Czechoslovak Mathematical Journal

Kamil John

Some remarks on compact maps in Banach spaces

Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 2, 177–182

Persistent URL: http://dml.cz/dmlcz/101158

Terms of use:

© Institute of Mathematics AS CR, 1973

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of Czechoslovak Academy of Sciences
V. *23* (98), PRAHA 14. 6. 1973, No 2

SOME REMARKS ON COMPACT MAPS IN BANACH SPACES

KAMIL JOHN, Praha

(Received January 15, 1971)

A characterization of compact maps is given similar to that of quasinuclear maps. Other related characterizations are stated in terms of factorization or expression of maps as countable series. Every compact map is a product of two compact maps. An equivalence between certain "into" extension properties of compact maps and a possibility of their factorization through c_0 or m is shown.

1. Notation. E, F, P will always be Banach spaces, G, H will be normed spaces. The usual spaces c_0, m, l_p are always understood with a countable indexset. If $\alpha_n > 0$, $\alpha_n \to 0$, we define a normed space $c^\alpha \subset c_0$, $c^\alpha = \{x; |x| < \infty\}$ with the norm $|x| = \sup |x_n| \alpha_n^{-1}$, where $x = \{x_n\}$. By I the identity map $c^\alpha \to c_0$ will be denoted. It will be shown later that I is compact. By a map we mean always a linear map. In c_0 we consider the usual norm $\|\{x_n\}\| = \sup |x_n|$.

To say that a map $T: E \to F$ can be factorized through G means that there are continuous maps $T_2 : E \to G$ and $T_1 : G \to F$, such that $T = T_1 T_2$.

$F \subset P$ means that F is norm-imbedded in P by inclusion. We say that F has $\lambda+$ extension property for compact maps ($\lambda+$ e.p.c.m.), if for every $\varepsilon > 0$, every compact map $T: E \to F$ and every $P \supset E$ there is a continuous (and thus compact in view of Proposition 4) extension $\tilde{T}$ of T, $\tilde{T}: P \to F$ with $|\tilde{T}| \leqq (\lambda + \varepsilon) |T|$. The space F is called a $\mathscr{P}_\lambda$ space if for every bounded map $T: E \to F$ and for every $P \supset E$ there is a linear extension $\tilde{T}$ of T, $\tilde{T}: P \to F$ with $|\tilde{T}| \leqq \lambda |T|$. For the definition of the $\mathscr{N}$. space see [3, p. 20]. We shall use the fact that every $\mathscr{N}_\lambda$ space has $\lambda+$ e.p.c.m, [3, p. 24].

We start with a strengthened form of a known fact.

2. Lemma. *Let $A \subset E$ be compact, such that $x \in A \Rightarrow |x| \leqq \beta$. Then for every $\varepsilon > 0$ there is a sequence $\{a_n\}$, $a_n \in E$, $a_n \to 0$, $|a_n| \leqq \beta + \varepsilon$, such that every point $x \in A$ can be expressed in the form $x = \sum \lambda_n a_n$, where $\lambda_n \geqq 0$, $\sum \lambda_n \leqq 1$.*

Proof. We use the following statement: For every $\varepsilon > 0$ and every $a > 0$ there are sequences $\{\alpha_n\}, \{\beta_n\}$, such that $\alpha_n > 0$, $a \geqq \beta_n > 0$, $\beta_n \to 0$, $\sum \alpha_n \beta_n = a$ and $\sum \alpha_n \leqq 1 + \varepsilon$. In fact, let $\varepsilon > 0$ and choose $\alpha_n > 0$, $\sum \alpha_n = 1 + \varepsilon$. Let p be such that $\alpha_n \leqq a$ for all $n > p$, $\sum_{n>p} \alpha_n^2 < a$ and $b = \sum_{n \leqq p} \alpha_n > 1$. Then $0 < c = a - \sum_{n>p} \alpha_n^2 < a$ and $c/b < a$. Put $\beta_n = c/b$ if $n \leqq p$ and $\beta_n = \alpha_n$ if $n > p$. Then $a \geqq \beta_n > 0$, $\beta_n \to 0$ and we have $\sum \alpha_n \beta_n = c/b \sum_{n \leqq p} \alpha_n + \sum_{n>p} \alpha_n^2 = a$. This completes the proof of our statement.

