Ján Jakubík

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CARDINAL SUMS OF LINEARLY ORDERED GROUPS

JÁN JAKUBÍK, Košice

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A lattice ordered group will be said to have the property (s) if each l-subgroup of G is a cardinal sum of linearly ordered groups. MARTINEZ [5] proposed the problem whether each cardinal product of linearly ordered groups has the property (s).

In this note the following theorem will be proved:

Theorem. *Let $G = \Sigma A_i$ $(i \in I)$ be a cardinal sum of linearly ordered groups A_i.*

(i) *If at most one l-group A_i is non-archimedean, then G has the property* (s).

(ii) *If there exist two distinct elements $j, k \in I$ such that A_j and A_k are not archimedean, then there exists an infinite set $\{H_t\}$ of l-subgroups of G such that no H_t is a cardinal sum of linearly ordered groups; thus G has not the property* (s).

For the terminology, cf. BIRKHOFF [1], Chap. XIV, and FUCHS [2]. The group operation in a lattice ordered group will be written additively, though it is not assumed to be commutative. Let us recall the following notions.

Let $\{A_i\}$ $(i \in I)$ be a system of l-subgroups of a lattice ordered group G such that

(i) the group G is a direct sum of its subgroups A_i;

(ii) if $0 \leqq g \in G$, $g = a_{i(1)} + \ldots + a_{i(n)}$, $a_{i(j)} \in A_{i(j)}$ with $i(j) \neq i(k)$ for $j \neq k$ $(j, k = 1, \ldots, n)$, then $a_{i(j)} \geqq 0$ for $j = 1, \ldots, n$.

Under these assumptions G is said to be a cardinal sum of its l-subgroups A_i $(i \in I)$ and in such case we write

$$G = \Sigma A_i \ (i \in I)\,. \tag{1}$$

Each l-subgroup A_i will be called a cardinal summand of G. If $I = \{1, \ldots, n\}$, then we write also $G = A_1 \oplus \ldots \oplus A_n$.

Let (1) be valid and let $0 \neq g \in G$. Then there are uniquelly determined distinct elements $i(1), \ldots, i(n) \in I$ and uniquelly determined elements $a_1 \in A_{i(1)}, \ldots, a_n \in A_{i(n)}$ such that $a_j \neq 0$ for $j = 1, \ldots, n$ and

$$g = a_1 + \ldots + a_n\,. \tag{2}$$

The element a_j is said to be the projection of g into $A_{i(j)}$ and it will be denoted by $g(A_{i(j)})$; if $i \in I$ and $i \neq i(j)$ for $j = 1, \ldots, n$, then we put $g(A_i) = 0$.

Let G be a lattice ordered group. If A is a cardinal summand of G and $0 \leqq g \in G$, then $g(A)$ is the greatest element of the set $\{a \in A : 0 \leqq a \leqq g\}$. If A, B are cardinal summands of G with $A \subseteqq B$ and $g \in G$, then $g(A) = (g(B))(A)$.

Let $X \subseteqq G$. We put

$$X^\delta = \{y \in G : |y| \wedge |x| = 0 \text{ for each } x \in X\}\,.$$

The set X^δ is called a polar of G; each polar of G is a closed convex l-subgroup of G. (Cf. Šik [6].) If A is a cardinal summand of G, then

$$G = A \oplus A^\delta\,.$$

G is said to be archimedean if there does not exist any pair of strictly positive elements $x, y \in G$ such that $nx \leqq y$ for each positive integer n.

Let (1) be valid and suppose that each A_i is linearly ordered and $A_i \neq \{0\}$. For $g \in G$ we put $S(g) = \{i \in I : g(A_i) \neq 0\}$. Let H be an l-subgroup of G and let M be the set of all $0 < g \in H$ such that $S(g_1) = S(g)$ for each $0 < g_1 \leqq g$, $g_1 \in H$. Obviously $g_1, g \in H$, $0 < g_1 \leqq g$ implies $S(g_1) \subseteqq S(g)$. Thus for each $0 < g \in H$ there exists $g_1 \in H$ with $0 < g_1 \leqq g$ such that $g_1 \in M$.

