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ON A PARTIAL COMPLEX STRUCTURE

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Let $M^3 \subset \mathscr{C}^2$ be a real hypersurface. $\mathscr{C}^2$ being identified with $\mathscr{R}^4$ endowed with an endomorphism $J : \mathscr{R}^4 \to \mathscr{R}^4$ satisfying $J^2 = -\mathrm{id}.$, we get, on M^3, a structure consisting of a field of tangent planes $\tau_m = T_m(M^3) \cap JT(M^3)$ and the restrictions $J_m : \tau_m \to \tau_m$ of J to τ_m; see [3] and [4] respectively. Such a structure is called the partial complex structure. An attempt to solve the equivalence problem for the structures of this type has been made by E. CARTAN [1]; unfortunately, his treatment is not a very clear one. In what follows, I am going to present a more simple method for solving the mentioned problem.

Let M be a 3-dimensional manifold; in what follows, all the considered manifolds and maps are supposed to be of class C^∞.

Definition. Let G be the group of matrices of the type

$$(1)\qquad \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & \varphi \end{pmatrix}; \quad \alpha, \beta, \gamma, \delta, \varphi \in \mathscr{R}\,; \quad (\alpha^2 + \beta^2)\,\varphi \neq 0\,.$$

A G-structure B_G on M is called a *partial complex structure.*

Let (v_1, v_2, v_3), (w_1, w_2, w_3) be two sections of B_G over a domain $U \subset M$; we have

$$(2)\qquad v_1 = \alpha w_1 - \beta w_2\,, \quad v_2 = \beta w_1 + \alpha w_2\,, \quad v_3 = \gamma w_1 + \delta w_2 + \varphi w_3\,.$$

At each point $m \in M$ there are induced a plane $\tau_m = \{v_1, v_2\}$ and an endomorphism $J_m : \tau_m \to \tau_m$ given by $J_m(v_1) = v_2$, $J_m(v_2) = -v_1$; obviously, $J_m^2 = -\mathrm{id}$. We are going to suppose that *the field of planes τ is non-integrable.*

Definition. A vector field v on M is called a *τ-field* if $v_m \in \tau_m$ for each $m \in M$. A vector field u is called an *infinitesimal motion* of B_G if: (i) $\mathscr{L}_u v$ is a τ-field for each τ-field v, (ii) $J(\mathscr{L}_u v') = \mathscr{L}_u(Jv')$ for each τ-field v'; here, $\mathscr{L}_u v = [u, v]$ is the Lie derivative of v. The structure B_G is said to be *locally transitive* if, for each $m \in M$ and $t \in T_m(M)$, there is a neighbourhood $U \subset M$ of m and an infinitesimal motion u over U such that $u_m = t$.

Our main problem is to determine all locally transitive G-structures. Let (v_1, v_2, v_3), (w_1, w_2, w_3) be two sections of B_G over $U \subset M$. Then

$$(3)\qquad \begin{aligned} [v_1, v_2] &= a_1 v_1 + a_2 v_2 + a_3 v_3\,, & [w_1, w_2] &= A_1 w_1 + A_2 w_2 + A_3 w_3\,,\\ [v_1, v_3] &= b_1 v_1 + b_2 v_2 + b_3 v_3\,, & [w_1, w_3] &= B_1 w_1 + B_2 w_2 + B_3 w_3\,,\\ [v_2, v_3] &= c_1 v_1 + c_2 v_2 + c_3 v_3\,, & [w_2, w_3] &= C_1 w_1 + C_2 w_2 + C_3 w_3\,, \end{aligned}$$

the functions $a_1, \ldots, c_3, A_1, \ldots, C_3$ satisfying the Jacobi identities

$$(4)\qquad \begin{aligned} &[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0\,,\\ &[w_1, [w_2, w_3]] + [w_2, [w_3, w_1]] + [w_3, [w_1, w_2]] = 0\,. \end{aligned}$$

