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The adjacency graphs of a complex

THE ADJACENCY GRAPHS OF A COMPLEX

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The concept of the line-graph $L(G)$ of a graph G is generalized here to the (n, m)-adjacency graph $A_{n,m}(K)$ of a simplicial complex K. Both Krausz [6] and Beineke [1] have given characterizations of those graphs which are themselves line-graphs. In the present work, the possibilities for generalizing these results are examined. The structural characterization of Krausz is directly generalized to complexes. However, it is shown that no very direct generalization of Beineke's characterization involving forbidden subgraphs can exist.

Let K be a collection of subsets called *simplexes* of a nonempty finite set $V = V(K)$ called *vertices*. Then K is a *complex* if it satisfies the following conditions:

(i) every nonempty subset of a simplex is a simplex.

(ii) Every vertex is a simplex.

This definition is equivalent to that of an "abstract simplicial complex", as given in [5, p. 41]. If x is a simplex in K, the *dimension of x* is the number $n = |x| - 1$ and x is called an *n-simplex*. The *dimension of K* is the maximum of the dimensions of its simplexes. A complex whose dimension is n will be called an *n-complex*.

Clearly, a *graph* is just a complex whose dimension is not greater than 1. Any graph-theoretic terms not defined explicitly here may be found in [4]. A 1-simplex in a graph is called an *edge* of the graph. The number of edges in which a vertex v of a graph G lies is the *degree* of that vertex and is written $d(v)$. A subgraph H of a graph G is said to be *induced* if for every edge $\{u, v\}$ of G for which $u, v \in V(H)$, $\{u, v\}$ also lies in H.

As usual, two complexes K and L are said to be *isomorphic*, written $K \simeq L$, if there is a $1 - 1$ onto mapping $f : V(K) \to V(L)$ such that x is a simplex in K if and only if $f(x)$ is a simplex in L. The complex L is a *subcomplex* of the complex K if L is a complex and $L \subseteq K$. Given a vertex v in a complex K, the *star of v*, written $\mathrm{st}(v, K)$, is the subcomplex $\{x \in K : v \in x\}$. Two simplexes x and y of different dimensions in a complex K are said to be *incident* if $x \subset y$ or $y \subset x$. Two n-simplexes of K are called *m-adjacent* $(m < n)$ if they are incident with a common m-simplex. The

(n, m)-*adjacency graph*, $m < n$, of a complex K, written $A_{n,m}(K)$, is the graph whose vertices are the n-simplexes of K, two vertices of $A_{n,m}(K)$ being adjacent if the corresponding n-simplexes of K are m-adjacent. If K is a 1-complex, then $A_{1,0}(K)$ is just the *line graph* of K, see [4, p. 71].

STRUCTURAL CHARACTERIZATION OF ADJACENCY GRAPHS

The characterization problem for (n, m)-adjacency graphs was posed by GRÜN-BAUM in [3] for the case $m = n - 1$. His discussion of this problem asked, in effect, whether either the characterization of Krausz [6] or of Beineke [1] could be generalized to the $(n, n - 1)$-adjacency graphs of a complex.

The first of these results can be stated as a theorem about 1-complexes.

Theorem A (Krausz). *The graph G is a $(1,0)$-adjacency graph if and only if G can be expressed as the union of complete subgraphs $\{G_i : i \in I\}$ satisfying the following two conditions*:

(i) *Each edge of G lies in exactly one subgraph G_i.*
(ii) *Each vertex of G lies in at most two subgraphs G_i.*

This theorem is generalized in straightforward fashion.

Theorem 1. *The graph G is an (n, m)-adjacency graph, $n > m$, if and only if there is a set $\{G_i : i \in I\}$ of subgraphs of G satisfying the following three conditions*:

(i) *Each edge of G lies in at most n and at least $m + 1$ of the graphs G_i.*
(ii) *Each vertex of G lies in at most $n + 1$ of the graphs G_i.*
(iii) *The intersection of any $m + 1$ of the graphs G_i is either empty or a complete graph.*

Proof. We first establish the necessity of these three conditions, which is the easier "half". Let $G = A_{n,m}(K)$ for some complex K.

For each vertex $v_i \in V(K)$ let $G_i = A_{n,m}(\text{st}(v_i, K))$. Let $\{u, v\}$ be an arbitrary edge of G and let y_u, y_v be the n-simplexes of K corresponding to u and v respectively. Clearly, $\{u, v\}$ lies in those graphs G_i which contain both u and v, and in those graphs only. The number of such graphs is precisely the number k of vertices shared by y_u and y_v whence $m + 1 \leq k \leq n$, establishing condition (i).