To prove the lemma choose $\varepsilon > 0$ and put $a = \beta + \varepsilon$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences provided by the statement at the beginning of this proof. By U_r we mean the open ball of radius r centred at zero. For every integer p put $V_p = U_{r_p}$, where $r_p = a - \sum_{n \leqq p} \alpha_n \beta_n$. We have $r_n \to 0$, $r_n > 0$ and thus $\{V_p\}$ forms a complete system of neighbourhoods of zero. We have $A \subset U_a$. The open set U_a is obviously the union of all open balls of radius r_1 contained in U_a. Because of the compactness of A, we may cover A by a finite number of them. Let B_1 be the finite set of their centres. Obviously $A \subset B_1 + V_1$ and $B_1 \subset \overline{U}_{s_1}$, where we put $s_n = r_{n-1} - r_n = \alpha_n \beta_n$. Now put $A_1 = (A - B_1) \cap V_1$. Then A_1 is compact, $A_1 \subset V_1$ and we may proceed with A_1 and V_1 as above with A and U_a: Cover A_1 by a finite number of open balls of radius r_2 contained in V_1. Their centres form a finite set $B_2 \subset \overline{U}_{s_2}$ and obviously $A_1 \subset B_2 + V_2$. Proceeding by induction we may find compact sets A_n and finite sets B_n, such that $A_n = (A_{n-1} - B_n) \cap V_n$, $A_{n-1} \subset B_n + V_n$ and $B_n \subset \overline{U}_{s_n}$. Every $x \in A$ can be expressed as $x = b_1 + a_1$, where $b_1 \in B_1$ and $a_1 \in A_1$. Every $a_1 \in A_1$ can again be expressed as $a_1 = b_2 + a_2$, where $b_2 \in B_2$ and $a_2 \in A_2$. Proceeding again by induction, we see that $x \in A$ can be expressed for every p in the form $x = a_p + \sum_{n \leqq p} b_n$, where $a_p \in A_p$ and $b_n \in B_n$. But $a_p \to 0$, because $A_p \subset V_p$ and thus $x = \sum b_n = \sum \alpha_n c_n$, where $c_n = \alpha_n^{-1} b_n$. Now let us order the elements of the sets $\alpha_1^{-1} B_1, \alpha_2^{-1} B_2, \ldots$ in a sequence $\{d_n\}$, such that the elements from $\alpha_n^{-1} B_n$ are before the elements from $\alpha_{n+1}^{-1} B_{n+1}$. We have $\alpha_n^{-1} B_n \subset \alpha_n^{-1} \overline{U}_{s_n} = \overline{U}_{\beta_n}$ and because of $\beta_n \to 0$, $|\beta_n| \leqq a$, we see that $d_n \to 0$ and $|d_n| \leqq \beta + \varepsilon$. Thus $x = \sum \alpha_n c_n = \sum \gamma_n d_n$, where $\sum \gamma_n = \sum \alpha_n \leqq 1 + \varepsilon$. We showed: For every $\varepsilon > 0$ there are $d_n \in E$, $|d_n| \leqq \beta + \varepsilon$, such that every $x \in A$ can be expressed as $\sum \gamma_n d_n$ with $\sum \gamma_n \leqq 1 + \varepsilon$, $\gamma_n \geqq 0$. Now putting $a_n = (1 + \varepsilon) d_n$ and $\lambda_n = (1 + \varepsilon)^{-1} \gamma_n$, we have $x = \sum \gamma_n d_n = \sum \lambda_n a_n$, where $|a_n| \leqq (1 + \varepsilon)(\beta + \varepsilon)$ and $\sum \lambda_n \leqq 1$. If we notice that $(1 + \varepsilon) . (\beta + \varepsilon) \to \beta$ for $\varepsilon \to 0$, we obtain our lemma.

The following proposition gives a characterization of compact maps. The equivalence a) and b) is implicitly contained in [6] and is a generalisation of the definition of the quasinuclear map in [4].