We get

$$\begin{aligned} [v_1, v_2] &= \{v_1\beta - v_2\alpha + (\alpha^2 + \beta^2) A_1\}\, w_1 + \{v_1\alpha + v_2\beta + (\alpha^2 + \beta^2) A_2\}\, w_2 +\\ &\quad + (\alpha^2 + \beta^2) A_3 w_3 =\\ &= (\alpha a_1 + \beta a_2 + \gamma a_3)\, w_1 + (\alpha a_2 - \beta a_1 + \delta a_3)\, w_2 + \varphi a_3 w_3\,,\\ [v_1, v_3] &= (\cdot)\, w_1 + (\cdot)\, w_2 + \{v_1\varphi + (\alpha\delta + \beta\gamma) A_3 + \alpha\varphi B_3 - \beta\varphi C_3\}\, w_3 =\\ &= (\cdot)\, w_1 + (\cdot)\, w_2 + \varphi b_3 w_3\,,\\ [v_2, v_3] &= (\cdot)\, w_1 + (\cdot)\, w_2 + \{v_2\varphi + (\beta\delta - \alpha\gamma) A_3 + \beta\varphi B_3 + \alpha\varphi C_3\}\, w_3 =\\ &= (\cdot)\, w_1 + (\cdot)\, w_2 + \varphi c_3 w_3\,. \end{aligned}$$

From this, we get the existence of sections satisfying $a_3 = 1$. Suppose $a_3 = A_3 = 1$, i.e., $\varphi = \alpha^2 + \beta^2$. Let us look for the existence of a section (w_1, w_2, w_3) satisfying $A_1 = A_2 = B_3 = C_3 = 0$. This amounts to the existence of $\alpha, \beta, \gamma, \delta$ such that

$$\begin{aligned} &v_1\beta - v_2\alpha = a_1\alpha + a_2\beta + \gamma\,, && 2\alpha v_1\alpha + 2\beta v_1\beta + \alpha\delta + \beta\gamma = (\alpha^2 + \beta^2)\, b_3\,,\\ &v_1\alpha + v_2\beta = a_2\alpha - a_1\beta + \delta\,, && 2\alpha v_2\alpha + 2\beta v_2\beta + \beta\delta - \alpha\gamma = (\alpha^2 + \beta^2)\, c_3\,. \end{aligned}$$

It is easy to see that this system has (at least locally) solutions such that $\alpha^2 + \beta^2 \neq 0$. Let $a_1 = a_2 = b_3 = c_3 = 0$, $a_3 = 1$ for (v_1, v_2, v_3). From the Jacobi identity (4_1), we get $v_1 c_1 - v_2 b_1 = v_1 c_2 - v_2 b_2 = c_2 + b_1 = 0$. A τ-field v is called special if the section $(v_1 = v, v_2 = Jv, v_3 = [v, Jv])$ has the just described property.

Theorem. *Let B_G be a partial complex structure over M. Let v, w be its special τ-fields, and let*

$$(5)\qquad \begin{gathered} [v, [v, Jv]] = av + bJv\,, \quad [Jv, [v, Jv]] = cv - aJv\,,\\ (Jv)\, b + va = 0\,, \quad (Jv)\, a - vc = 0\,; \end{gathered}$$

(6) $$[w,[w,Jw]] = Aw + BJw\,, \quad [Jw,[w,Jw]] = Cw - AJw\,,$$
$$(Jw)\,B + wA = 0\,, \quad (Jw)\,A - wC = 0\,;$$

(7) $$v = \alpha w - \beta Jw\,.$$

Consider the functions

(8) $$j_1 = (vv - Jv\,.\,Jv)(c - b) + 8[v, Jv]\,a - 3(c^2 - b^2)\,,$$
$$j_2 = (v\,.\,Jv + Jv\,.\,v)(c - b) + 4[v, Jv](b + c) + 6a(c - b)\,;$$

J_1 and J_2 be defined similarly. Then:

(i) *We have*

(9) $$j_1^2 + j_2^2 = (\alpha^2 + \beta^2)^4\,(J_1^2 + J_2^2)\,.$$

(ii) *If $j_1 = j_2 = 0$, B_G is locally transitive. For each point $m \in M$, there is its neighbourhood $U \subset M$ and special τ-fields v over U satisfying*

(10) $$[v,[v,Jv]] = 0\,, \quad [Jv,[v,Jv]] = 0\,.$$

Choose such a τ-field v. Further, choose arbitrary real numbers $R_1, \ldots, R_8$. Then there is exactly one field $u \in \mathscr{L}(B_G)$ over asuitable neighbourhood $m \in U_1 \subset U$ such that

(11) $$u_m = R_1 v_m + R_2 (Jv)_m + R_3 [v, Jv]_m\,,$$
$$[v, u]_m = R_4 v_m + R_5 (Jv)_m\,,$$
$$[v.\,[v, u]]_m = R_6 v_m + R_7 (Jv)_m + R_5 [v, Jv]_m\,,$$
$$[v,[v,[v, u]]]_m = R_8 (Jv)_m + 2R_7 [v, Jv]_m\,.$$

(iii) *Let $j_1^2 + j_2^2 \neq 0$. To each point $m \in M$, there is its neighbourhood U and exactly two special τ-fields $v, v' = -v$ over U satisfying (5) and*

(12) $$j_1 = 1\,, \quad j_2 = 0\,.$$

B_G being transitive, v satisfies

(13) $$[v,[v,Jv]] = bJv\,, \quad [Jv,[v,Jv]] = cv\,;$$
$$b, c \in \mathscr{R}\,, \quad 3(c^2 - b^2) + 1 = 0\,.$$

For each vector $t \in T_m(M)$, there is exactly one field $u \in \mathscr{L}(B_G)$ — defined over a suitable neighbourhood U_1 of m — such that $u_m = t$.

Proof. Let $T^C(M) = T(M) \oplus i\, T(M)$ be the complexification of the tangent bundle $T(M)$ of M. The bracket operation in $T^C(M)$ be introduced, quite naturally, by

$$(14) \qquad [v + iv', w + iw'] = [v, w] - [v', w'] + i([v', w] + [v, w']) .$$

In B_G, consider two special sections (v_1, v_2, v_3), (w_1, w_2, w_3) satisfying

$$(15) \quad \begin{aligned} &[v_1, v_2] = v_3 , \quad [v_1, v_3] = av_1 + bv_2 , \quad [v_2, v_3] = cv_1 - av_2 ; \\ &[w_1, w_2] = w_3 , \quad [w_1, w_3] = Aw_1 + Bw_2 , \quad [w_2, w_3] = Cw_1 - Aw_2 . \end{aligned}$$

In $T^C(M)$, consider the vector fields

$$(16) \quad \begin{aligned} &V_1 = v_1 + iv_2 , \quad V_2 = v_1 - iv_2 , \quad V_3 = -2iv_3 ; \\ &W_1 = w_1 + iw_2 , \quad W_2 = w_1 - iw_2 , \quad W_3 = -2iw_3 . \end{aligned}$$

We get

$$(17) \quad \begin{gathered} [V_1, V_2] = V_3 , \quad [V_1, V_3] = pV_1 + qV_2 , \quad [V_2, V_3] = rV_1 - pV_2 , \\ p = c - b , \quad q = b + c - 2ia , \quad r = -(b + c + 2ia) , \\ V_2 q = -V_1 p , \quad V_1 r = V_2 p ; \\ [W_1, W_2] = W_3 , \quad [W_1, W_3] = PW_1 + QW_2 , \quad [W_2, W_3] = RW_1 - PW_2 , \\ P = C - B , \quad Q = B + C - 2iA , \quad R = -(B + C + 2iA) , \\ W_2 Q = -W_1 P , \quad W_1 R = W_2 P . \end{gathered}$$