For $a, b \in H$, $a \leqq b$ we denote by $[a, b]$ the set $\{h \in H : a \leqq h \leqq b\}$.

Lemma 1. *Let $g \in M$. Then $[0, g]$ is a chain.*

Proof. Suppose that $[0, g]$ fails to be a chain. Then there are elements $g_1, g_2 \in \in [0, g]$ such that $g_1 > 0$, $g_2 > 0$ and

$$g_1 \wedge g_2 = 0\,. \tag{3}$$

From (3) it follows

$$S(g_1) \cap S(g_2) = \emptyset\,. \tag{4}$$

Because $\emptyset \neq S(g_i) \subseteqq S(g)$ $(i = 1, 2)$, both $S(g_1)$ and $S(g_2)$ are proper subsets of $S(g)$, which is a contradiction.

For $X \subseteqq H$ we denote by X^σ the polar of X with respect to the lattice ordered group H; i.e.,

$$X^\sigma = \{y \in H : |y| \wedge |x| = 0 \text{ for each } x \in X\}\,.$$

If $g \in M$, we put

$$B(g) = \{g\}^{\sigma\sigma}\,.$$

Each $B(g)$ is a convex l-subgroup of H. Let M_1 be the set of all l-subgroups $B(g)$ $(g \in M)$.

Lemma 2. *Each l-subgroup $B(g) \in M_1$ is linearly ordered.*

Proof. Let $g \in M$. For each $0 < z \in B(g)$ we have $g \wedge z > 0$. Suppose that $B(g)$ is not a chain. Hence there are elements $x, y \in B(g)$ with $x > 0$, $y > 0$, $x \wedge y = 0$. Then $x \wedge g = x_1 > 0$, $y \wedge g = y_1 > 0$ and according to Lemma 1, $x_1 \wedge y_1 = \min \{x_1, y_1\} > 0$. Thus $x \wedge y > 0$, which is a contradiction.

Assume that at most one lattice ordered group A_i fails to be archimedean; if such A_i does exist, then it will be denoted by A_{i_0}.

Lemma 3. *Let $g \in M$. Then $B(g)$ is not upper bounded in H.*

Proof. Let $0 < h \in H$. It suffices to show that there is $g' \in B(g)$ such that g' non $\leqq h$.

At first suppose that $B(g)$ is not a subset of A_{i_0}. Then there is $g_1 \in B(g)$ and $i \in I$, $i \neq i_0$ such that $g_1(A_i) \neq 0$. Since A_i is archimedean, there exists a positive integer n with

$$n(g(A_i)) \text{ non} \leqq h(A_i)$$

and hence $g' = ng_1 \in H$, g' non $\leqq h$.

Assume that $B(g) \subseteqq A_{i_0}$. Thus, in particular, $g \in A_{i_0}$. Hence $g_2(A_{i_0}) = 0$ for each $g_2 \in \{g\}^\sigma$. The element h can be expressed as

$$h = h(A_{i_0}) + h(A_{i(1)}) + \ldots + h(A_{i(n)}),$$

where $i_0, i(1), \ldots, i(n)$ are distinct elements of I, $n \geqq 0$, $h(A_{i(j)}) > 0$ for $j = 1, \ldots, n$. Suppose that h is an upper bound for $B(g)$ and let n be the least non-negative integer with this property.

We have $g \in B(g)$, hence $B(g) \neq \{0\}$ and thus $B(g)$ has no maximal element. Therefore $h \notin B(g)$. This implies that $h \wedge g_2 = x > 0$ for some $g_2 \in \{g\}^\sigma$. Then $x \in \{g\}^\sigma$, hence $x(A_{i_0}) = 0$. Since $0 < x \leqq h$, we get $0 \leqq x(A_i) \leqq h(A_i)$ for each $i \in I$. Therefore without loss of generality we can suppose that