3. Proposition. *Let $T: E \to F$ be a map. The following assertions are equivalent.*

a) *T is compact and $|T| \leqq \beta$.*

b) *For every* $\varepsilon > 0$ *there is a sequence* $\{a_n\}$, $a_n \in E'$, $|a_n| \to 0$, $\sup |a_n| \leqq \beta + \varepsilon$, *such that*

$$|T(x)| \leqq \sup |\langle x, a_n \rangle| \quad \textit{for every} \quad x \in E .$$

c) *For every* $\varepsilon > 0$ *there is* $\alpha \in c_0$, *a subspace* $H \subset c^\alpha$, *continuous maps* $A : E \to H$ *and* $B : I(H) \to F$, *such that* $T = B \circ I \circ A$ *and* $|A| \leqq 1$, $|B| \leqq 1$, $|I| = \sup |\alpha_n| \leqq \beta + \varepsilon$.

d) *For every* $\varepsilon > 0$ *and every* $\mathscr{P}_\lambda$ *space* $P \supset F$ *there is a sequence* $\{a_n\}$, $a_n \in E'$, $\sup |a_n| \leqq \beta + \varepsilon$ *and a sequence* $\{y_n\}$, $y_n \in P$, *such that the map* T *can be expressed in the form*

$$T(x) = \sum \langle x, a_n \rangle \, y_n \,, \quad x \in E \,, \quad \textit{and} \quad \left|\sum b_n y_n\right| \leqq \lambda \sup |b_n|$$

for every $\{b_n\} \in c_0$.

Proof. a) $\Rightarrow$ b).The dual map $T' : F' \to E'$ is compact, $|T'| \leqq \beta$. Let U denote the unit ball in F'. Then the set $A = T'(U)$ is compact. Now according to Lemma 2 for every $\varepsilon > 0$ there are $a_n \in E'$, $\sup |a_n| \leqq \beta + \varepsilon$, $|a_n| \to 0$, such that for every $f \in U$ there is a sequence $\{\lambda_n\}$, $\sum |\lambda_n| \leqq 1$, such that $T'(f) = \sum \lambda_n a_n$. But for every $x \in E$ we have $|f \circ T(x)| = |T'(f)(x)| = |\sum \lambda_n \langle x, a_n \rangle| \leqq \sup |\langle x, a_n \rangle|$. Thus $|T(x)| = \sup \{|f \circ T(x)| \mid f \in U\} \leqq \sup |\langle x, a_n \rangle|$.

b) $\Rightarrow$ c). We can assume that in b) all $a_n \neq 0$. Let $\alpha = \{\alpha_n\}$, $\alpha_n = |a_n|$. We define A by $A(x) = \{\langle x, a_n \rangle\} \in c^\alpha$. A is continuous because $|A(x)| = \sup |\langle x, a_n \rangle| \,.\, \alpha_n^{-1} \leqq |x|$, $|A| \leqq 1$. Let $H = A(E) \subset c^\alpha$. The map $I : H \to H \subset c_0$ is obviously continuous, $\|I(x)\| = \sup |x_n| \leqq \sup |x_n| \alpha_n^{-1} \sup \alpha_n \leqq (\beta + \varepsilon) |x|$, $\|\alpha\| = |I| \leqq \beta + \varepsilon$. Now put for every $b = \{\langle x, a_n \rangle\} \in H \subset c_0$, $B(b) = T(x)$. B is actually a mapping, because if $\{\langle x, a_n \rangle\} = \{\langle y, a_n \rangle\}$, then $|T(x - y)| \leqq \sup |\langle x - y, a_n \rangle| = 0$. We have $|B(b)| = |T(x)| \leqq \sup |\langle x, a_n \rangle| = \|b\|$, thus $|B| \leqq 1$. Obviously $T(x) = B \circ I \circ A(x)$.

