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MULTIFUNCTIONS WITH CONVEX CLOSED GRAPH

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In this paper we shall establish some properties of multifunctions with convex closed graph which are in connection with "open mapping theorem" and "closed graph theorem".

Let X, Y be real linear topological spaces, and let $\mathscr{U}$, $\mathscr{V}$ be the families of all neighbourhoods of the origins in X, Y, respectively.

Let C be a non-empty subset of X, and let $F : C \to Y$ be a multifunction with non-empty values.

Throughout the paper: core, lin, int, cl denote algebraic interior, algebraic closure, topological interior, topological closure (see [4]); gr denotes graph.

The main result of the paper is the following

Theorem (see [5]). *Let X be a locally convex, complete, semi-metrizable space, and let Y be a barelled space. Let* gr F *be a convex, closed set, and let* core $F(C) \neq \emptyset$. *Then*

$$F(x) \cap \operatorname{core} F(C) \subseteq \operatorname{int} F(C \cap (x + U)),$$

$$F(x) \subseteq \operatorname{lin} \operatorname{int} F(C \cap (x + U))$$

for all $x \in C$ and $U \in \mathscr{U}$.

Before to make good the theorem, let us prove some lemmas.

Lemma 1. *The set* gr F *is convex if and only if the set C is convex and $t_1 F(x_1) + t_2 F(x_2) \subseteq F(t_1x_1 + t_2x_2)$ for all $x_1 \in C$, $x_2 \in C$, $t_1 \geqq 0$, $t_2 \geqq 0$, and $t_1 + t_2 = 1$.*

Proof. The demonstration is not difficult.

Lemma 2. *Let X be a locally convex space, and let Y be a barrelled space. Let* core $F(C) \neq \emptyset$. *Then*

$$F(x) \cap \operatorname{core} F(C) \subseteq \operatorname{int} \operatorname{cl} F(C \cap (x + U))$$

for all $x \in C$ and $U \in \mathscr{U}$.

Proof. Let $U \in \mathscr{U}$. There exists $\tilde{U} \in \mathscr{U}$ convex such that $\tilde{U} \subseteq U$. Let $x \in C$. Denote $C_n = C \cap (x + n\tilde{U})$. Then $C = \bigcup_{n=1}^{\infty} C_n$ and $((n-1)/n)\, x + (1/n)\, C_n \subseteq C_1$, hence $F(C) = \bigcup_{n=1}^{\infty} F(C_n)$ and $((n-1)/n)\, F(x) + (1/n)\, F(C_n) \subseteq F(C_1)$.

Let $y \in F(x) \cap \operatorname{core} F(C)$. Then $F(C_n) - y \subseteq n(F(C_1) - y)$ and $0 \in \operatorname{core}(F(C) - y)$. But $F(C) - y = \bigcup_{n=1}^{\infty} (F(C_n) - y) \subseteq \bigcup_{n=1}^{\infty} n(F(C_1) - y)$, hence $\bigcup_{n=1}^{\infty} n(F(C_1) - y) = Y$ and $V = \operatorname{cl}(F(C_1) - y) \in \mathscr{V}$ (recall that $F(C_1)$ is a convex set) (see [2], p. 3). Consequently $y + V = \operatorname{cl} F(C_1)$ and $y \in \operatorname{int} \operatorname{cl} F(C_1) \subseteq \operatorname{int} \operatorname{cl} F(C \cap (x + U))$.

Lemma 3. (see [3], p. 202). *Let X be a complete, semi-metrizable space. Let* gr F *be a convex, closed set. Let $x \in C$, let $y \in Y$, and suppose that*

$$y \in \operatorname{int} \operatorname{cl} F(C \cap (x + U))$$

for all $U \in \mathscr{U}$. Then

$$y \in \operatorname{int} F(C \cap (x + U))$$

for all $U \in \mathscr{U}$.

Proof. First, let us prove that

$$t \operatorname{cl} F(C \cap (x + U)) + (1 - t)\, y \subseteq F(C \cap (x + t(U + \tilde{U})))$$

for all $t \in (0, 1)$, $U \in \mathscr{U}$, and $\tilde{U} \in \mathscr{U}$.

Let $t \in (0, 1)$. Denote $t_n = t^{(1/2^n)}$. Then $\lim_{n \to \infty} t_n \dots t_1 = t$.

Let $\tilde{U} \in \mathscr{U}$. There exists a fundamental sequence $U_n \in \mathscr{U}$ of closed neighbourhoods such that $U_1 + U_1 \subseteq \tilde{U}$ and $U_{n+1} + U_{n+1} \subseteq U_n$ (see [1], p. 23). Denote $\tilde{U}_n = (t_n \dots t_1/(1 - t_n))\, U_n$. There exists $\tilde{V}_n \in \mathscr{V}$ such that $y + \tilde{V}_n \subseteq \operatorname{cl} F(C \cap (x + \tilde{U}_n))$. Denote $V_n = ((1 - t_n)/t_n)\, \tilde{V}_n$.