Let v be an arbitrary vertex of G. Then v appears in as many graphs G_i as the corresponding n-simplex y_v has vertices, namely $n + 1$, proving condition (ii).

Let $G_{i_1}, G_{i_2}, \ldots, G_{i_{m+1}}$ be any collection of $m + 1$ of the graphs G_i. If $\bigcap_{j=1}^{m+1} G_{i_j} \neq \emptyset$, then this intersection either contains a single vertex or at least two vertices, say u and v. In the former case, the intersection is the complete graph with just one vertex (the trivial graph). In the latter case, the n-simplexes y_u and v_v both contain all the

vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_{m+1}}$ corresponding to the $m + 1$ subgraphs. Therefore, y_u and y_v are m-adjacent and $\{u, v\}$ must appear in each subgraph $G_{i_j}, j = 1, 2, \ldots$
$\ldots, m + 1$. Since u and v may be arbitrarily chosen, $\bigcap\limits_{j=1}^{m+1} G_{i_j}$ must be a complete graph. Thus in both of the two cases considered, this intersection is a complete graph, so condition (iii) holds and the necessity is proved.

To prove the sufficiency, let G be a graph having a collection of subgraphs $\{G_i : i \in I\}$ satisfying conditions (i), (ii), and (iii). Our first step is to augment this collection if necessary: for each vertex v of G lying in just k of the graphs where $k < n + 1$, create $n + 1 - k$ additional trivial graphs consisting of the single vertex v. Add all the graphs so created to the collection $\{G_i : i \in I\}$, thus forming the augmented collection $\{G_i : i \in I'\}, I \subseteq I'$. Clearly conditions (i) and (ii) continue to hold for the augmented collection. It is claimed that condition (iii) also holds for this collection: let $G_{i_1}, G_{i_2}, \ldots, G_{i_{m+1}}$ be any $m + 1$ graphs in $\{G_i : i \in I'\}$. If $i_j \in I, j = 1, 2, \ldots$ $\ldots, m + 1$, then these subgraphs satisfy condition (iii) already. Otherwise there is at least one graph, say G_{i_1}, such that $i_1 \in I' - I$ and, therefore, G_{i_1} has just one vertex, say v. If $\bigcap\limits_{i=1}^{m+2} G_{i_j} \neq \emptyset$, then this intersection consists of a graph on just one vertex and is, therefore, a complete graph.

A complex K is now constructed such that $G \simeq A_{n,m}(K)$. For each $w \in V(G)$ let $f(w) = \{i \in I' : w \in V(G_i)\}$. Define K to be the pure n-complex whose n-simplexes are the sets $f(w), w \in V(G)$. Suppose that u and v are distinct vertices of G. If $f(u) = f(v)$, then u and v lie in precisely the same $n + 1$ subgraphs in the augmented collection $\{G_i : i \in I'\}$. Therefore, the intersection of these $n + 1$ graphs contains both u and v, whence, by condition (iii), it contains the edge $x = \{u, v\}$. But by condition (i), x lies in at most n of these graphs, a contradiction. It follows that f is a $1-1$ function from $V(G)$ onto the n-simplexes of K. If u is adjacent to v in G, then by condition (i) $\{u, v\}$ lies in at least $m + 1$ of the graphs $G_i, i \in I$. Moreover, for each graph G_i containing $\{u, v\}, i \in f(u) \cap f(v)$. It follows that $f(u)$ and $f(v)$ are m-adjacent. If, on the other hand, $f(u)$ and $f(v)$ are m-adjacent, then there is a set of $m + 1$ graphs in the collection containing both u and v. The intersection of these $m + 1$ graphs is a complete graph by condition (iii) and, therefore, contains $\{u, v\}$. Thus u is adjacent to v and the proof that $G = A_{n,m}(K)$ is complete.

Remark. Examining the statement of Theorem 1 with $n = 1$ and $m = 0$, it is found that condition (iii) implies that the subgraphs G_i must be complete graphs. It is also found that G is the union of the subgraphs G_i. These two observations together imply that Theorem A is just this special case of Theorem 1.

Two interesting corollaries can be drawn from the theorem:

Corollary 1a. *The graph G is an $(n, 0)$-adjacency graph if and only if there is a set $\{G_i : i \in I\}$ of complete subgraphs of G satisfying the following two conditions:*

(i) *Each edge of G lies in at most n and at least 1 subgraph G_i.*

(ii) *Each vertex of G lies in at most n + 1 of the subgraphs G_i.*

Proof. With $m = 0$, condition (iii) of Theorem 2 implies that each of the subgraphs G_i is complete.