$$x = x(A_{i(1)}) + \ldots + x(A_{i(k)}),$$

$1 \leqq k \leqq n$, $x(A_{i(j)}) > 0$ for $j = 1, \ldots, k$. Because $A_{i(1)}, \ldots, A_{i(k)}$ are archimedean, there is a positive integer n such that

$$n\, x(A_{i(j)}) > h(A_{i(j)})$$

is valid for $j = 1, \ldots, k$. Thus $h_1 = h \wedge nx \in H$ and

$$h_1 = h(A_{i(j)}) + \ldots + h(A_{i(k)}) \cdot$$

Then $h_2 = h - h_1 \in H$ and

$$h_2 = h(A_{i_0}) + h(A_{i(k+1)}) + \dots + h(A_{i(n)}).$$

Let $g' \in B(g)$. Since $g' \leqq h$ and $B(g) \subseteqq A_{i_0}$, we infer that $g' \leqq h(A_{i_0})$ and hence $g' \leqq h_2$. Since the number of elements $h(A_{i(k+1)}), \dots, h(A_{i(n)})$ is less then n we have a contradiction.

Lemma 4. *The l-subgroup $B(g)$ is a cardinal summand of H for each $g \in M$. If $g_1, g_2 \in M$, $B(g_1) \neq B(g_2)$, then $B(g_1) \cap B(g_2) = \{0\}$.*

This follows from Lemma 3 and [3], 17.1.

Proof of the Theorem:

(i) Let at most one A_i be non-archimedean and let H be an l-subgroup of G. By Lemma 4, each l-subgroup $B(g)$ $(g \in M)$ is a cardinal summand of H. Let $0 < h \in H$ and let $\overline{M}_1$ be the set of all $B(g) \in M_1$ such that

$$B(g) \cap [0, h] \neq \{0\}.$$

This is equivalent with

$$h(B(g)) = \max \{g_1 \in B(g) : 0 \leqq g_1 \leqq h\} > 0.$$

The set $\overline{M}_1$ is nonempty, because there is $g \in M$ with $g \leqq h$ and then $B(g)$ belongs to $\overline{M}_1$.

Let $B(g) \in \overline{M}_1$. Because $h(B(g)) \neq 0$, we have $S(h(B(g))) \neq \emptyset$. Let $i \in S(h(B(g)))$. From $h(B(g)) \leqq h$ we get

$$h(A_i) \geqq (h(B(g)))(A_i) > 0. \tag{5}$$

If $B(g_1), B(g_2) \in \overline{M}_1$ and $B(g_1) \neq B(g_2)$, then according to Lemma 4,

$$h(B(g_1)) \wedge h(B(g_2)) = 0 \tag{6}$$

and hence

$$S(h(B(g_1))) \cap S(h(B(g_2))) = \emptyset. \tag{7}$$

Because the set $S(h)$ is finite, from (5) and (7) it follows that the set $\overline{M}_1$ is finite as well. Put

$$h_1 = \bigvee h(B(g)) \, (B(g) \in \overline{M}_1).$$

We have $0 \leqq h_1 \leqq h$. Denote $h_2 = h - h_1$. According to (6),

$$h_1 = h(B(g_1)) + \dots + h(B(g_n)), \tag{8}$$

where $\overline{M}_1 = \{B(g_1), \ldots, B(g_n)\}$. If $B(g_j)$, $B(g_k)$ are distinct elements of $\overline{M}_1$, then by (6),

$$(h(B(g_j)))\,(B(g_k)) = 0\,,$$

and clearly

$$(h(B(g_j)))\,(B(g_j)) = h(B(g_j))\,.$$

Thus we get from (8)

$$h_1(B(g_j)) = h(B(g_j)) \tag{9}$$

for each $B(g_j) \in \overline{M}_1$.