c) $\Rightarrow$ d). P being a $\mathscr{P}_\lambda$ space, we may extend $B : H \to P$ to a map $\tilde{B} : c_0 \to P$, with $|\tilde{B}| \leqq \lambda |B| \leqq \lambda$. Let π_n be linear forms on c^α, $\pi_n(\{x_n\}) = x_n$. If $x = \{x_n\} \in c^\alpha$, then $|\pi_n(x)| = |x_n| = \alpha_n |x_n| \alpha_n^{-1} \leqq \alpha_n |x|$. Thus $|\pi_n| \leqq \alpha_n$. Now put $\langle x, a_n \rangle = \pi_n \circ A(x)$. Obviously $a_n \in E'$, $|a_n| \leqq \alpha_n |A| \leqq \alpha_n$. If $b = \{b_n\} \in c_0$, then $b = \sum b_n e_n$, where $e_n = \{\delta_{nj}\} \in c_0$. Put $y_n = \tilde{B}(e_n)$. Then $\tilde{B}(b) = \sum b_n y_n$ and $|\sum b_n y_n| = |\tilde{B}(b)| \leqq \lambda \sup |b_n|$. Thus $T(x) = B \circ I \circ A(x) = B(\{\pi_n \circ A(x)\}) = \sum \pi_n \circ A(x) \, y_n = \sum \langle x, a_n \rangle \, y_n$. We have $\sup |a_n| \leqq \sup \alpha_n \leqq \beta + \varepsilon$.

d) $\Rightarrow$ a). There is a $\mathscr{P}_1$ space $P \supset F$ (for example a suitable m_I space). The maps $T_p : E \to P$, $T_p(x) = \sum_{n \leqq p} \langle x, a_n \rangle \, y_n$ are obviously compact maps. We have

$$|(T_p - T(x)| = \left|\sum_{n > p} \langle x, a_n \rangle \, y_n\right| \leqq \sup_{n > p} |\langle x, a_n \rangle| \leqq |x| \sup_{n > p} |a_n| \underset{p}{\to} 0 \,.$$

Thus $T = \lim T_p$ is compact. Putting $\lambda = 1$ in d), we have $|T(x)| = |\sum \langle x, a_n \rangle \, y_n| \leqq \sup |\langle x, a_n \rangle| \leqq |x| \sup |a_n| \leqq |x| \, (\beta + \varepsilon)$ for every $\varepsilon > 0$, and thus $|T| \leqq \beta$.

Remarks. a) If we are not interested in the norm of the map T, then of course we have the equivalence of

a') T is compact

b') There is a sequence $\{a_n\}$, $a_n \in E'$, $|a_n| \to 0$, such that $|T(x)| \leqq \sup |\langle x, a_n\rangle|$ for every $x \in E$.

Similarly we obtain c') and d') by omitting the β's and ε's. In this case the a_n can be taken linearly independent, if E has infinite dimension. This follows by a slight modification of the proof of the known weaker form of Lemma 2 (for instance [5, Lemma VII. 2.2.]). By choosing B_n in that proof we use repeatedly a trivial observation: In every neighbourhood of zero there is an element which does not belong to a fixed finitedimensional subspace. Thus every element $b_p \in B_n = \{b_1, \ldots, b_i\}$ can be chosen linearly independent of all B_m, $m < n$ and of $b_1, \ldots, b_{p-1}$. Then $\bigcup_n B_n$ is linearly independent.

b) The speed of convergence of $\{a_n\}$ may offer some information on the "compactness" of the map T. If we denote by S_T the set of all sequences $\{x_n\}$, such that there are $a_n \in E'$, with $|a_n| \leqq |x_n|$, $|T(x)| \leqq \sup |\langle x, a_n\rangle|$, then S_T delivers some information about the measure of the compactness of T. Various classes of maps T may be defined using S_T. For example:

$$T \text{ is compact} \Leftrightarrow S_T \cap c_0 \neq \emptyset,$$
$$T \text{ is quasinuclear} \Leftrightarrow S_T \cap l_1 \neq \emptyset,$$
$$T \text{ is ``}p\text{-quasinuclear''} \Leftrightarrow S_T \cap l_p \neq \emptyset.$$

It is not difficult to see that a finite product of p-quasinuclear maps is nuclear (see [4]).

c) In view of Proposition 4 it is not difficult to see that the space P in Proposition 4 d) can also be a space which has $\lambda+$ e.p.c.m., especially $\mathcal{N}_\lambda$ space (see [3]). As a consequence we have:

d) Every $\mathcal{P}$ space, $\mathcal{N}_\lambda$ space, space with $\lambda+$ e.p.c.m. has the approximation property of Grothendieck [2]. In fact it suffices that a p-th dual space has $\lambda+$ e.p.c.m.