Corollary 1b. *The graph G is an $(n, n - 1)$-adjacency graph if and only if there is a set $\{G_i : i \in I\}$ of subgraphs of G satisfying the following three conditions:*

(i) *Each edge of G lies in exactly n of the subgraphs G_i.*

(ii) *Each vertex of G lies in at most n + 1 of the subgraphs G_i.*

(iii) *The intersection of any n of the subgraphs G_i is either empty or a complete graph.*

Corollary 1a may be applied more or less immediately to hypergraphs, especially those hypergraphs all of whose edges have the same cardinality, say k. Such hypergraphs are called *k-graphs*. A $(k - 1)$-complex gives rise to a k-graph quite naturally by taking the set of its $(k - 1)$-simplexes. The $(k - 1, 0)$-adjacency graph becomes the "representing graph" (as defined by BERGE [2, p. 383]) of the corresponding k-graph and Corollary 2a yields immediately a theorem characterizing representing graphs for k-graphs.

Both corollaries above generalize Krausz's theorem, but in opposite directions, so to speak. Corollary 1b gives a more direct generalization along the lines posed by Grünbaum in [3].

FORBIDDEN SUBGRAPHS OF ADJACENCY GRAPHS

The result of Beineke [1] is now stated as a theorem about 1-complexes.

Theorem B (Beineke): *The graph G is a $(1,0)$-adjacency graph if and only if it contains none of the graphs shown in Figure 1a as an induced subgraph.*

Observe that Figure 1a displays a finite number of graphs, as there are just nine. This theorem obviously offers a more efficient test for whether G is a $(1,0)$-adjacency graph (line graph) than does Theorem A.

A graph G will be called (n, m)-*forbidden*, $n > m$, if G is not an (n, m)-adjacency graph, but every proper induced subgraph of G is such an adjacency graph. Clearly, a graph is an (n, m)-adjacency graph if and only if it contains no (n, m)-forbidden induced subgraphs. Beineke's theorem shows that the number of $(1,0)$-forbidden graphs is finite. Figures 1b and 1c display several $(2,1)$- and $(2,0)$-forbidden graphs respectively. The verification that these graphs are forbidden is tedious and has been omitted.

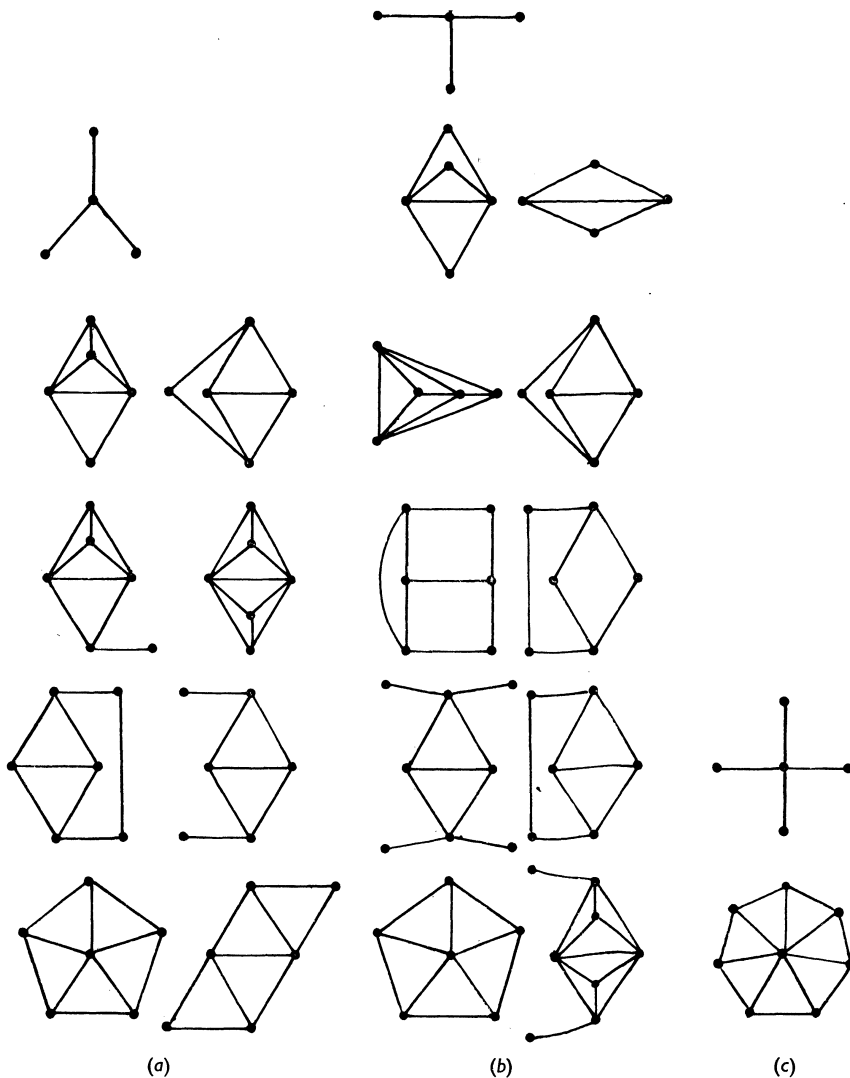It will now be shown that the number of $(2,1)$-forbidden graphs is infinite.

