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ON A GENERALIZED HEAT POTENTIAL

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1.

In this article a special heat potential is introduced and its properties are investigated. Similar questions were studied in [3] for the Newtonian potential and in [6] for the heat potential. Methods, notation and some results from those papers and especially from [4] are used. The regions studied in [4], [6] were of a special shape (generalized cylinders). In this article we shall prove similar results for more general regions with time-dependent boundaries. The acquaintance with the above mentioned articles is not generally supposed except the proof of Lemma 1.3. Similar proofs in those articles are sometimes reminded of.

Let R^k be the Euclidean k-space where $k = m$ or $k = m + 1$ and $m \geqq 3$ (for the case $m = 1$ see [1], [2]). To distinguish the notation in R^m and R^{m+1} we shall use the following convention (see [6]). For example, we put for $a \in R^m$, $\varrho > 0$,

$$(1.1) \qquad \Omega(a, \varrho) = \{b \in R^k;\ |b - a| < \varrho\}\,,$$

where $|\ldots|$ denotes the Euclidean norm. Let us denote by Γ the boundary $\partial\Omega(0, 1)$. Now the symbols $\Omega(0, 1)$ and Γ denote the ball and the sphere in R^{m+1} and $\Omega^*(0, 1)$, Γ^* have the same meaning in R^m. For $z = [z_1, \ldots, z_{m+1}] \in R^{m+1}$ we write $z = [\hat{z}, z_{m+1}] = [x, t]$ where $x \in R^m$, $t \in R^1$. The same convention is applied to differential operators ∇, Δ, for example $\nabla = [\partial_1, \ldots, \partial_{m+1}]$ and $\hat{\nabla} = [\partial_1, \ldots, \partial_m]$.

Let us define for $z \in R^{m+1}$

$$(1.2) \qquad \begin{aligned} \mathscr{G}(z) &= z_{m+1}^{-m/2} \exp\left(-|\hat{z}|^2/4z_{m+1}\right) && \text{for} \quad z_{m+1} > 0\,, \\ \mathscr{G}(z) &= 0 && \text{for} \quad z_{m+1} \leqq 0\,. \end{aligned}$$

Thus $\mathscr{G}(z - w)$ is for any $w \in R^{m+1}$ an infinitely differentiable function of z on $R^{m+1} - \{w\}$ which is caloric on this set, i.e.,

$$(1.3) \qquad \hat{\Delta}_z \mathscr{G}(z - w) - \frac{\partial}{\partial z_{m+1}} \mathscr{G}(z - w) = 0\,.$$

For $\tau \in R^1$ we put $R_\tau^{m+1} = \{[x, t] \in R^{m+1};\ t < \tau\}$. Let D be an open set in R^{m+1} and $\partial D = B$. We denote

$$(1.4) \qquad D_\tau = D \cap R_\tau^{m+1}\,, \quad B_\tau = \overline{B \cap R_\tau^{m+1}}\,.$$

The initial assumptions about D are the following:

(A) B_τ is compact for all $\tau \in R^1$ and

$B_\tau = \emptyset$ for $\tau = 0$.

We denote the spaces of all bounded Baire functions or bounded continuous functions on B by $\mathscr{B}$ or $\mathscr{C}$, respectively. $\mathscr{D}$ is the space of all infinitely differentiable functions φ with compact support spt φ in R^{m+1}. We shall use the symbol $\|\ldots\|$ for the norm defined in all those spaces in the usual way by means of supremum.

If D is a region with "sufficiently smooth" boundary one can introduce heat potentials in the following way: Let us denote by H_k the k-dimensional Hausdorff measure and by $n(w)$ the exterior normal to D at the point $w \in B$. We put for all $f \in \mathscr{C}$ and $z \in R^{m+1} - B$ (we denote by $\circ$ the scalar product)

$$(1.5) \qquad Vf(z) = \int_{B_t} f(w)\, \mathscr{G}(z - w)\, \mathrm{d}H_m(w)\,,$$

$$Wf(z) = \int_{B_t} f(w)\, \hat{n}(w) \circ \hat{\nabla}_w \mathscr{G}(z - w)\, \mathrm{d}H_m(w)\,.$$

The function Wf is the heat potential of the double distribution while the function Vf is the heat potential of the single distribution. Both functions are studied in connection with the boundary value problems for the heat equation (see for example [5]) etc. The following approach is motivated by the fact that sometimes it is useful to study the sum of those potentials.

We denote

$$(1.6) \qquad \mathscr{D}(z) = \{\varphi \in \mathscr{D};\ z \notin \operatorname{spt} \varphi\}$$

and supposing (A) for D we define for $z \in R^{m+1}$, $\varphi \in \mathscr{D}(z)$ the linear functional T over $\mathscr{D}(z)$ by

$$(1.7) \qquad T\varphi(z) = -\int_D (\hat{\nabla}_w \mathscr{G}(z - w) \circ \hat{\nabla}\varphi(w) + \mathscr{G}(z - w)\, \partial_{m+1}\varphi(w))\, \mathrm{d}w\,.$$

The integral is finite for all $\varphi \in \mathscr{D}(z)$; one can see it easily from estimates (10) and (11) in [4]. It was shown in [6] for time-independent regions (generalized cylinders) that $T\varphi$ reduces in this case to the heat potential of the double distribution provided the boundary of the region is sufficiently smooth. So we shall start now to study relations between $T\varphi$ and potentials $V\varphi$, $W\varphi$.

For a moment we assume not only (A) but also that the boundary B is a C^1-surface. Let us write $z = [x, t]$, $w = [y, u]$. We replace $\mathscr{G}(z - w)$ in (1.7) by a function $\tilde{\mathscr{G}} \in \mathscr{D}$ for which $\mathscr{G}(z - w) = \tilde{\mathscr{G}}(w)$ in some neighborhood of spt φ. Using the fact that

$$\hat{\Delta}\tilde{\mathscr{G}}(w) + \partial_{m+1}\tilde{\mathscr{G}}(w) = 0$$

holds in a neighborhood of spt φ and applying the Fubini theorem we obtain

$$T\varphi(z) = T\varphi(x, t) =$$

$$= -\int_0^t \left(\int_{D(u)} \left(\sum_{j=1}^m \partial_j \tilde{\mathscr{G}}(y, u)\, \partial_j \varphi(y, u) + \varphi(y, u)\, \partial_j^2 \tilde{\mathscr{G}}(y, u) \right) \mathrm{d}y \right) \mathrm{d}u -$$

$$- \int_0^t \left(\int_{D(u)} \left(\varphi(y, u) \,.\, \partial_u \tilde{\mathscr{G}}(y, u) + \partial_u \varphi(y, u)\, \tilde{\mathscr{G}}(y, u) \right) \mathrm{d}y \right) \mathrm{d}u$$

where $D(u) = \{y \in R^m;\ [y, u] \in D\}$. Let us write

$$n(w) = [\hat{n}(w), n_\tau(w)]\,,$$

i.e., n_τ is the last coordinate of n. Making use of the Gauss integral theorem we get after a simple calculation

$$T\varphi(z) = -\int_{B_t} \varphi(w)\, \hat{n}(w) \circ \hat{\nabla}_w \mathscr{G}(z - w)\, \mathrm{d}H_m(w) - \tag{1.8}$$

$$- \int_{B_t} \varphi(w) \,.\, n_\tau(w)\, \mathscr{G}(z - w)\, \mathrm{d}H_m(w)\,.$$

Comparing (1.8) and (1.5) one can see that $T\varphi$ is the sum of potentials where the single-layer potential has the density φn_τ.