Let $U \in \mathscr{U}$, let $\tilde{y} \in \operatorname{cl} F(C \cap (x + U))$, and let us show that

$$t\tilde{y} + (1 - t)\, y \in F(C \cap (x + t(U + \tilde{U})))\,.$$

We shall construct, step by step, a sequence $u \in U$, $u_1 \in U_1, \dots, u_n \in U_n, \dots$ with the following property:

$$t_n \dots t_1 \tilde{y} + (1 - t_n \dots t_1)\, y \in \operatorname{cl} F(C \cap (x + t_n \dots t_1(u + \dots + u_{n-1} + U_n)))\,.$$

Since $(\tilde{y} - V_1) \cap F(C \cap (x + U)) \neq \emptyset$, there exist $v_1 \in V_1$ and $u \in U$ such that $x + u \in C$ and $\tilde{y} - v_1 \in F(x + u)$, hence $t_1\tilde{y} + (1 - t_1)\, y = t_1(\tilde{y} - v_1) + (1 - t_1) \,.\, (y + (t_1/(1 - t_1))\, v_1) \in t_1 F(x + u) + (1 - t_1)\,(y + (t_1/(1 - t_1))\, V_1)$. But $y + (t_1/(1 - t_1))\, V_1 = y + \tilde{V}_1 \subseteq \operatorname{cl} F(C \cap (x + \tilde{U}_1)) = \operatorname{cl} F(C \cap (x + (t_1/(1 - t_1)) \,.\, U_1))$ which means that $t_1\tilde{y} + (1 - t_1)\, y \in \operatorname{cl} F(C \cap (x + t_1(u + U_1)))$.

Suppose that we have $u \in U, \ldots, u_{n-1} \in U_{n-1}$ with the desired property. Since

$$(t_n \ldots t_1 \tilde{y} + (1 - t_n \ldots t_1) y - V_{n+1}) \cap$$
$$\cap F(C \cap (x + t_n \ldots t_1(u + \ldots + u_{n-1} + U_n))) \neq \emptyset,$$

there exist $v_{n+1} \in V_{n+1}$ and $u_n \in U_n$ such that

$$x + t_n \ldots t_1(u + \ldots + u_n) \in C$$

and

$$t_n \ldots t_1 \tilde{y} + (1 - t_n \ldots t_1) y - v_{n+1} \in F(x + t_n \ldots t_1(u + \ldots + u_n)),$$

hence $t_{n+1} \ldots t_1 \tilde{y} + (1 - t_{n+1} \ldots t_1) y = t_{n+1}(t_n \ldots t_1 \tilde{y} + (1 - t_n \ldots t_1) y - v_{n+1}) + (1 - t_{n+1})(y + (t_{n+1}/(1 - t_{n+1})) v_{n+1}) \in t_{n+1} F(x + t_n \ldots t_1(u + \ldots + u_n)) + (1 - t_{n+1})(y + (t_{n+1}/(1 - t_{n+1})) V_{n+1})$. But $y + (t_{n+1}/(1 - t_{n+1})) V_{n+1} = y + \tilde{V}_{n+1} \subseteq \operatorname{cl} F(C \cap (x + \tilde{U}_{n+1})) = \operatorname{cl} F(C \cap (x + (t_{n+1} \ldots t_1/(1 - t_{n+1})) U_{n+1}))$ which means that $t_{n+1} \ldots t_1 \tilde{y} + (1 - t_{n+1} \ldots t_1) y \in \operatorname{cl} F(C \cap (x + t_{n+1} \ldots t_1(u + \ldots + u_n + U_{n+1}))$ and the desired sequence is obtained.

Since $u_{p+1} + \ldots + u_q \in U_{p+1} + \ldots + U_q \subseteq U_p$, the series $\sum_{n=1}^{\infty} u_n$ is convergent. Denote by $\tilde{u}$ its limit. Since $u_1 + \ldots + u_n \in u_1 + U_1$, $\tilde{u} \in u_1 + U_1 \subseteq \tilde{U}$.

