a^j(x) = a^\eta_{j_1 \ldots j_k}(x) = 0$ for $0 \leqq k \leqq l + 1$. The projectability of X_1 into $\tilde{J}^l$ implies the projectability of a^j, $a^\eta_{j_1 \ldots j_k}$ for $0 \leqq k \leqq l$. Thus X_1 can be expressed as a sum of differentiable vector fields of type aW_1 where a and W_1 are projectable into $\tilde{J}^l$, $a(x) = 0$, and differentiable vector fields of type bW_2 where $\mathrm{d}\pi_l^{l+1} W_2 = 0$, $b(x) = 0$. We have

$$\chi^i_x(j^1_x(aW_1)) = (\partial^i_\# a)(x) \cdot \mathrm{d}\pi_l^{l+1} W_{1x}, \quad \chi^i_x(j^1_x(bW_2)) = 0 .$$

Using this we get

$$\{\partial^i_\# [\omega(\mathrm{d}\pi(aW_1), \mathrm{d}\pi X_2, \ldots, \mathrm{d}\pi X_r)]\} (x) - \omega_x(\chi^i_x(j^1_x(aW_1), \mathrm{d}\pi X_2, \ldots, \mathrm{d}\pi X_r) - \\ - \sum_{u=2}^{r} \omega_x(a \, \mathrm{d}\pi W_1, \mathrm{d}\pi X_2, \ldots, \chi^i_x(j^1_x(X_u)), \ldots, \mathrm{d}\pi X_r) = \\ = (\partial^i_\# a)(x) \cdot \omega_x(\mathrm{d}\pi W_1, \mathrm{d}\pi X_2, \ldots, \mathrm{d}\pi X_r) - \\ - (\partial^i_\# a)(x) \cdot \omega_x(\mathrm{d}\pi W_1, \mathrm{d}\pi X_2, \ldots, \mathrm{d}\pi X_r) = 0$$

and the same result in the second case. Thus we have shown that our definition is good. Very simple calculation gives.

Proposition 7. *Let $\omega, \omega_1, \omega_2$ be differentiable r-linear froms on V, Ω_1 and Ω_2 differentiable r_1 and r_2-linear forms on V respectively. Let c be a differentiable function on V. There is*

$$\partial^i_\#(\omega_1 + \omega_2) = \partial^i_\# \omega_1 + \partial^i_\# \omega_2$$

$$\partial^i_\#(c\omega) = \partial^i_\# c \cdot (\pi_l^{l+1})^* \omega + (c \circ \pi_l^{l+1}) \, \partial^i_\# \omega$$

$$\partial^i_\#(\Omega_1 \otimes \Omega_2) = \partial^i_\# \Omega_1 \otimes (\pi_l^{l+1})^* \Omega_2 + (\pi_l^{l+1})^* \Omega_1 \otimes \partial^i_\# \Omega_2 .$$

Now we should like to prove the following

Proposition 8. *Let* $y = \pi_l^{l+1}(x)$, $x \in U$, *and let* $g_y(\omega) \in \tilde{\mathscr{R}}^l_{(0,r)}$. *Then* $g_x(\partial_\#^i \omega) \in \tilde{\mathscr{R}}^{l+1}_{(0,r)}$ *for all* $1 \leqq i \leqq n$.

But for its proof we must first develop some necessary tools. We start with

Definition 3. *A differentiable vector field X on U is called admissible with respect to* $(x^i, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_{l+1}})$ if it is projectable into $\tilde{J}^l$ and for any differentiable function f defined on an open set $V_1 \subset V$ the function $(\delta_\#^i X) f$ is projectable into $\tilde{J}^l$ for all $i = 1, \ldots, n$.

Clearly if X is admissible and $x_1, x_2 \in U$ such that $\pi_l^{l+1}(x_1) = \pi_l^{l+1}(x_2) = y$ then $\chi^i_{x_1}(j^1_{x_1}(X)) = \chi^i_{x_2}(j^1_{x_2}(X))$ and we can define a differentiable vector fields $\chi^i X$ on V setting for every $y \in V$

$$(\chi^i X)_y = \chi^i_x(j^1_x(X)) = [(\delta^i_\# X) f](x)$$

where x is any element of U such that $\pi_l^{l+1}(x) = y$. For such vector fields we shall prove

Proposition 9. *Let Y be an admissible differentiable vector field on U. Let* X^{l+1} *be a differentiable vector field on U which is the* $(l+1)$*-th prolongation of a vector field X defined on an open subset of M. For any* $x \in U$ *there is*

$$\chi^i_x[X^{l+1}, Y] = [X^l, \chi^i Y]_y$$

where X^l *is the l-th prolongation of X and* $y = \pi_l^{l+1}(x)$.