In case spt $\varphi \cap B_t = \emptyset$ there is a $\tilde{D} \subset D$ with smooth boundary $\tilde{B}$ and the process described leads us to $T\varphi(z) = 0$. Thus $T\varphi(z)$ depends only on the values of φ in a neighbourhood of B (or, more precisely, B_t where $z = [x, t]$). If $z \in R^{m+1} - B$, we can use this fact and extend $T\varphi(z)$ from $\mathscr{D}(z)$ to $\mathscr{D}$ defining

$$T\varphi(z) \overset{\text{def}}{=} T\tilde{\varphi}(z) \tag{1.9}$$

where $\tilde{\varphi} \in \mathscr{D}(z)$ coincides with the given $\varphi \in \mathscr{D}$ in a neighborhood of B. One can see that $T\varphi(z)$ may be considered a distribution over $\mathscr{D}$ and the following simple lemma holds:

1.1 Lemma. *Define $T\varphi(z)$ by (1.7) and (1.9) provided D has properties (A). Then for any $z = [x, t] \in R^{m+1} - B$ the support of the distribution $T\varphi(z)$ over $\mathscr{D}$ is contained in B_t. For fixed $\varphi \in \mathscr{D}$ the function $T\varphi(z)$ of the variable z is a caloric function on $R^{m+1} - B$.*

We shall say that $T\varphi(z)$ is a measure provided there exists a measure ν_z (Borel signed measure with a finite variation $|\nu_z|(B_t)$) with the property

$$(1.10) \qquad T\varphi(z) = \int_{R^{m+1}} \varphi \, d\nu_z$$

for every $\varphi \in \mathscr{D}(z)$. Then we can easily extend $T\varphi(z)$ from $\mathscr{D}(z)$ to $\mathscr{B}$ — it is sufficient to put for all $f \in \mathscr{B}$

$$(1.11) \qquad Tf(z) = \int_{R^{m+1}} f \, d\nu_z \, .$$

The first natural question of representability may be formulated as follows: what necessary and sufficient conditions are to be imposed on D to secure that $T\varphi(z)$ is a measure?

For this purpose we shall find similarly as in [4] (see Lemma 1.6) and [6] (see Lemma 2.2) another representation of $T\varphi(z)$. First we shall introduce a useful notation. For $z \in R^{m+1}$ we define the mapping S_z on $(0, \infty) \times (0, \infty) \times \Gamma^*$ by

$$(1.12) \qquad S_z(\varrho, \gamma, \theta^*) = \left[\hat{z} + \varrho\theta^*, z_{m+1} - \frac{\varrho^2}{4\gamma}\right]$$

where $\varrho, \gamma \in (0, \infty)$, $\theta^* \in \Gamma^*$. Instead of $\varphi * S_z$ we shall write $S_z\varphi$.

1.2 Lemma. *Suppose that a fixed $D \in R^{m+1}$ has the properties* (A). *Let $z \in R^{m+1}$, $\varphi \in \mathscr{D}(z)$. We put*

$$(1.13) \qquad \tilde{D}(\gamma, \theta^*) = \{\varrho > 0;\ S_z(\varrho, \gamma, \theta^*) \in D\}.$$

Then $T\varphi(z)$ is represented in the integral form

$$(1.14) \quad T\varphi(z) = 2^{m-1} \int_{\Gamma^*} dH_{m-1}(\theta^*) \int_0^\infty e^{-\gamma} \gamma^{m/2-1} \, d\gamma \int_{\tilde{D}(\gamma,\theta^*)} \partial_\varrho S_z \varphi(\varrho, \gamma, \theta^*) \, d\varrho \, .$$

Proof: In (1.7) we put $z = [x, t]$, $w = [y, u]$. In virtue of (1.2) we obtain

$$\begin{aligned} T\varphi(z) &= T\varphi(x, t) = \\ &= - \int_{D_t} \left[(t-u)^{-m/2} \exp\left(-\frac{|x-y|^2}{4(t-u)}\right) \cdot \frac{(x-y)}{2(t-u)} \circ \hat{\nabla}\varphi(y, u) + \right. \\ &\quad \left. + (t-u)^{-m/2} \exp\left(-\frac{|x-y|^2}{4(t-u)}\right) \frac{\partial\varphi}{\partial u}(y, u) \right] dy \, du = \\ &= -\frac{1}{2} \int_{D_t} (t-u)^{-m/2-1} \exp\left(-\frac{|x-y|^2}{4(t-u)}\right) \cdot \\ &\quad \cdot \left[(x-y) \circ \hat{\nabla}\varphi(y, u) + 2(t-u) \frac{\partial\varphi(y, u)}{\partial u} \right] dy \, du \, . \end{aligned}$$

Now we apply the Fubini theorem and use the substitution

$$y = x + \varrho\theta^* , \quad u = t - \frac{\varrho^2}{4\gamma} ,$$

which may be considered the composition of substitutions I, II

$$\text{I:} \quad \begin{aligned} y &= x + r\Phi , \\ u &= v , \end{aligned} \qquad \text{II:} \quad \begin{aligned} \Phi &= \theta^* , \\ r &= \varrho , \\ v &= t - \varrho^2/4\gamma . \end{aligned}$$

This is useful in order to determine the value of the Jacobian. For the terms under the integral sign we have

$$\mathrm{d}y \, \mathrm{d}u \to^{\mathrm{I}} r^{m-1} \, \mathrm{d}H_{m-1}(\Phi) \, \mathrm{d}v \to^{\mathrm{II}} \frac{\varrho^{m+1}}{4\gamma^2} \, \mathrm{d}H_{m-1}(\theta^*) \, \mathrm{d}\gamma \, \mathrm{d}\varrho .$$

After a short calculation we get

$$T\varphi(x, t) = -\frac{1}{2} \int_\Delta \left(\frac{\varrho^2}{4\gamma}\right)^{-m/2-1} \exp(-\gamma) \, .$$

$$\cdot \left[-\varrho \frac{\partial S_z \varphi(\varrho, \gamma, \theta^*)}{\partial \varrho} - 2 \left(\frac{\varrho^2}{4\gamma}\right) \frac{\partial S_z \varphi(\varrho, \gamma, \theta^*)}{\partial \varrho} \cdot \left(\frac{2\gamma}{\varrho}\right) \right] \cdot$$

$$\cdot \frac{\varrho^{m+1}}{4\gamma^2} \, \mathrm{d}H_{m-1}(\theta^*) \, \mathrm{d}\gamma \, \mathrm{d}\varrho$$

and then

$$T\varphi(z) = 2^{m-1} \int_\Delta \exp(-\gamma) \, \gamma^{m/2-1} \, \partial_\varrho S_z \varphi(\varrho, \gamma, \theta^*) \, \mathrm{d}H_{m-1}(\theta^*) \, \mathrm{d}\gamma \, \mathrm{d}\varrho$$

where $\Delta = \tilde{D}(\gamma, \theta^*) \times (0, \infty) \times \Gamma^*$. Lemma is proved.