Suppose that $h_2 \neq 0$. Then $h_2 > 0$ and hence there exists $g \in M$ such that $g \leqq h_2$. Therefore

$$h_2(B(g)) \geqq g > 0\,. \tag{10}$$

Since $h \geqq h_2$,

$$h(B(g)) \geqq h_2(B(g)) > 0$$

and thus $B(g) \in \overline{M}_1$. But then it follows from (9) that

$$h_2(B(g)) = 0$$

and this is a contradiction. Hence $h_2 = 0$ and therefore by (8),

$$h = h(B(g_1)) + \ldots + h(B(g_n))\,. \tag{11}$$

Because for each $h' \in H$ there are h'', $h''' \in H$ with $h'' \geqq 0$, $h''' \geqq 0$, $h' = h'' - h'''$, it follows from (11) that the group H is generated by the set $\bigcup B(g_i)$ $(B(g_i) \in \overline{M}_1)$.

Let $B(g_i)$ be a fixed element of $\overline{M}_1$ and let H_1 be the subgroup of H generated by the union of all $B(g_j) \in \overline{M}_1$, $B(g_j) \neq B(g_i)$. By Lemma 4 we have

$$B(g_j) \subseteqq (B(g_i))^\sigma$$

and since $(B(g_i))^\sigma$ is a subgroup of H, $H_1 \subseteqq (B(g_i))^\sigma$. Hence $B(g_i) \cap H_1 = \{0\}$. Because $B(g_i)$ is a cardinal summand of H, it is a normal subgroup of the group H. Therefore the group G is a direct sum of its subgroups $B(g_i) \in M_1$.

For each $0 \neq h \in H$ the representation of h as a sum of elements $0 \neq a_i \in B(g_i)$ is unique; thus from (11) it follows that if $h > 0$, then $a_i > 0$. Hence the lattice ordered group H is a cardinal sum of its linearly ordered l-subgroups $B(g_i) \in M_1$.

(ii) Assume that i, j are distinct elements of I, and that A_i and A_j are not archimedean. There exist elements $a_1, a_2 \in A_i$, $b_1, b_2 \in A_j$ such that

$$0 < na_1 \leqq a_2\,, \quad 0 < nb_1 \leqq b_2$$

holds for each positive integer n. Let C be the set of all elements $x \in A_i$ such that

$n|x| < a_2$ for each positive integer n. Then C is an l-subgroup of A_i. Analogously we define the set D with B_j and b_2 instead of A_i and a_2. For $x \in C$ we have

$$n(-a_2 + x + a_2) = -a_2 + nx + a_2 < a_2$$

for each positive integer n, hence $-a_2 + x + a_2 \in C$. From this it follows that the set C_k consisting of all elements of A_i that can be written as

$$c + nka_2$$

where c runs over the set C, n is any integer and k is a fixed positive integer, is a subgroup of A_i. Because A_i is linearly ordered, C_k is an l-subgroup of A_i. Similarly, the set D_k consisting of all elements

$$d + mkb_2 \quad (m = 0, \pm 1, \pm 2, \ldots)$$

with $d \in D$ is an l-subgroup of A_j. Hence the set E_k of all elements

$$c + d + nk(a_2 + b_2)$$

is a subgroup of the group G. We shall show that E_k is an l-subgroup of G. It suffices to verify that, for each $e \in E_k$, the element $e \vee 0$ belongs to E_k. Let $e = c + d + nk(a_2 + b_2)$. Obviously $e = (c + nka_2) + (d + nkb_2)$ and

$$e \vee 0 = ((c + nka_2) \vee 0) + ((d + nkb_2) \vee 0).$$

If $n \neq 0$, then $e \vee 0 = e$ for $n > 0$ and $e \vee 0 = 0$ for $n < 0$. Let $n = 0$; then $c \vee 0 \in \{c, 0\} \subseteqq C$, $d \vee 0 \in \{d, 0\} \subseteqq D$. Therefore $e \vee 0 = (c + d) \vee 0 = (c \vee d) \vee 0 = (c \vee 0) \vee (d \vee 0) = (c \vee 0) + (d \vee 0) \in E_k$.

If $e = c + d + nk(a_2 + b_2) \in E_k$ and $n > 0$, then $e > c_1$ and $e > d_1$ for each $c_1 \in C$ and each $d_1 \in D$, thus the interval $[0, e]$ of the l-group E_k fails to be a chain.