Proof. In view of Prop. 6 we get

$$\begin{aligned}
&\chi^i_x[X^{l+1}, Y] f - [X^l, \chi^i Y]_y f = \\
&= [(\delta^i_\#[X^{l+1}, Y]) f](x) - X^l_y(\chi^i Y) f + (\chi^i Y)_y X^l f = \\
&= [\partial^i_\#(X^l(\mathrm{d}\pi Y) f)](x) - [\partial^i_\#((\mathrm{d}\pi Y) X^l f)](x) - X^{l+1}_x Y(\partial^i_\# f) + \\
&+ Y_x X^{l+1}(\partial^i_\# f) - X^{l+1}_x((\delta^i_\# Y) f) + [(\delta^i_\# Y)(X^l f)](x) = \\
&= [\partial^i_\#(X^l(\mathrm{d}\pi Y) f)](x) - [\partial^i_\#((\mathrm{d}\pi Y) X^l f)](x) - X^{l+1}_x Y(\partial^i_\# f) + \\
&+ Y_x X^{l+1}(\partial^i_\# f) - X^{l+1}_x[\partial^i_\#((\mathrm{d}\pi Y) f)] + X^{l+1}_x[Y(\partial^i_\# f)] + \\
&+ [\partial^i_\#((\mathrm{d}\pi Y)(X^l f))](x) - Y_x[\partial^i_\#(X^l f)] = \\
&= [(\delta^i_\# X^{l+1})((\mathrm{d}\pi Y) f)](x) - Y_x[(\delta^i_\# X^{l+1}) f] = 0\,.
\end{aligned}$$

Corollary. $[X^{l+1}, X]$ *is an admissible vector field and there is* $\chi^i[X^{l+1}, X] = [X^l, \chi^i X]$.

For a projectable vector field

$$X = a^j \frac{\partial}{\partial x^j} + \sum_{k=0}^{l+1} a^\eta_{j_1 \ldots j_k} \frac{\partial}{\partial y^\eta_{j_1 \ldots j_k}}$$

an easy calculation gives

$$(\delta^i_\# X) f = \partial^i_\# a^j \frac{\partial f}{\partial x^j} + \sum_{k=0}^{l} (\partial^i_\# a^\eta_{j_1 \ldots j_k} - a^\eta_{j_1 \ldots j_k i}) \frac{\partial f}{\partial y^\eta_{j_1 \ldots j_k}}$$

and from this we can conclude that X is admissible if for example the functions a^j, $a^\eta_{j_1 \ldots j_k}$ for $0 \leqq k \leqq l$ are projectable into $\tilde{J}^{l-1}$ and the functions $a^\eta_{j_1 \ldots j_{l+1}}$ are projectable into $\tilde{J}^l$. From this trivially follows .

Proposition 10. *Let $x \in U$ be an arbitrary point and $Y \in T^{l+1}_x$ an arbitrary vector. Then there exists on U a differentiable vector field X admissible with respect to $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_{l+1}})$ and such that $Y = X_x$.*

Now we are in position for

Proof of Prop. 8: Let $\xi = q\pi^{l+1}_0(x)$ and $X_1, \ldots, X_k$ be differentiable vector fields defined on an open neighborhood of ξ such that $g_\xi(X_1), \ldots, g_\xi(X_k)$ are generators of $\mathscr{F}_\xi$. As $g_y(\omega) \in \tilde{\mathscr{R}}^l_{(0,r)}$ there exists an open neighborhood $U' \subset U$ of y on which $L_{X_1{}^l}\omega = \ldots = L_{X_k{}^l}\omega = 0$. We are going to prove the equality $L_{X_i{}^{l+1}}(\partial^j_\# \omega) = \partial^j_\#(L_{X_i{}^l}\omega)$ from which our assertion immediately follows. In view of Prop. 10 it is quite sufficient to prove the equality

$$(L_{X_i{}^{l+1}}(\partial^j_\# \omega))(Y_1, \ldots, Y_r) = (\partial^j_\#(L_{X_i{}^l}\omega))(Y_1, \ldots, Y_r)$$

for admissible vector fields $Y_1, \ldots, Y_r$. We omit the subscript i and write $\bar{Y}$ instead of $d\pi^{l+1}_l Y$. We get

$$(L_{X^{l+1}}(\partial^j_\# \omega))_z (Y_1, \ldots, Y_r) = X^{l+1}_z[(\partial^j_\# \omega)(Y_1, \ldots, Y_r)] -$$

$$- \sum_{u=1}^{r} (\partial^j_\# \omega)_z (Y_1, \ldots, [X^{l+1}, Y_u], \ldots, Y_r) =$$