Let us denote for any r, $0 < r \leqq \infty$

$$(1.15) \qquad \delta(\gamma) = \min(r, 2\sqrt{r\gamma})$$

and let $\mathscr{S}_z(r)$ stand for the system of all parabolas which are described by

$$(1.16) \qquad S_z(\cdot, \gamma, \theta^*) : (0, \delta(\gamma)) \to R^{m+1}$$

where $(\gamma, \theta^*) \in (0, \infty) \times \Gamma^*$. An element of the set $\mathscr{S}_z(r)$ is called an r-ray from z (a ray from z in case $r = \infty$). Let $A \subset R^{m+1}$ be an open set. A point $b \in S$ where $S \in \mathscr{S}_z(r)$ will be called a hit of S on A provided

$$(1.17) \qquad H_1(\Omega(b, \varrho) \cap S \cap A) > 0 , \quad H_1((\Omega(b, \varrho) \cap S) - A) > 0$$

for any $\varrho > 0$. If we compare this notion with that from [6] we see that arcs of parabolas are used now instead of open segments or half-lines.

The number of all hits of an $S \in \mathscr{S}_z(r)$ described by (1.16) on D will be denoted by $n_r(z, \gamma, \theta^*)$. We put for all $z \in R^{m+1}$, $r \in (0, \infty\rangle$

$$(1.18) \qquad v_r(z) = \int_{\Gamma^*} \mathrm{d}H_{m-1}(\theta^*) \int_0^\infty e^{-\gamma} \gamma^{m/2-1} n_r(z, \gamma, \theta^*)\, \mathrm{d}\gamma\,.$$

The function v_r defined on R^{m+1} in this way will be called the parabolic variation with the radius r or the parabolic r-variation. We shall use this notation very often for the special case $r = \infty$. In this case we shall omit the index r in all symbols just introduced and we shall write only $\mathscr{S}_z$, $n(z, \gamma, \theta^*)$, $v(z)$ etc.

The definition of the parabolic r-variation by the formula (1.18) is not quite correct. To remove the gap we shall prove the measurability of the "hit-function" $n_r(z, \gamma, \theta^*)$ — we shall do it in the course of proof of the following

1.3 Lemma. *For fixed* $z = [x, t] \in R^{m+1}$ *and* $0 < r \leqq \infty$ *put*

$$(1.19) \qquad \mathscr{D}^1 = \{\varphi \in \mathscr{D};\ \|\varphi\| \leqq 1,\ \operatorname{spt} \varphi \subset (\Omega^*(x, r) - \{x\}) \times (t - r, t)\}\,.$$

Then

$$(1.20) \qquad \sup \{T\varphi(z);\ \varphi \in \mathscr{D}^1\} = 2^{m-1}\, v_r(z)\,.$$

1.4 Remark. As it was mentioned above the proof depends on Lemma 1.3 from [4] which is too long to be presented here. We remind the reader also of the similar proof of Proposition 1.8 in [4]. The author tried to find a proof which would not depend on the mentioned lemma but in vain. It should not be difficult to understand all other proofs without knowledge of [4].

Proof. We apply the lemma mentioned in Remark 1.4 and define

$$\mathrm{d}\mu(\gamma) = 2^{m-1} e^{-\gamma} \gamma^{m/2-1}\, \mathrm{d}H_1(\gamma)\,.$$

For this measure we put $\lambda = \mu \times H_{m-1}$; we consider this measure λ on the σ-algebra of all Borel subsets of $Z = (0, \infty) \times \Gamma^*$. The mapping S_z defined in (1.12) maps $(0, \infty) \times (0, \infty) \times \Gamma^* = (0, \infty) \times Z$ homeomorphically onto

$$(R^m - \{x\}) \times (-\infty, t)\,.$$

Further, we define for $\gamma > 0$

$$r(\gamma) = \min \{r, \sqrt{2r\gamma}\}$$

and put

$$\tilde{E} = (\Omega^*(x, r) - \{x\}) \times (t - r, t)\,.$$

We have

$$G = S_z^{-1}(\tilde{E}) = \{[\varrho, \gamma];\ \gamma > 0,\ 0 < \varrho < r(\gamma)\} \times \Gamma^* .$$

Now we choose a decreasing sequence of positive numbers ε_k where $\varepsilon_1 < r/2$, $\varepsilon_k \searrow 0$ and define

$$G_k = \{[\varrho, \gamma];\ \gamma > \varepsilon_k/r,\ \varepsilon_k < \varrho < r(\gamma) - \varepsilon_k\} \times \Gamma^* .$$

Let Ψ_k stand for the class of all functions ψ defined on $X = R^1 \times Z$ for which there is a $\varphi \in \mathscr{D}^1$ with the properties

$$\psi = S_z\varphi \quad \text{on} \quad G\,, \quad \psi(X - G) = \{0\}, \quad \operatorname{spt} \psi \subset G_k .$$

The class of all point-wise limits of sequences of elements of Ψ_k is just the class of all functions g of the first class of Baire on X such that

$$\|g\| \leqq 1\,, \quad X - G_k \subset g^{-1}(0) .$$

So the conditions (P_5), (P_6) from the above mentioned lemma are satisfied and we pass to (P_4). We fix an infinitely differentiable function ω on R^1 such that

$$\operatorname{spt} \omega \subset (-1, 1)\,, \quad \omega(t) = \omega(-t)\,, \quad t \in R^1\,, \quad \int_{R^1} \omega(t)\, \mathrm{d}H_1(t) = 1 .$$

We define for $\psi \in \Psi_k$ (this operator does not change too much the supports of functions)

$$A_n\psi(\varrho, \gamma, \theta^*) = n \int_{R^1} \psi(\varrho - a, \gamma, \theta^*)\, \omega(na)\, \mathrm{d}a .$$

Let us fix $n_k > \varepsilon_k^{-1}$ and consider $n \geqq n_k$. For these n $A_n\psi$ has compact support contained in G. Let $\psi \in \mathscr{D}^1$ be the function for which we have $\psi = S_z\varphi$ on G. The value of the function $A_n\psi * S_z^{-1}$ at the point $[y, u] \in (R^m - \{x\}) \times (-\infty, t)$ is given by the integral

$$n \int_{R^1} \varphi(x + (y - x)\, h(a, x),\, t - (t - u)\, h^2(a, x))\, \omega(na)\, \mathrm{d}a \tag{1.20}$$

where

$$h(a, x) = \frac{|y - x| - a}{|y - x|} .$$

Now define $\tilde{\varphi}(y, u)$ for any $[y, u] \in \tilde{E}$ by (1.20) and let $\tilde{\varphi}(R^{m+1} - \tilde{E}) = \{0\}$. Then we have $\tilde{\varphi} \in \mathscr{D}^1$,

$$A_z\psi = \tilde{\varphi} * S_z = S_z\tilde{\varphi} \quad \text{on} \quad G .$$