From (2),

$$(18) \quad \begin{gathered} V_1 = \varrho W_1 , \quad V_2 = \sigma W_2 , \quad V_3 = \mu W_1 + \nu W_2 + \varphi W_3 ; \\ \varrho = \alpha + i\beta , \quad \sigma = \alpha - i\beta , \quad \mu = -(\delta + i\gamma) , \quad \nu = \delta - i\gamma ; \quad \varrho\sigma\varphi \neq 0 . \end{gathered}$$

Now,

$$(19) \qquad \varphi = \varrho\sigma ,$$

$$(20) \quad \begin{gathered} V_1 \varrho = -2\varrho\sigma^{-1}\nu , \quad V_2 \varrho = -\mu ; \quad V_1 \sigma = \nu , \quad V_2 \sigma = 2\varrho^{-1}\sigma\mu ; \\ V_2 \mu = \varrho r - \varrho\sigma^2 R , \quad V_1 \nu = \sigma q - \varrho^2 \sigma Q , \\ V_1 \mu - V_3 \varrho = \varrho p - \varrho^2 \sigma P , \quad V_2 \nu - V_3 \sigma = -\sigma p + \varrho\sigma^2 P . \end{gathered}$$

It is known [2] that the integrability conditions of (20) imply

$$(21) \qquad k_1 = \varrho^3 \sigma K_1 , \quad k_2 = \varrho\sigma^3 K_2$$

with

$$(22)\qquad k_1 = V_1V_1p - 2V_3q - 3pq\,, \quad k_2 = V_2V_2p - 2V_3r + 3pr$$

and similar definition of K_1 and K_2 respectively. We get

$$(23)\qquad k_1 = j_1 + ij_2\,, \quad k_2 = j_1 - ij_2\,,$$

j_1 and j_2 being given by (8). Thus (21) reduce to $k_1 = \varrho^3\bar{\varrho}K_1$; the equation $k_1\bar{k}_1 = \varrho^4\bar{\varrho}^4K_1\bar{K}_1$ is exactly (9). The equation $k_1 = \varrho^3\bar{\varrho}$ has solutions $\varrho \neq 0$ for each $k_1 \neq 0$; the equation $1 = \varrho^3\bar{\varrho}$ has exactly two solutions $\varrho = \pm 1$. Thus (i) and the first part of (iii) have been proved.

Suppose $j_1 = j_2 = 0$, i.e., $k_1 = k_2 = 0$, and consider the system (19) + (20), $P = Q = R = 0$. According to [2], this system is completely integrable, and we have proved the first part of (ii).

Now, consider the structure B_G, given by a section (v_1, v_2, v_3) satisfying

$$(24)\qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = 0\,, \quad [v_2, v_3] = 0\,.$$

Let $u \in \mathscr{L}(B_G)$,

$$(25)\qquad u = Av_1 + Bv_2 + Cv_3\,.$$

From

$$[v_1, u] = v_1A\,.\,v_1 + v_1B\,.\,v_2 + (v_1C + B)\,v_3\,,$$
$$[v_2, u] = v_2A\,.\,v_1 + v_2B\,.\,v_2 + (v_2C - A)\,v_3\,,$$

we get

$$(26)\qquad v_1A - v_2B = v_2A + v_1B = v_1C + B = v_2C - A = 0\,.$$

For $D := v_1A$, $E := v_2A$, we have

$$(27)\qquad v_1A = D\,, \quad v_2A = E\,;$$
$$v_1B = -E\,, \quad v_2B = D\,; \quad v_1C = -B\,, \quad v_2C = A\,.$$

From $(27_{5,6})$,

$$(28)\qquad v_3C = 2D\,;$$

the integrability conditions of $(27_1) + (27_2)$, $(27_3) + (27_4)$, $(27_5) + (28)$ and $(27_6) + (28)$ are

$$v_3A = v_1E - v_2D\,, \quad v_3B = v_1D + v_2E\,,$$
$$0 = 2v_1D + v_3B\,, \quad 0 = 2v_2D - v_3A$$

respectively; for $3F := v_1 E$, $3G := v_2 E$, they become

$$(29) \qquad v_3 A = 2F\,; \quad v_3 B = 2G\,;$$
$$v_1 D = -G\,, \quad v_2 D = F\,; \quad v_1 E = 3F\,, \quad v_2 E = 3G\,.$$