$$= X^{l+1}_z[\partial^j_\#(\omega(\bar{Y}_1, \ldots, \bar{Y}_r) - \sum_{u=1}^{r} \omega(\bar{Y}_1, \ldots, \chi^j Y_u, \ldots, \bar{Y}_r) -$$

$$- \sum_{u=1}^{r} [\partial^j_\#(\omega(\bar{Y}_1, \ldots, [X^l, \bar{Y}_u], \ldots, Y_r))](z) +$$

$$+ \sum_{u=1}^{r} \sum_{v=1}^{r} \omega_z(\bar{Y}_1, \ldots, [X^l, \bar{Y}_u], \ldots, \chi^j Y_v, \ldots, \bar{Y}_r) +$$

$$+ \sum_{u=1}^{r} \omega_z(\bar{Y}_1, \ldots, \chi^j[X^l, \bar{Y}_u], \ldots, \bar{Y}_r)$$

$$(\partial^j_\#(L_{X^l}\omega))_z\,(Y_1, \ldots, Y_r) = [\partial^j_\#((L_{X^l}\omega)\,(\bar{Y}_1, \ldots, \bar{Y}_r))]\,(z) -$$

$$- \sum_{u=1}^{r} (L_{X^l}\omega)_z\,(\bar{Y}_1, \ldots, \chi^j Y_u, \ldots, \bar{Y}_r) =$$

$$= \{\partial^j_\#[X^l(\omega(\bar{Y}_1, \ldots, \bar{Y}_r)) - \sum_{u=1}^{r} \omega(\bar{Y}_1, \ldots, [X^l, \bar{Y}_u], \ldots, \bar{Y}_r)]\}\,(z) -$$

$$- \sum_{u=1}^{r} X^l_z(\omega(\bar{Y}_1, \ldots, \chi^j Y_u, \ldots, \bar{Y}_r)) +$$

$$+ \sum_{u=1}^{r} \sum_{v=1}^{r} \omega_z(\bar{Y}_1, \ldots, [X^l, \bar{Y}_v], \ldots, \chi^j \bar{Y}_u, \ldots, \bar{Y}_r) +$$

$$+ \sum_{u=1}^{r} \omega_z(\bar{Y}_1, \ldots, [X^l, \chi^j Y_u], \ldots, \bar{Y}_r)$$

and these two expressions coincide in view of Prop. 6 and Prop. 9 with its Corollary.

Proposition 11. *Let f be a differentiable function on V. There is $\partial^i_\#(\mathrm{d}f) = \mathrm{d}(\partial^i_\# f)$.*

Proof. Let $x \in U$, $Y \in T^{l+1}_x$, $y = \pi^{l+1}_l(x)$ and let X be a differentiable vector field defined on an open neighborhood of x, projectable into $\tilde{J}^l$, and such that $X(x) = Y$. We have

$$[\partial^i_\#(\mathrm{d}f)]_x\,(Y) = [\partial^i_\#(\mathrm{d}f(\mathrm{d}\pi X))]\,(x) - (\mathrm{d}f)_y\,(\chi^i_x j^1_x(X)) =$$

$$= [\partial^i_\#((\mathrm{d}\pi X)\,f)]\,(x) - (\chi^i_x j^1_x(X))\,f = [\partial^i_\#((\mathrm{d}\pi X)\,f)]\,(x) - ((\delta^i_\# X)\,f)\,(x) =$$

$$= [\partial^i_\#((\mathrm{d}\pi X)\,f)]\,(x) - [\partial^i_\#((\mathrm{d}\pi X)\,f)]\,(x) + X_x(\partial^i_\# f) =$$

$$= [\mathrm{d}(\partial^i_\# f)]_x\,(Y)\,.$$

Let us define a subset $K \subseteq \tilde{\mathcal{T}}^{l+1}_{(0,r)}$ in this way: $g_x(\omega) \in \tilde{\mathcal{T}}^{l+1}_{(0,r)}$, where ω is a differentiable r-linear form defined on an open neighborhood of $x \in \tilde{J}^{l+1}$, belongs to K if and only if there exist either a differentiable r-linear form ω' defined on an open neighborhood of $y = \pi^{l+1}_l(x)$ such that $g_y(\omega') \in \tilde{\mathcal{R}}^l_{(0,r)}$ and $g_x(\omega) = g_x((\pi^{l+1}_l)^*\,\omega')$ or a differentiable r-linear form ω'' defined on an open neighborhood V of y and an associated coordinate system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$ on V such that $g_y(\omega'') \in \tilde{\mathcal{R}}^l_{(0,r)}$ and for some $1 \leqq i \leqq n$ there is $g_x(\omega) = g_x(\partial^i_\# \omega'')$. $K \subseteq \tilde{\mathcal{T}}^{l+1}_{(0,r)}$ is clearly a subsheaf of sets. Let us denote by $p\tilde{\mathcal{R}}^l_{(0,r)}$ the smallest subsheaf of $\mathcal{A}^{l+1}$ – algebras of $\tilde{\mathcal{T}}^{l+1}_{(0,r)}$ containing K. The subsheaf $p\tilde{\mathcal{R}}^l_{(0,r)}$ will be called the *formal prolongation of* $\tilde{\mathcal{R}}^l_{(0,r)}$. Proposition 8 gives us immediately the inclusion $p\tilde{\mathcal{R}}^l_{(0,r)} \subseteq \tilde{\mathcal{R}}^{l+1}_{(0,r)}$. Moreover for $y = \pi^{l+1}_l(x) \in J^l \subset \tilde{J}^l$ (for the definition of J^l see [2], Def. 10, p. 464) we get