Consequently, $A_n\psi \in \Psi = \bigcup_{k=1}^{\infty} \Psi_k$ and (P_4) is verified. The remaining conditions (P_1),

(P_2), (P_3) are readily verified and so we can apply Lemma 1.3 from [4] to the characteristic function f of G. We use Lemma 2.2 and obtain

$$\sup \{T\varphi(z);\ \varphi \in \mathscr{D}^1\} = \sup \left\{ \int_X f\, \partial_1 \psi \,\mathrm{d}(H_1 \times \lambda);\ \psi \in \Psi \right\} = \int_Z F \,\mathrm{d}\lambda$$

where for a fixed $(\gamma, \theta^*) \in Z$ we have

$$F(\gamma, \theta^*) = \sup \left\{ \int_{R^1} f(\varrho, \gamma, \theta^*) \frac{\partial \psi}{\partial \varrho} (\varrho, \gamma, \theta^*) \,\mathrm{d}H_1(\varrho);\ \psi \in \Psi \right\}.$$

F is a non-negative λ-measurable function. For $(\gamma, \theta^*) \in Z$ the system of all functions $\{\psi(\cdot, \gamma, \theta^*);\ \psi \in \Psi\}$ coincides with the system of all infinitely differentiable functions η on R^1 with compact support with the property

$$\|\eta\| \leqq 1\,, \quad \operatorname{spt} \eta \subset (0, r(\gamma))\,.$$

Taking into account that the function $f(\cdot, \alpha, \theta^*)$ is the characteristic function of

$$\tilde{D}(\gamma, \theta^*) \cap (0, r(\gamma))$$

(compare (1.13)) we apply 1.9 from [3] and obtain the equality

$$F(\gamma, \theta^*) = n_r(z, \gamma, \theta^*)\,. \tag{1.21}$$

Now the above mentioned gap is removed and the definition of v_r is correct. From the definition of the measure λ and (1.21) we obtain easily (1.20) and the lemma is proved.

1.5 Proposition. *Let us denote for $z \in R^{m+1}$*

$$\mathscr{D}^1(z) = \{\varphi \in \mathscr{D}(z);\ \|\varphi\| \leqq 1\}\,.$$

Then we have

$$\sup \{T\varphi(z);\ \varphi \in \mathscr{D}^1(z)\} = 2^{m-1}\, v(z)\,. \tag{1.22}$$

Proof. Put $r = +\infty$ in the preceding lemma. The rest follows from the relation between $\mathscr{D}^1$ from Lemma 1.3 and $\mathscr{D}^1(z)$.

Using some ideas from Král's papers we obtain the solution of our problem.

1.6 Proposition. *Suppose that D fulfils (A) and let $z \in R^{m+1}$. Then for all uniformly convergent sequences $\{\varphi_k\}$, $\varphi_k \in \mathscr{D}(z)$ converging to $\varphi \in \mathscr{D}(z)$ the equality*

$$\lim_{k \to \infty} T\varphi_k(z) = T\varphi(z) \tag{1.23}$$

holds if and only if

$$v(z) < \infty \tag{1.24}$$

is valid.

Proof. Supposing (1.24) we can obtain from (1.22) the estimate

$$|T\varphi_k(z) - T\varphi(z)| \leqq 2^{m-1}\, v(z)\, \|\varphi_k - \varphi\| \,. \tag{1.25}$$

If $v(z) = \infty$ it is possible to find a sequence of functions $\varphi_k \in \mathscr{D}^1(z)$ such that $\lim_{k\to\infty} T\varphi_k(z) = \infty$. From this sequence we can easily construct another sequence converging uniformly to 0 for which $\lim_{k\to\infty} T\varphi_k(z) = \infty$ and so (1.23) does not hold for it.

The well-known theorem about integral representation of linear functionals and Proposition 1.5 yield the following

1.7 Theorem. *Let $\mathscr{D}$ be as above, $z \in R^{m+1}$. Then the linear functional $T\varphi(z)$ over $\mathscr{D}(z)$ is a measure v_z if and only if (1.24) holds. The equality (1.10) for all $\varphi \in \mathscr{D}(z)$ together with any of the following conditions*

$$v_z(\{z\}) = 0\,, \quad |v_z|\,(B) = 2^{m-1}\, v(z)$$

determine v_z uniquely.

1.8 Remark. Now we can define $Tf(z)$ for all $f \in \mathscr{B}$ by

$$Tf(z) \overset{\text{def}}{=} \int_{R^{m+1}} f \,\mathrm{d}v_z \tag{1.26}$$

provided (1.24) is valid. We shall give another formula for $Tf(z)$ which will be useful in the sequel.

1.9 Lemma. *Suppose $D \subset R^{m+1}$ has properties (A), $z \in R^{m+1}$, $v(z) < \infty$. For any $(\gamma, \theta^*) \in Z = (0, \infty) \times \Gamma^*$ we define the function $s_z(\cdot, \gamma, \theta^*)$ of the variable $\varrho > 0$ by*

$$s_z(\varrho, \gamma, \theta^*) = \sigma\,, \quad \sigma \in \{-1, 1\}$$

provided there is $\delta > 0$ such that for H_1-almost every $u \in (0, \delta)$

$$S_z(\varrho + \sigma u, \gamma, \theta^*) \in R^{m+1} - D\,, \quad S_z(\varrho - \sigma u, \gamma, \theta^*) \in D$$

hold (for the notation see (1.12)). In all other cases we put

$$s_z(\varrho, \gamma, \theta^*) = 0\,.$$

Given $f \in \mathscr{B}$ we introduce the function Σ_f by

$$\Sigma_f(z, \gamma, \theta^*) = \sum_{\varrho > 0} S_z f(\varrho, \gamma, \theta^*)\, s_z(\varrho, \gamma, \theta^*) \tag{1.27}$$

provided $n(z, \gamma, \theta^*) < \infty$ (it is easily seen that we deal with all $\varrho > 0$ related to hits of the ray on D). If $n(z, \gamma, \theta^*) = \infty$ we put $\Sigma_f(z, \gamma, \theta^*) = 0$.

Then we have for any $f \in \mathscr{B}$

$$Tf(z) = 2^{m-1} \int_Z e^{-\gamma} \gamma^{m/2-1}\, \Sigma_f(z, \gamma, \theta^*)\, \mathrm{d}H_m((\gamma, \theta^*))\,. \tag{1.28}$$

Proof. Suppose (1.24). If $f \in \mathscr{D}(z)$ we obtain from (1.14)

$$Tf(z) = 2^{m-1} \int_Z e^{-\gamma} \gamma^{m/2-1}\, \mathrm{d}H_m((\gamma, \theta^*)) \int_{\tilde{D}(\gamma, \theta^*)} \partial_\varrho S_z f(\varrho, \gamma, \theta^*)\, \mathrm{d}\varrho\,.$$

Note that for H_m-almost every (γ, θ^*) we have $n(z, \gamma, \theta^*) < \infty$ which implies for those (γ, θ^*)

$$\int_{\tilde{D}(\gamma, \theta^*)} \partial_\varrho S_z f(\varrho, \gamma, \theta^*)\, \mathrm{d}\varrho = \Sigma_f(z, \gamma, \theta^*)\,.$$

The rest is obvious and (1.28) for $f \in \mathscr{D}(z)$ is proved.