Figure 1. Some (1,0)-, (2,1)- and (2,0)-forbidden graphs.

Lemma 2a. *Let G be a graph having no triangles and let $\{G_i : i \in I\}$ be a collection of subgraphs of G satisfying conditions* (i), (ii), *and* (iii) *of Corollary* 1b *with $n = 2$. Then G also satisfies the following conditions:*

(a) *G has no vertices of degree 4 or more.*

141

(b) *If v has degree 3 in G, then v lies in exactly 3 graphs G_i and has degree 2 in each of them.*

(c) *If v has degree 2 in G, then v lies in exactly 3 graphs G_i and has degree 2 in one of them and degree 1 in two of them.*

Proof. Let v be a vertex of G and let the edges of G incident with v be denoted by $\{v, v_i\}$, $i = 1, 2, ..., k$, $k \geq 2$. By condition (i) of Corollary 1b, $\{v, v_1\}$ lies in exactly two subgraphs G_i, say G_1 and G_2. If $\{v, v_2\} \in G_1 \cap G_2$, then $G_1 \cap G_2$ contains at least three vertices and thus, by condition (ii), at least one triangle. Since G contains no triangles, $\{v, v_2\} \in G_3$. Then v lies in G_1, G_2 and G_3 and hence, by condition (ii), precisely these subgraphs. As already observed, no two edges $\{v, v_1\}$ lie in the same pair of subgraphs G_i. It follows that $k \leq 3$ and this establishes condition (a).

There is no loss of generality in assuming that $\{v, v_2\} \in G_1 \cap G_3$. If $k = 3$, then $\{v, v_3\} \in G_2 \cap G_3$ and condition (b) at v follows immediately. If $k = 2$, then condition (c) at v also follows at once.

Theorem 2. *There are an infinite number of* $(2,1)$*-forbidden graphs.*

Proof. Let G be any graph having no triangles and all but two of its vertices of degree 3. Let the two exceptional vertices, u and v, be non-adjacent and let each have degree 2. Then it is claimed that G is not a $(2,1)$-adjacency graph: if it is, let $\{G_i : i \in I\}$ be a collection of subgraphs of G as in Corollary 1b with $n = 2$. By the foregoing lemma, u has degree 1 in exactly 2 subgraphs of this collection, say G_1 and G_2.
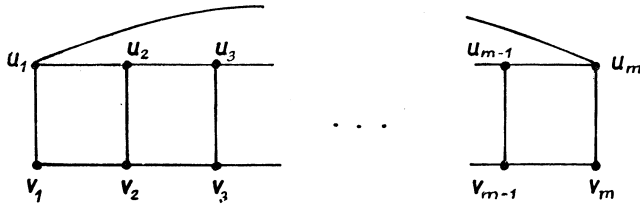

Figure 2. A family of (2,1)-forbidden graphs.

Using both the lemma and this corollary, it is readily inferred that no edge of G incident with u lies in $G_1 \cap G_2$. By condition (b) of the lemma, every vertex $w \neq u, v$ in G is either absent from G_i or has degree 2 there, $i = 1, 2$. Since the number of vertices in G_1 having odd degree is even, both u and v lie in G_1. Similarly, both u and v lie in G_2. By condition (iii) of the corollary, $G_1 \cap G_2$ is a complete graph and, therefore, contains the edge $\{u, v\}$. But $\{u, v\}$ is not even an edge of G and this contradiction proves the claim.