The absolutly summable maps, quasinuclear maps and the nuclear maps cannot generally be expressed as products of two maps of the same category. For compact maps we have:

4. Proposition. *Every compact map T is for every $\varepsilon > 0$ a product of two compact maps, $T = A \circ B$ with $|A|\,|B| \leqq |T| + \varepsilon$.*

Proof. Let $T: E \to F$ be a compact map. We shall repeatedly use the equivalence of a) and b) from Proposition 3. Choose $a_n \in E'$, $|a_n| \leqq 1$, $\alpha_n \geqq 0$, $\alpha_n \to 0$, $\sup \alpha_n \leqq$ $\leqq |T| + \varepsilon$, such that $|T(x)| \leqq \sup \alpha_n |\langle x, a_n\rangle|$. Put $B(x) = \{x_n\} \in c_0$, where $x_n =$ $= (\sqrt{\alpha_n}) \langle x, a_n\rangle$. We have $\|B(x)\| = \sup (\sqrt{\alpha_n}) |\langle x, a_n\rangle|$ and thus B is a compact map $B: E \to \bar{G}$, where $G = B(E) \subset c_0$. Now put $A(g) = T(x)$ for $g = \{x_n\} \in G$,

$x_n = (\sqrt{\alpha_n}) \langle x, a_n \rangle$. Then $A : G \to F$ is a mapping and we have $|A(g)| = |T(x)| \leqq \leqq \sup \alpha_n |\langle x, a_n \rangle| = \sup (\sqrt{\alpha_n}) (\sqrt{\alpha_n}) \langle x, a_n \rangle| = \sup (\sqrt{\alpha_n}) |\langle g, b_n \rangle|$, where $b_n \in c_0'$ are projections, i.e. $\langle \{x_n\}, b_n \rangle = x_n$. The same holds for the unique extension A of A to the Banach space $\bar{G}$ and for the extensions of b_n to $\bar{G}$. Thus the map A is compact. Obviously $T = A \circ B$ and $|A| \leqq \sup (\sqrt{\alpha_n}) \leqq (|T| + \varepsilon)^{1/2}$, $|B| \leqq \sup (\sqrt{\alpha_n}) \leqq \leqq (|T| + \varepsilon)^{1/2}$, e.g. $|A| |B| \leqq |T| + \varepsilon$.

The procedure employed here may be used to show some implications in the question of extending compact maps. We restrict our attention only to "into" extension questions.

5. Proposition. *The following assertion are equivalent.*

a) *The Banach space F has $\lambda+$ e.p.c.m.*

b) *For every $\varepsilon > 0$ every compact map T into F can be factorized through c_0, $T = T_1 \circ T_2$ with $|T_1| |T_2| \leqq (\lambda + \varepsilon) |T|$.*

c) *Every compact map $T : E \to F$ can be expressed for every $\varepsilon > 0$ in the form $T(x) = \sum \langle x, a_n \rangle y_n$, where $y_n \in F$, $a_n \in E'$, $|a_n| \to 0$, $|a_n| \leqq (|T| + \varepsilon)$ and $|\sum b_n y_n| \leqq \lambda \sup |b_n|$ for every $\{b_n\} \in c_0$.*

The same holds for spaces m or c^α instead of c_0.