Let $\{f_k\}$ be a (pointwise) convergent sequence of functions on B such that (1.28) holds for all f_k and f_k, $k = 1, 2, \ldots$ are uniformly bounded by $K > 0$. Than we have

$$|\Sigma_{f_k}(z, \gamma, \theta^*)| \leqq K\,.\, n(z, \gamma, \theta^*)$$

H_m-almost everywhere on Z. By the Lebesgue dominated convergence theorem, (1.28) holds for $f = \lim_{k \to \infty} f_k$ as well. We obtained that (1.28) is valid for every bounded Baire function f vanishing at z. As a consequence of 1.7 we get that (1.28) holds for every $f \in \mathscr{B}$.

1.10 Remark. We denote by K_z or L_z the set of all $(\gamma, \theta^*) \in Z$ for which there is an $\varepsilon = \varepsilon(\gamma, \theta^*) > 0$ such that

$$H_1(\{S_z(\varrho, \gamma, \theta^*);\ 0 < \varrho < \varepsilon\} - D) = 0$$

or

$$H_1(\{S_z(\varrho, \gamma, \theta^*);\ 0 < \varrho < \varepsilon\} \cap D) = 0$$

respectively. We supposed $v(z) < \infty$ at $z = [x, t]$; hence we obtain easily

$$H_m(Z - (K_z \cup L_z)) = 0\,.$$

Considering $f \equiv 1$ in (1.28) we obtain that Σ_f is H_m-almost everywhere equal to the characteristic function of one the sets K_z, L_z (may be up to the sign). Especially, both of them are measurable sets which shall be used later.

2.

Given a function $f \in \mathscr{C}$ we would like to study $Tf(z)$ as the function of z. For this purpose it is useful to use the notion of perimeter of a set (compare with Král's results from [3]).

We shall start with the notation. Let $M \subset R^{m+1}$ be a measurable set. We define its perimeter $P(M)$ by

$$(2.1) \qquad P(M) = \sup_{\omega} \int_M \operatorname{div} \omega(w) \, \mathrm{d}w$$

where $\omega = [\omega_1, \ldots, \omega_{m+1}]$ ranges over all vector valued functions with $(m + 1)$ components $\omega_i \in \mathscr{D}$ satisfying

$$\sum_{j=1}^{m+1} \omega_j^2(w) \leqq 1$$

for all $w \in R^{m+1}$. Similarly we define $P_i(M)$, $i = 1, 2, \ldots, m + 1$ by

$$(2.2) \qquad P_i(M) = \sup_{\varphi} \int_M \partial_i \varphi(w) \, \mathrm{d}w$$

where φ ranges over all $\varphi \in \mathscr{D}$, $\|\varphi\| \leqq 1$. It is easily seen that

$$(2.3) \qquad \varphi_i(M) \leqq P(M) \leqq \sum_{j=1}^{m+1} P_j(M)$$

holds for all $i = 1, 2, \ldots, m + 1$.

2.1 Proposition. *Let D be a set with properties (A) (see Part 1) and let for all τ, $0 < \tau < \infty$*

$$(2.4) \qquad P(D_\tau) < \infty$$

hold. Then for all $z \in R^{m+1} - B$ and all $r > 0$

$$(2.5) \qquad v_r(z) < \infty$$

is valid.

Proof. It is sufficient to show that $v(z)$ is finite. For a fixed $\varphi \in \mathscr{D}(z)$ we obtain for $z = [x, t]$ easily from (1.7)

$$(2.6) \quad T\varphi(z) = - \sum_{j=1}^{m} \int_{D_t} (\partial_j \mathscr{G}(z - w) \, \partial_j \varphi(w) + \partial_j^2 \mathscr{G}(z - w) \, \varphi(w)) \, \mathrm{d}w -$$

$$- \int_{D_t} (\partial_{m+1} \mathscr{G}(z - w) \, . \, \varphi(w) + \mathscr{G}(z - w) \, \partial_{m+1} \varphi(w)) \, \mathrm{d}w =$$

$$= - \sum_{j=1}^{m} \int_{D_t} \partial_j (\varphi(w) \, \partial_j \mathscr{G}(z - w)) \, \mathrm{d}w - \int_{D_t} \partial_{m+1} (\varphi(w) \, . \, \mathscr{G}(z - w)) \, \mathrm{d}w \, .$$

Supposing that $z \in R^{m+1} - B$ is fixed we can find $\varrho > 0$ for which dist $(z, B) > \varrho$. Since the value of $T\varphi(z)$ depends only on the behavior in a certain neighborhood of B we can suppose $\varphi(\Omega(z, \varrho)) = \{0\}$. For $\mathscr{G}(z - w)$ as a function of w on $R^{m+1} - \Omega(z, \varrho)$ we have the following estimates:

$$
(2.7) \qquad |\mathscr{G}(z - w)| \leqq \varrho^{-m/2},
$$

$$
|\partial_j \mathscr{G}(z - w)| \leqq \varrho^{-m/2} \cdot \tfrac{1}{2} \max_{t \in \langle 0, \infty)} \{t \exp(-\varrho t)\} = \frac{1}{2e} \varrho^{-m/2-1}.
$$

From (1.22) and (2.6) we obtain immediately

$$
\begin{aligned}
v(z) &= 2^{m-1} \sup \{T\varphi(z);\ \varphi \in \mathscr{D}^1(z)\} \leqq \\
&\leqq 2^{m-1} \Bigg[\sum_{j=1}^{m} \sup_{\varphi} \int_{D_t} \partial_j(-\varphi(w) \cdot \partial_j \mathscr{G}(z - w)) \, dw + \\
&\qquad + \sup_{\varphi} \int_{D_t} \partial_j(-\varphi(w) \cdot \mathscr{G}(z - w)) \, dw \Bigg].
\end{aligned}
$$

As mentioned above we can restrict our consideration to those φ which vanish on $\Omega(z, \varrho)$. From the definition of the perimeter and from (2.7) we get

$$
v(z) \leqq 2^{m-1} \left[\frac{1}{2e} \varrho^{-m/2-1} \sum_{i=1}^{m} P_i(D_t) + \varrho^{-m/2} P_{m+1}(D_t) \right] \leqq K(\varrho) P(D_t)
$$

and so the proposition is proved.