Let now $U' \in \mathscr{U}$, $V' \in \mathscr{V}$ be arbitrary open neighbourhoods. There exists n such that

$$x + t_n \ldots t_1(u + \ldots + u_{n-1} + U_n) \subseteq x + t(u + \tilde{u}) + U'$$

and

$$t_n \ldots t_1 \tilde{y} + (1 - t_n \ldots t_1) y \in t\tilde{y} + (1 - t) y + V'.$$

Since $t_n \ldots t_1 \tilde{y} + (1 - t_n \ldots t_1) y \in \operatorname{cl} F(C \cap (x + t(u + \tilde{u}) + U'))$, it follows that $(t\tilde{y} + (1 - t) y + V') \cap F(C \cap (x + t(u + \tilde{u}) + U')) \neq \emptyset$ hence there exist $u' \in U'$ and $v' \in V'$ such that $x + t(u + \tilde{u}) + u' \in C$ and $t\tilde{y} + (1 - t) y + v' \in F(x + t(u + \tilde{u}) + u'))$ that is $(x + t(u + \tilde{u}), t\tilde{y} + (1 - t) y) + (u', v') \in \operatorname{gr} F$. Consequently $(x + t(u + \tilde{u}), t\tilde{y} + (1 - t) y) \in \operatorname{gr} F$, i.e., $x + t(u + \tilde{u}) \in C$ and $t\tilde{y} + (1 - t) y \in F(x + t(u + \tilde{u})) \subseteq F(C \cap (x + t(U + \tilde{U})))$.

Finnaly, let us prove the lemma. Let $U \in \mathscr{U}$. There exists $\tilde{U} \in \mathscr{U}$ such that $\tilde{U} + \tilde{U} \subseteq 2U$. There exists $\tilde{V} \in \mathscr{V}$ such that $y + \tilde{V} \subseteq \operatorname{cl} F(C \cap (x + \tilde{U}))$. Denote $V = (\frac{1}{2}) \tilde{V}$. Then $y + V = (\frac{1}{2})(y + \tilde{V}) + (\frac{1}{2}) y \subseteq (\frac{1}{2}) \operatorname{cl} F(C \cap (x + \tilde{U})) + (\frac{1}{2}) y \subseteq F(C \cap (x + (\frac{1}{2})(\tilde{U} + \tilde{U}))) \subseteq F(C \cap (x + U))$ and $y \in \operatorname{int} F(C \cap (x + U))$.

Let now return to the theorem.

P r o o f of the theorem. Let $x \in C$ and $U \in \mathscr{U}$. The first inclusion follows by lemmas 2 and 3. Let us prove the second inclusion. Let $y \in F(x)$. Since $F(C) \subseteq \operatorname{lin\,core} F(C$ (recall that $F(C)$ is a convex set) there exist $\tilde{y} \in Y$ and $r_1 > 0$ such that $y + s\tilde{y} \in \operatorname{core} F(C)$ for all $s \in (0, r_1]$. Moreover there exists $\tilde{x} \in X$ such that $x + r_1 \tilde{x} \in C$

and $y + r_1\tilde{y} \in F(x + r_1\tilde{x})$. Then $y + s\tilde{y} = ((r_1 - s)/r_1)\, y + (s/r_1)(y + r_1\tilde{y}) \in$ $\in ((r_1 - s)/r_1)\, F(x) + (s/r_1)\, F(x + r_1\tilde{x}) \subseteq F(x + s\tilde{x})$ for all $s \in (0, r_1]$. Let $\tilde{U} \in \mathcal{U}$ such that $\tilde{U} + \tilde{U} \subseteq U$, let $r_2 > 0$ such that $s\tilde{x} \in \tilde{U}$ for all $s \in (0, r_2]$, and denote $r =$ $= \min(r_1, r_2)$. Let $s \in (0, r]$. Then $y + s\tilde{y} \in F(x + s\tilde{x}) \cap \operatorname{core} F(C) \subseteq \operatorname{int} F(C \cap$ $\cap (x + s\tilde{x} + \tilde{U})) \subseteq \operatorname{int} F(C \cap (x + U))$ and $y \in \operatorname{lin} \operatorname{int} F(C \cap (x + U))$.

Remark 1. The first inclusion of the theorem contains an "open mapping theorem" (i.e., the multifunction F is open at every $x \in C$ with $F(x) \subseteq \operatorname{core} F(C)$), and a "closed graph theorem" (i.e., the multifunction $y \in F(C) \to F^{-}(y) = \{x \in C;\ y \in F(x)\}$ is lower semi-continuous at every $y \in \operatorname{core} F(C)$).

Remark 2. The first inclusion of the theorem becomes uninteresting if, accidentely, $F(x) \cap \operatorname{core} F(C) = \emptyset$ (accidentaly, since, denoting $\tilde{C} = \{x \in C;\ F(x) \cap \operatorname{core} F(C) \neq$ $\neq \emptyset\}$, we have $C \subseteq \operatorname{lin} \tilde{C}$). The second inclusion of the theorem removes this trouble.

Remark 3. From the second inclusion of the theorem it follows $\operatorname{int} F(C \cap$ $\cap (x + U)) \neq \emptyset$ for all $x \in C$ and $U \in \mathcal{U}$, hence $\operatorname{int} F(C) \neq \emptyset$.

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