Proposition 12. *Let $x \in \tilde{J}^{l+1}$, $y = \pi^{l+1}_l(x) \in J^l$. Then there is $(p\mathcal{R}^l_{(0,r)})\,(x) = \mathcal{R}^{l+1}_{(0,r)}(x)$, where $(p\mathcal{R}^l_{(0,r)})\,(x)$ and $\mathcal{R}^{l+1}_{(0,r)}(x)$ denotes the fibers at the point x of the sheaves $p\mathcal{R}^l_{(0,r)}$ and $\mathcal{R}^{l+1}_{(0,r)}$ respectively.*

Proof. $y \in J^l \subset \bar{J}^l$ and thus we can find an open neighborhood $V \subset J^l$ of y with a coordinate system $(f_1, \ldots, f_{n_l})$ on it such that $g_z(f_{k+1}), \ldots, g_z(f_{n_l})$ is a φ-basis of $\mathscr{A}_z^l$ for every $z \in V$. In view of our considerations in the first part of this paper we can find differentiable 1-forms $\omega_1, \ldots, \omega_k$ which can be supposed without loss of generality to be defined again on V, such that the germs $g_z(\omega_1), \ldots, g_z(\omega_k), g_z(\mathrm{d}f_{k+1}), \ldots$ $\ldots, g_z(\mathrm{d}f_{n_l})$ form a basis of the $\mathscr{A}_z^l$-module $\mathscr{R}_{(0,1)}^l$ for every $z \in V$. Again without loss of generality we can suppose that there is given on V an associated coordinated system $(x^i, y^\alpha, y^\alpha_{i_1}, \ldots, y^\alpha_{i_1 \ldots i_l})$. Now because $y \in J^l$ we can according to Prop. 15 from [2] choose from the system $(f_j \circ \pi_l^{l+1}, \partial_{\#}^{x^i} f_j;\ i = 1, \ldots, n;\ j = k+1, \ldots, n_l)$ of differentiable functions on $U = (\pi_l^{l+1})^{-1} V$ a subsystem, which we denote $(f'_{k+1}, \ldots$ $\ldots, f'_{n_{l+1}})$ such that $(g_z(f'_{k+1}), \ldots, g_z(f'_{n_{l+1}})$ is a φ-basis of $\mathscr{A}_z^{l+1}$ for any z from a sufficiently small neighborhood of x. It is clear from our construction of $(f'_{k+1}, \ldots$ $\ldots, f'_{n_{l+1}})$ and from Prop. 11 that $g_x(\mathrm{d}f'_{k+1}), \ldots, g_x(\mathrm{d}f'_{n_{l+1}}) \in p\mathscr{R}^l_{(0,1)}$. As well we have $g_x((\pi_l^{l+1})^* \omega_1), \ldots, g_x((\pi_l^{l+1})^* \omega_k) \in p\mathscr{R}^l_{(0,1)}$. It is easy to see that the differentiable 1-forms $(\pi_l^{l+1})^* \omega_1, \ldots, (\pi_l^{l+1})^* \omega_k, \mathrm{d}f'_{k+1}, \ldots, \mathrm{d}f'_{n_{l+1}}$ are linearly independent at the point x, and thus because of their number n_{l+1} they form a basis of the $\mathscr{A}_x^{l+1}$-module $\mathscr{R}^{l+1}_{(0,1)}(x)$ from which our proposition immediately follows.

References

[1] *M. Kuranishi:* Lectures on Involutive Systems of Partial Differential Equations, Publiçacões da Sociedade de Matemática de São Paulo, 1967.

[2] *J. Vanžura:* Invariants of Submanifolds, Czech. Math. J. *19* (*94*) 1969, pp. 452—468.

Author's address: Praha 1, Malostranské nám. 25, ČSSR (Matematicko-fyzikální fakulta UK).