2.2 Remark. The condition (A) together with

$$
(2.8) \qquad P(D_\tau) < \infty, \quad 0 < \tau < \infty
$$

yield the finiteness of $v(z)$ on $R^{m+1} - B$ and so we can define $Tf(z)$ as a function of z for any function $f \in \mathscr{B}$. In the next part we shall study some properties of functions Tf (with fixed f) defined on D; for this purpose the condition (2.8) is "quite natural" as we shall show in the following

2.3 Proposition. *Let $D \subset R^{m+1}$ fulfil (A) and suppose that there exist points $z^1, \ldots, z^{m+2}$ in general position (i.e., not situated in a single hyperplane) such that*

$$
(2.9) \qquad \sum_{j=1}^{m+2} v(z^j) < \infty .
$$

For $z^i = [x^i, t^i]$, $i = 1, 2, \ldots, m + 2$ we denote

$$
t_0 = \min \{t^i;\ i = 1, 2, \ldots, m + 2\} .
$$

Then

$$P(D_\tau) < \infty \tag{2.10}$$

for all τ, $0 < \tau \leqq t_0$.

Proof. To prove the proposition for the nontrivial case we can suppose $t_0 > 0$, $D_{t_0} \neq \emptyset$. Let us define the vector valued function $\boldsymbol{G}$ (which is "nearly" equal to $\nabla_w \mathscr{G}(z - w)$ by the formula

$$\boldsymbol{G}(z - w) = (\hat{\nabla}_w \mathscr{G}(z - w), \mathscr{G}(z - w)) = \tag{2.11}$$

$$= \mathscr{G}(z - w) \left[\frac{z_1 - w_1}{2(z_{m+1} - w_{m+1})}, \ldots, \frac{z_m - w_m}{2(z_{m+1} - w_{m+1})}, \frac{z_{m+1} - w_{m+1}}{z_{m+1} - w_{m+1}} \right].$$

Thus $\boldsymbol{G}$ is a vector valued function of the variables z, w and for a fixed z all its components are infinitely differentiable functions on $R^{m+1} - \{z\}$. For a $z = [x, t] \in$ $\in R^{m+1}$ and any $\varphi \in \mathscr{D}(z)$ we can rewrite (1.7) in the form

$$T\varphi(z) = -\int_{D_t} \nabla\varphi(w) \circ \boldsymbol{G}(z - w) \,\mathrm{d}w\,. \tag{2.12}$$

We remember (1.22) which shows the relation between (2.12) and the parabolic variation. To prove (2.10) we shall show that $P(D_{t_0}) < \infty$ or $P_j(D_{t_0}) < \infty$ for all $j = 1, \ldots, m + 2$ (see (2.3)). We shall prove a little bit more, viz., that for any $\theta \in \Gamma$

$$\sup \left\{ \int_{D_{t_0}} \theta \circ \nabla\varphi(w) \,\mathrm{d}w;\ \varphi \in \mathscr{D},\ \|\varphi\| \leqq 1 \right\} < \infty \tag{2.13}$$

holds. We shall do it similarly as in the proof of Proposition 2.10 in [3]. Let Π_j, $j = 1, 2, \ldots, m + 2$ stand for the hyperplane determined by z^k, $k = 1, 2, \ldots, m + 2$, $k \neq j$. Then

$$\bigcup_{j=1}^{m+2} (R^{m+1} - \Pi_j) = R^{m+1}$$

and so there are functions α_j, $j = 1, 2, \ldots, m + 2$ such that

$$\alpha_j \in \mathscr{D}\,, \quad \Pi_j \cap \operatorname{spt} \alpha_j = \emptyset\,, \quad 0 \leqq \alpha_j \leqq 1\,,$$

$$\alpha = \sum_{j=1}^{m+2} \alpha_j = 1$$

in a neighborhood of B_{t_0}. Now we can write α as a coefficient in the integrand in (2.13) and thus it is sufficient to prove

$$\sup \left\{ \int_{D_{t_0}} \alpha_j(w)\, \theta \circ \nabla\varphi(w) \,\mathrm{d}w;\ \varphi \in \mathscr{D},\ \|\varphi\| \leqq 1 \right\} < \infty \tag{2.14}$$

for all $j = 1, \ldots, m + 2$. Let us fix a j, for example $j = 1$. For any $w \in \operatorname{spt} \alpha_1$, the vectors $z^2 - w, \ldots, z^{m+2} - w$ are linearly independent. Components of $\mathbf{G}(z^k - w)$ are infinitely differentiable functions in a neighborhood of $\operatorname{spt} \alpha_1$ and we can express θ in the form

$$\theta = \sum_{k=2}^{m+2} a_k(w) \,.\, \mathbf{G}(z^k - w)$$

with infinitely differentiable coefficients a_k, $k = 2, \ldots, m + 2$ in a neighbourhood of $\operatorname{spt} \alpha_1$. Extending a_k arbitrarily to R^{m+1} we get the integral in (2.14) by replacing θ by a sum of terms

$$\int_{D_{t_0}} \alpha_1(w)\, a_k(w)\, \nabla\varphi(w) \circ \mathbf{G}(z^k - w)\, \mathrm{d}w \,.$$

Instead of proving (2.14) for $j = 1$ we shall prove

$$\text{(2.15)} \quad \sup \left\{ \int_{D_{t_0}} \alpha_1(w)\, a_k(w)\, \nabla\varphi(w) \circ \mathbf{G}(z^k - w)\, \mathrm{d}w;\ \varphi \in \mathscr{D},\ \|\varphi\| \leqq 1 \right\} < \infty$$

for any $k = 2, \ldots, m + 2$. Let us fix k and put $F(w) = \alpha_1(w)\, a_k(w)$. Then $F \in \mathscr{D}(z^k)$ and so it is a bounded function. We have

$$\int_{D_{t_0}} F(w)\, \nabla\varphi(w) \circ \mathbf{G}(z^k - w)\, \mathrm{d}w =$$

$$= \int_{D_{t_0}} \nabla[F(w)\, \varphi(w)] \circ \mathbf{G}(z^k - w)\, \mathrm{d}w - \int_{D_{t_0}} \varphi(w)\, \nabla F(w) \circ \mathbf{G}(z^k - w)\, \mathrm{d}w \,.$$

The first integral on the right-hand side can be estimated by K $v(z^k)$ with a suitable $K > 0$ while the other is finite and uniformly bounded on $\mathscr{D}^1(z^k)$. The proposition is proved.

2.4 Remark. Denoting by

$$M = \{z \in R^{m+1};\ v(z) < \infty\}$$

we obtain for any $\tau \in R^1$

$$\overline{R_\tau^{m+1} - M} = \overline{R_\tau^{m+1}} \quad \text{or} \quad R_\tau^{m+1} - B_\tau \subset M \,.$$

The latter case takes place if and only if $P(D_\tau) < \infty$. In the rest of the paper we suppose that D has properties (A) and the property (B), i.e.,

$$\text{(2.16)} \qquad P(D_\tau) < \infty$$

for all $\tau \in R^1$.

3.

In what follows we shall study some properties of $Tf(z)$ for fixed f and $z \in R^{m+1} - B$. Thus Tf can be considered for any $f \in \mathscr{B}$ to be a function and we pointed out in Part 2 of the paper that for this purpose it is natural to suppose (A), (B) for D.