The integrability conditions of $(27_1) + (29_1)$, $(27_2) + (29_1)$, $(27_3) + (29_2)$, $(27_4) + (29_2)$, $(29_3) + (29_4)$ and $(29_5) + (29_6)$ being

$$2v_1 F = v_3 D\,, \quad 2v_2 F = v_3 E\,, \quad 2v_1 G = -v_3 E\,, \quad 2v_2 G = v_3 D\,,$$
$$v_3 D = v_1 F + v_2 G\,, \quad v_3 E = 3v_1 G - 3v_2 F\,,$$

we have

$$(30) \qquad v_3 D = 2H\,; \quad v_3 E = 0\,; \quad v_1 F = H\,, \quad v_2 F = 0\,; \quad v_1 G = 0\,, \quad v_2 G = H$$

for $H := v_1 F$. The integrability conditions of $(29_3) + (30_1)$, $(29_4) + (30_1)$, $(29_5) + (30_2)$, $(29_6) + (30_2)$, $(30_3) + (30_4)$ and $(30_5) + (30_6)$ are

$$2v_1 H = -v_3 G\,, \quad 2v_2 H = v_3 F\,, \quad v_3 F = 0\,, \quad v_3 G = 0\,,$$
$$v_3 F = -v_2 H\,, \quad v_3 G = v_1 H\,;$$

from these, we get

$$(31) \qquad v_3 F = 0\,; \quad v_3 G = 0\,; \quad v_1 H = 0\,, \quad v_2 H = 0\,.$$

The integrability conditions of $(30_3) + (31_1)$, $(30_4) + (31_1)$, $(30_5) + (31_2)$, $(30_6) + (31_2)$ and $(31_3) + (31_4)$ reduce to

$$(32) \qquad v_3 H = 0\,.$$

Thus the system (27)–(32) is completely integrable. For $u \in \mathscr{L}(B_G)$ given by (25), we get

$$(33) \qquad [v_1, u] = Dv_1 - Ev_2\,, \quad [v_1, [v_1, u]] = -Gv_1 - 3Fv_2 - Ev_3\,,$$
$$[v_1, [v_1, [v_1, u]]] = -3Hv_2 - 6Fv_3\,,$$

this completing the proof of (ii).

Finally, let B_G be transitive with $j_1^2 + j_2^2 \neq 0$. Then they are exactly two special sections of B_G satisfying $j_1 = 1$, $j_2 = 0$; let (v_1, v_2, v_3) be one of them. The functions a, b, c being now invariants of B_G, they have to be constants, and we get $-3(c^2 - b^2) = 1$, $a(c - b) = 0$. Because of $c - b \neq 0$, $a = 0$, i.e.,

$$(34) \qquad [v_1, v_2] = v_3\,, \quad [v_1, v_3] = bv_2\,, \quad [v_2, v_3] = cv_1\,;$$
$$b, c \in \mathscr{R}\,, \quad 3(c^2 - b^2) + 1 = 0\,.$$

Let $u \in \mathscr{L}(B_G)$, u being given by (25). From

$$[v_1, u] = v_1 A \,.\, v_1 + (v_1 B + bC)\, v_2 + (v_1 C + B)\, v_3 \,,$$
$$[v_2, u] = (v_2 A + cC)\, v_1 + v_2 B \,.\, v_2 + (v_2 C - A)\, v_3 \,,$$

we get

$$v_1 A - v_2 B = v_2 A + v_1 B + (b + c)\, C = v_1 C + B = v_2 C - A = 0 \,.$$

For $D := v_1 A$, $E := v_1 B + bC$,

$$(35) \qquad v_1 A = D \,, \quad v_2 A = -E - cC \,; \quad v_1 B = E - bC \,, \quad v_2 B = D \,;$$
$$v_1 C = -B \,, \quad v_2 C = A \,.$$