Proof. a) $\Rightarrow$ b) Let F have $\lambda+$ e.p.c.m. and let $T : E \to F$ be a compact map. According to Proposition 4 $T = A \circ B$ where A, B are compact maps and $|A| |B| \leqq \leqq |T| + \varepsilon$. Now factorize the compact map B through a subspace H of c_0 according to Proposition 3c), $B = C \circ T_2$ with $|C| |T_2| \leqq (1 + \varepsilon) |B|$. Let $T_1 : c_0 \to F$ be an extension of the map $A \circ C : H \to F$ with $|T_1| \leqq (\lambda + \varepsilon) |A \circ C|$. We have $T = = A \circ B = A \circ C \circ T_2 = T_1 \circ T_2$ with $|T_1| |T_2| \leqq (\lambda + \varepsilon) |A \circ C| |T_2| \leqq (\lambda + \varepsilon) |A| . . |C| |T_2| \leqq (\lambda + \varepsilon)(1 + \varepsilon) |A| |B| \leqq (\lambda + \varepsilon)(1 + \varepsilon)(|T| + \varepsilon)$. But $(\lambda + \varepsilon)(1 + \varepsilon) . . (|T| + \varepsilon) \to \lambda |T|$ for $\varepsilon \to 0$, which proves b).

b) $\Rightarrow$ a) Conversely suppose b) and let $T : E \to F$ be a compact map, $P \supset E$. T is a product of two compact maps $T = A \circ B$, where $|A| |B| \leqq |T| + \varepsilon$. According to the assumption the map A can be factorized through c_0, $A = C \circ D$ with $|C| |D| \leqq \leqq (\lambda + \varepsilon) |A|$. The space c_0 being $\mathscr{N}_1$ space, the map $D \circ B : E \to c_0$ can be extended to a map $M : P \to c_0$ with $|M| \leqq (1 + \varepsilon) |D \circ B|$. Thus $C \circ M$ is an extension of T to a map $P \to F$ with the norm $|C \circ M| \leqq (1 + \varepsilon) |C| |D| |B| \leqq (\lambda + \varepsilon)(1 + \varepsilon) . . |A| |B| \leqq (\lambda + \varepsilon)(1 + \varepsilon)(|T| + \varepsilon) \to \lambda |T|$.

a) $\Rightarrow$ c) follows from Remark 3c).

c) $\Rightarrow$ a) Let $T : E \to F$ be a compact map, $P \supset E$ and let for the map T the expression of c) hold. By Hahn-Banach theorem there are $b_n \in P'$ extending a_n, such that $|b_n| = |a_n|$. Then the expression $U(x) = \sum \langle x, b_n \rangle y_n$ defines a continuous map $U : : P \to F$, which is an extension of T. We have $|U(x)| \leqq \lambda \sup |\langle x, b_n \rangle| \leqq \lambda |x| . . (|T| + \varepsilon)$, showing that $|U| \leqq \lambda(|T| + \varepsilon) \to \lambda |T|$ for $\varepsilon \to 0$.

The second part of the proposition can be proved similarly, but instead of using that c_0 is $\mathscr{N}_1$ space, we use the easily seen fact that m and c^α are $\mathscr{P}_1$ spaces.

Corollary. *For $\lambda = 1$ we obtain a condition equivalent to those expressed by Theorem* 6.1 *in* [3]. *Thus for example the conditions* a), b), c) *for $\lambda = 1$ are equivalent to the statement that F'' is $\mathscr{P}_1$ space.*

Added in proof. After this paper had been submitted for publication, several works appeared, viz. [7], [8], [9].

References

[1] *N. Bourbaki,* Espaces vectoriels topologiques, Hermann, Paris, 1953/55.
[2] *A. Grothendieck,* Produits tensoriels topologiques et espaces nucléaires, Memoirs Amer. Math. Soc., *16* (1955).
[3] *J. Lindenstrauss,* Extension of compact operators, Memoirs Amer. Math. Soc. *48* (1964).
[4] *A. Pietsch,* Nukleare lokalkonvexe Räume, Akademie-Verlag-Berlin, 1965.
[5] *A. Robertson,* Topological vector spaces, Cambridge university press, 1964.
[6] *T. Terzioğlu,* On Schwartz spaces, Math. Ann. *182* (1969), 236—242.
[7] *D. J. Randtke,* Characterizations of pracompact maps, Schwartz spaces and nuclear spaces. Trans. Amer. Math. Soc. *165,* (1972), 87—101.
[8] *D. J. Randtke,* A factorization theorem for compact operators, Proc. Amer. Math. Soc. *34.* 1 (1972), 201—202.
[9] *T. Terzioğlu,* A characterization of compact linear mappings, Arch. Math., *22* (1972), 76—78.

Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).