In this part we shall change a little the notation. We denote by $n(w)$ the exterior normal to D at a point w in the sense of Federer, i.e., $n(w) = \theta \in \Gamma$ provided the symmetric difference of D and the half-space

$$\{z \in R^{m+1};\ (z - w) \circ \theta < 0\}$$

has $(m + 1)$-dimensional density 0 at w; in all other cases we put as usual $n(w) = 0$. The set of all $w \in R^{m+1}$ with $n(w) \neq 0$ is called the reduced boundary of D and will be denoted by $\hat{B}$. We shall write $\hat{B}_t$ for $B_t \cap \hat{B}$. If the classical normal exists it is equal to Federer's one. We have

$$H_m(\hat{B}_t) < \infty \tag{3.1}$$

for any t (compare with [3] and also with other papers mentioned there). The generalized Gauss theorem helps us to derive another expression for $Tf(z)$. Let us denote

$$n(w) = [\hat{n}(w), n_\tau(w)]\,. \tag{3.2}$$

We get an analog of (1.8), i.e.

$$Tf(z) = -\int_{B_t} f(w)\, \hat{n}(w) \circ \hat{\nabla}_w \mathscr{G}(z - w)\, \mathrm{d}H_m(w) - \\ - \int_{B_t} f(w)\, n_\tau(w) \,.\, \mathscr{G}(z - w)\, \mathrm{d}H_m(w) \tag{3.2}$$

for any $f \in \mathscr{B}$ and $z = [x, t] \in R^{m+1} - B$. By means of the vector-valued function $\mathbf{G}(z - w)$ we can express $Tf(z)$ similarly in the form

$$Tf(z) = \int_{B_t} f(w) \,.\, n(w) \circ \mathbf{G}(z - w)\, \mathrm{d}H_m(w)\,. \tag{3.3}$$

3.1 Lemma. *For every* $z \in [x, t] \in R^{m+1}$

$$v(z) = 2^{1-m} \int_{B_t} |n(w) \circ \mathbf{G}(z - w)|\, \mathrm{d}H_m(w)\,. \tag{3.4}$$

If $v(z) < \infty$ *and* $A \subset B$ *is a Borel set then*

$$v_z(A) = \int_{B_t \cap A} n(w) \circ \mathbf{G}(z - w)\, \mathrm{d}H_m(w)\,. \tag{3.5}$$

Proof is similar to that of Lemma 2.12 from [3]. For fixed z we put $l(z) = 0 \ (\in R^{m+1})$,

$$l(w) = \varphi(w) \circ \mathbf{G}(z - w), \quad z \neq w$$

with $\varphi \in \mathscr{D}(z)$. Applying the Gauss theorem to D_t we obtain after some calculation

$$T\varphi(z) = -\int_{B_t} \varphi(w)\, n(w) \circ \mathbf{G}(z - w)\, \mathrm{d}H_m(w).$$

To prove the rest we use first (1.22) to obtain (3.4). For z with $v(z) < \infty$ we continue similarly as in the proof of Lemma 1.9. Applying the obtained formula for any $f \in \mathscr{B}$ to the characteristic function of A we get (3.5).

Supposing (A), (B) we shall seek for the necessary and sufficient condition for the extensibility of Tf from D to $D \cup B$ with any $f \in \mathscr{C}$. Such a function is caloric on D and so our question is obviously connected with the Dirichlet boundary value problem. We shall derive some other properties of the parabolic variation v.

3.2 Lemma. *Parabolic variation of D fulfilling (A), (B) is a lower semicontinuous function of z on R^{m+1} which is finite for any $z \in R^{m+1} - B$.*

Proof. Let us fix $z \in R^{m+1}$ and choose $K < v(z)$. Then there is a $\varphi \in \mathscr{D}^1(z)$ (see Proposition 1.5) such that $T\varphi(z) > K$. Then we have

$$\liminf_{w \to z} v(w) \geqq \lim_{w \to z} T\varphi(w) = T\varphi(z) > K.$$

The rest follows from Proposition 2.1.

3.3 Remark. The following simple geometrical fact will be used in the proof of the next lemma. Given $z \in R^{m+1}$ and a closed interval $I \subset R^{m+1}$ the intersection of $S \equiv S_z(\cdot, \gamma, \theta^*)$ on $(0, \infty)$ is either empty or it is described by $S_z(\cdot, \gamma, \theta^*)$ on some one-dimensional interval.

3.4 Lemma. *Let D fulfil (A), (B) and $v(z) < \infty$ for a fixed $z = [x, t]$. Then for any closed interval $I \subset R^{m+1}$,*

$$|v_z(B_t \cap I)| \leqq 2^{m-1}\, \Gamma\left(\frac{m}{2}\right) H_{m-1}(\Gamma^*). \tag{3.6}$$

Proof. We denote by f the characteristic function of $B_t \cap I$. Let us consider (1.27) for f. We obtain on the right-hand side the sum

$$\sum_{\varrho} s_z(\varrho, \gamma, \theta^*)$$

where it is sufficient to consider ϱ from a certain interval. It is easily seen that for the corresponding expression Σ_f we have

$$|\Sigma_f(z, \gamma, \theta^*)| \leqq 1 \tag{3.7}$$

for all $(\gamma, \theta^*) \in Z$. The rest is a consequence of (1.28).

3.5 Proposition. *Let us denote for fixed $t \in R^1$*

$$V_t = \sup \{v(w);\ w \in B_t\}\,. \tag{3.8}$$

Then for any $w = [y, u]$ with $u \leqq t$ we have

$$v(w) \leqq V_t + 2^{m-1}\,\Gamma\left(\frac{m}{2}\right) H_{m-1}(\Gamma^*) = V_t + A\,. \tag{3.9}$$

Proof. We suppose $V_t < \infty$ for given $t \in R^1$, and fix $w = [y, u]$ with $u \leqq t$, $w \notin B_t$. Then for an arbitrarily fixed $d < v(w)$ there exists a system of mutually disjoint closed parallelepipeds $K_1, \ldots, K_q$ such that

$$\sum_{j=1}^{q} |v_w(B_t \cap K_j)| > d\,.$$

Let us denote $\sigma_j \equiv \operatorname{sign} v_w(B_t \cap K_j)$, $j = 1, \ldots, q$ and define the function

$$h(z) = \sum_{j=1}^{q} \sigma_j v_z(B_t \cap K_j)$$

which is caloric on

$$R^{m+1} - \bigcup_{j=1}^{q} B_t \cap K_j \supset R^{m+1} - B_t\,.$$

Fix an arbitrary $\zeta \in B_t$. If $\zeta \notin \bigcup_{j=1}^{q} K_j$, then

$$\lim_{z \to \zeta} h(z) = h(\zeta) \leqq \|v_\zeta\| \leqq V_t\,.$$

In the opposite case we can suppose for example $\zeta \in K_1$. Then we have again

$$\lim_{z \to \zeta} \sum_{j=2}^{q} \sigma_j v_z(B_t \cap K_j) = \sum_{j=2}^{q} \sigma_j v_\zeta(B_t \cap K_j) \leqq \|v_\zeta\| \leqq V_t\,.$$

From Lemma 3.4 we have for any $z \in R^{m+1}$

$$|v_z(B_t \cap K_1)| \leqq A$$

and so we obtain

$$\limsup_{z \to \zeta\ z \notin B_t} h(z) \leqq V_t + A$$

for $\zeta \in B_t$. According to (A) B_t is a compact set. Using (2.7) we conclude that the function $|\mathbf{G}(z - w)|$ of the variable z can be estimated outside $\Omega(0, \varrho)$ with sufficiently large $\varrho > 0$ uniformly with respect to $w \in B_t$. This estimate converges to 0 for $\varrho \to \infty$. Condition (B) implies that $H_m(\hat{B}_t) < \infty$ so that we obtain easily (see (3.4)) that $h(z) \to 0$ for $|z| \to \infty$. Applying the maximum principle for caloric functions we get

$$h \leqq V_t + A$$

on $R^{m+1} - B$ and, in particular, $d < h(w) \leqq V_t + A$. The proposition is proved.