The integrability condition of (35_5) and (35_6) is

$$(36) \qquad v_3 C = 2D \,.$$

The integrability conditions of $(35_1) + (35_2)$, $(35_3) + (35_4)$, $(35_5) + (36)$ and $(35_6) + (36)$ are

$$v_3 A + v_1 E + v_2 D = cB \,, \quad v_3 B - v_1 D + v_2 E = bA \,,$$
$$2v_1 D + v_3 B = bA \,, \quad v_3 A - 2v_2 D = cB \,.$$

For $3F := v_1 E$, $3G := v_2 A$, we get

$$(37) \qquad v_3 A = -2F + cB \,; \quad v_3 B = -2G + bA \,;$$
$$v_1 D = G \,, \quad v_2 D = -F \,; \quad v_1 E = 3F \,, \quad v_2 E = 3G \,.$$

The integrability conditions of $(35_1) + (37_1)$, $(35_2) + (37_1)$, $(35_3) + (37_2)$, $(35_4) + (37_2)$, $(37_3) + (37_4)$ and $(37_5) + (37_6)$ are

$$2v_1 F + v_3 D = (b + c)\, E \,, \quad 2v_2 F - v_3 E = 2cD \,,$$
$$2v_1 G + v_3 E = 2bD \,, \quad 2v_2 G + v_3 D = -(b + c)\, E \,,$$
$$v_3 D + v_1 F + v_2 G = 0 \,, \quad v_3 E - 3v_1 G + 3v_2 F = 0 \,.$$

For $H := v_1 F - bE$, they are

$$(38) \qquad v_3 D = -2H + (c - b)\, E \,; \quad v_3 E = 2(b - c)\, D \,;$$
$$v_1 F = H + bE \,, \quad v_2 F = \tfrac{1}{3}(3b + 5c)\, D \,;$$
$$v_1 G = \tfrac{1}{3}(5b + 3c)\, D \,, \quad v_2 G = H - cE \,.$$

The integrability conditions of $(37_3) + (38_1)$, $(37_4) + (38_1)$, $(37_5) + (38_2)$, $(37_6) + (38_2)$, $(38_3) + (38_4)$ and $(38_5) + (38_6)$ are

$$(39) \qquad 2v_1H + v_3G = (3c - 2b)\,F\,, \quad 2v_2H - v_3F = (2c - 3b)\,G\,;$$

$$(40) \qquad v_3F = -\tfrac{1}{3}(b + 2c)\,G\,; \quad v_3G = -\tfrac{1}{3}(2b + c)\,F\,;$$

$$(41) \qquad v_2H = \tfrac{1}{3}(7c - 5b)\,G\,, \quad v_1H = \tfrac{1}{3}(5c - 7b)\,F\,.$$

Substituting from (40) and (41) into (39), we get

$$(42) \qquad bF = 0\,, \quad cG = 0\,.$$

Let $b \neq 0 \neq c$, i.e., $F = G = 0$. From $(38_{4,5})$, $D = 0$. The equations $(38_{3,6})$ imply $(b + c)\,E = 0$; $b + c = 0$ being impossible because of (34), we have $E = 0$. We get $H = 0$ from (38_3). Thus we obtain the completely integrable system

$$(43) \qquad \begin{aligned} v_1A &= 0\,, & v_2A &= -cC\,, & v_3A &= cB\,, \\ v_1B &= -bC\,, & v_2B &= 0\,, & v_3B &= bA\,, \\ v_1C &= -B\,, & v_2C &= A\,, & v_3C &= 0\,. \end{aligned}$$

Next, suppose $b \neq 0$, $c = 0$ (the case $b = 0$, $c \neq 0$ being analoguous). Then $F = 0$ because of (42_1). We get $D = 0$ and $G = 0$ from (38_4) and (40_1) respectively. From $(38_{1,3})$, $v_3D + v_1F = -H$, $v_3D + 2v_1F = bE$, i.e., $E = H = 0$. Thus we obtain the system (43) with $c = 0$. This proves the second part of (iii) and the Theorem.

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