Suppose that E_k is a direct sum of linearly ordered groups. Then each $0 < e \in E_k$ can be written as

$$e = e_1 + \ldots + e_m$$

such that each $0 < e_i$ and the interval $[0, e_i]$ of E_k is a chain for $i = 1, \ldots, m$. Hence

$$e_i = c_i + d_i \quad (c_i \in C,\ d_i \in D)$$

and therefore

$$e = c + d \quad (c \in C,\ d \in D).$$

If we choose $e \in E_k$ such that $e = nk(a_2 + b_2)$, $n > 0$, then we have $c + d < e$ for each $c \in C$ and each $d \in D$, which is a contradiction. Hence E_k is not a direct sum of linearly ordered groups.

If k, k' are positive integers with $k < k'$, then $k(a_2 + b_2) \in E_k$ and $k(a_2 + b_2)$ non $\in$ $\in E_{k'}$, thus $E_k \neq E_{k'}$. The proof is complete.

An l-subgroup H of a lattice ordered group G is said to be convex if from $h \in H$, $g \in G$, $0 \leqq g \leqq h$ it follows $g \in H$. By investigating cardinal summands of G it suffices to consider only convex l-subgroups of G, since each cardinal summand is convex. A lattice ordered group B will be called stricly cyclic, if, for each $0 \neq b \in B$, the convex l-subgroup of B generated by the element b equals B. A lattice ordered group will be called cyclic if it is generated by one element. The following two statements are analogous to the well-known KULIKOV's theorem on subgroups of direct sums cyclic groups (cf. [3], Thm. 18.1).

Proposition 1. *Each l-subgroup of a cardinal sum of strictly cyclic l-groups is again a cardinal sum of strictly cyclic l-groups.*

Proof. Let B be a strictly cyclic l-group, $B \neq \{0\}$. For each $b \in B$ we denote by $[b]$ the convex l-subgroup of B generated by b. Suppose that B is not linearly ordered. Then there are $b_1, b_2 \in B$ with $0 < b_1$, $0 < b_2$, $b_1 \wedge b_2 = 0$. Hence $[b_1] \neq [b_2]$, which is a contradiction. Thus B is linearly ordered. If B is not archimedean, then there are $b_1, b_2 \in B$ with $0 < nb_1 < b_2$ for each positive integer n; in this case we would have $b_2 \notin [b_1]$, a contradiction. Therefore B is archimedean. Conversely, each archimedean linearly ordered group is strictly cyclic. Now the assertion immediately follows from the Theorem.

Proposition 2. *Let H be an l-subgroup of a cardinal sum of cyclic lattice ordered groups. Then H is again a cardinal sum of cyclic lattice ordered groups.*

Proof. Let us denote by Z and R the additive group of all integers or all reals, respectively, with the natural linear order. Let G_0 be the free lattice ordered group with one free generator. Then G_0 is isomorphic to $Z \oplus Z$ (cf. [1], p. 297, Ex. 6). Each cyclic l-group is a homomorphic image of G_0. Hence a lattice ordered group is cyclic if and only if it is isomorphic to some of the following l-groups: $\{0\}$, Z, $Z \oplus Z$. Therefore each l-group $G \neq \{0\}$ that is a cardinal sum of cyclic groups is a cardinal sum of l-groups isomorphic to Z.

Let $H \neq \{0\}$ be an l-subgroup of G. Since Z is linearly ordered and archimedean, we have

$$H = \sum H_i \quad (i \in I)$$

where each H_i is linearly ordered and $H_i \neq \{0\}$. Because G is archimedean, each H_i is archimedean and hence H_i is isomorphic to an l-subgroup of R. Obviously G^+ fulfils the descending chain condition and thus H_i^+ fulfils the descending chain condition as well. Therefore there exists $e_i \in H_i$ such that e_i covers 0 in H_i. From this we immediately obtain (cf. [1], Chap. XIII, Thm. 10) that H_i is a cyclic group generated by e_i.

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Author's address: 040 01 Košice, Švermova 5, ČSSR (Vysoké učení technické).