3.6 Remark. We shall not prove here the other facts concerning properties of the parabolic variation v. For example by a straightforward but a little bit complicated computation we could obtain that v is a locally Lipschitzian function on $R^{m+1} - B$. We shall not use this fact in the sequel.

3.7 Proposition. *Suppose again* (A), (B) *for* D. *Let* $L \subset D$ *be such a bounded set that* Tf *is bounded on it for every* $f \in \mathscr{C}$. *Then* v *is bounded on* $\bar{L}$.

Proof. We shall consider the system of linear functionals $Tf(z) = \langle f, v_z \rangle$, $z \in L$ on $\mathscr{C}$. We can consider those functionals on a subspace of all $f \in \mathscr{C}$ with the support $\operatorname{spt} f$ contained in a B_t with a suitably fixed $t > 0$. Applying the Banach-Steinhaus theorem we obtain that those functionals are uniformly bounded on L. Lower semicontinuity of the function v proved in Lemma 3.2 implies that v is a bounded function on $\bar{L}$.

We can apply the proposition to a special situation obtaining easily

3.8 Proposition. *Let* D *be as above and* $\zeta = [\xi, \eta] \in B$. *If for every* $f \in \mathscr{C}$ *there exists finite limit*

$$\lim_{z \to \zeta, z \in D} Tf(z) \tag{3.10}$$

then there is $\varrho > 0$ *such that*

$$\sup \{v(z);\ z \in B \cap \Omega(\zeta, \varrho)\} < \infty \tag{3.11}$$

holds.

3.9 Remark. We shall not prove that the condition from the preceding proposition is not only necessary but also sufficient for the existence of the limit (3.10) for every $f \in \mathscr{C}$. We are interested in the "global condition" only.

3.10 Theorem. *Let D fulfil* (A), (B). *Then condition* (C), *i.e.*,

$$\sup \{v(\zeta);\ \zeta \in B_t\} < \infty \tag{3.12}$$

for all $t \in R^1$ is satisfied if and only if Tf can be continuously extended from D to $D \cup B$ for any $f \in \mathscr{C}$.

Proof. The necessity of (C) results from Proposition 3.8 and (A) guaranteeing the compactness of B_t for any $t \in R^{m+1}$. To complete the proof we shall show that for arbitrarily chosen $\zeta \in B$ and $f \in \mathscr{C}$ there exists finite limit (3.10). Let us fix $\zeta = [\xi, \eta] \in B$ and $f \in \mathscr{C}$.

For a moment let us suppose $f \equiv k$ (const.). Then we obtain from (1.28) easily that Tf is also a constant function on D and on $R^{m+1} - \bar{D}$ (its value will be determined later). Now we shall pass to the general case. The linearity of T permits us to suppose $f(\zeta) = 0$. We fix $t > \eta$ and define V_t by (3.8). According to Proposition 3.5 we have

$$v(z) \leqq V_t + A < \infty \tag{3.13}$$

for any $z \in R_t^{m+1}$ (see the proposition for the notation). Now we shall construct the decompositions of f in the form $f = f_n + g_n$ where f_n vanish in a fixed neighborhood of ζ and $\|g_n\| \leqq 1/n$. We have $Tf = Tf_n + Tg_n$, where all Tf_n are continuous functions at ζ and $|Tg_n| \leqq (V_t + A) \,.\, 1/n$. This implies easily the continuity of Tf at $\zeta \in B$ with respect to D.

3.11 Remark. Now we shall formally simplify our assumptions. Let D_1 be an open bounded set in R^{m+1} fulfilling (A), (B), (C), or – which is the same – the condition.

$$\sup \{v(\zeta);\ \zeta \in B\} = V < \infty\,. \tag{3.14}$$

We denote $D_2 = R^{m+1} - \bar{D}_1$. Following again the whole argument one can find that this assumption is quite natural (minimal in a sense) provided we want to use the results obtained to solve the classical Dirichlet boundary value problem for the heat equation for D_1 by means of generalized potentials Tf, $f \in \mathscr{C}$. As a consequence of (3.14), the function Tf is well-defined in the whole space R^{m+1} for any f continuous on $B = \partial D_1$ (or $f \in \mathscr{B}$). The function v is finite in R^{m+1} and we can calculate the value of Tf for $f \equiv 1$ at any $z \in R^{m+1}$ by means of (1.28). We shall denote Tf for $f \equiv 1$ by d. We obtain easily

$$\begin{aligned} d(w) &= 2^{m-1}\, \Gamma\left(\frac{m}{2}\right) H_{m+1}(\Gamma^*) \quad &&\text{for} \quad w \in D_1\,, \\ d(w) &= 0 &&\text{for} \quad w \in D_2\,; \end{aligned} \tag{3.15}$$

we shall write $d_1 = d(w)$, $w \in D_1$. Using the notation and some facts from 1.9 and 1.10 we obtain for $w \in B$

$$d(w) = 2^{m-1} \int_{K_w} e^{-\gamma} \gamma^{m/2-1} \, \mathrm{d}H_m(\gamma, \theta^*) . \tag{3.16}$$

In the notation just introduced we can reformulate the Dirichlet boundary value problem for the heat equation for D_1 and continuous "boundary condition" g provided (3.14) holds in the following form: on the space $\mathscr{C}$, solve the equation

$$\lim_{z \to \zeta, z \in D_1} Tf(z) = Tf(\zeta) + (d_1 - d(\zeta)) f(\zeta) = g(\zeta) \tag{3.17}$$

with respect to the unknown function f. This requires the other information on this equation.

References

[1] *M. Dont:* On a Heat Potential, Czech. Mat. J. 1975, 84–109.
[2] *M. Dont:* On a Boundary Value Problem for the Heat Equation, Czech. Mat. J. 1975, 110–133.
[3] *J. Král:* The Fredholm Method in Potential Theory, TAMS 1966, 511–547.
[4] *J. Král:* Flows of Heat and the Fourier Problem, Czech. Mat. J. 1970, 556–598.
[6] *J. Veselý:* On the Heat Potential of the Double Distribution, Čas. pro pěst. mat. 1973, 181–198.

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