It will now be shown that all graphs represented in Figure 2 below are $(2,1)$-forbidden. For each $m = 4, 5, 6, ...$ there is a graph $G = G(m)$ which has $2m$
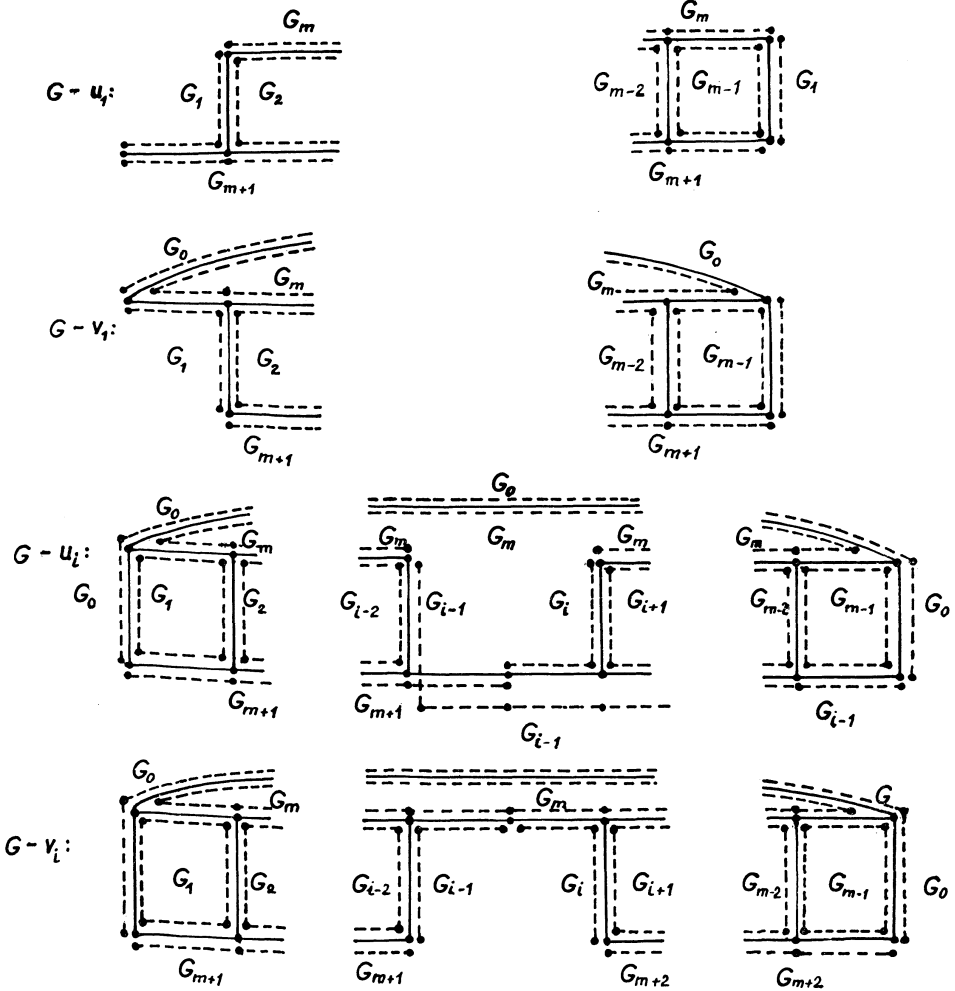
Figure 3. The proper induced subgraphs of G are $(2,1)$-adjacency graphs.

vertices u_i, v_i, $i = 1, 2, \ldots, m$. The edges of G are given by $\{u_i, v_i\}$, $i = 1, 2, \ldots, m$, $\{u_i, u_{i+1}\}$ and $\{v_i, v_{i+1}\}$, $i = 1, 2, \ldots, m - 1$ and $\{u_1, u_m\}$. Clearly G contains no triangles, G contains exactly two (non-adjacent) vertices of degree 2 and all other vertices of G have degree 3. According to the observation above, G is, therefore, not a $(2,1)$-adjacency graph.

Recall that $G - w$ is the subgraph of G induced by all its vertices except w, i.e., the subgraph of G resulting from the removal of vertex w. To show that every proper induced subgraph of G is a $(2,1)$-adjacency graph, we observe first that every proper

143

induced subgraph is isomorphic to one of the following kinds of induced subgraph: $G - u_1$, $G - v_1$, $G - u_i$, $G - v_i$, $1 < i < m$. We observe next that an induced subgraph of a $(2,1)$-adjacency graph is itself a $(2,1)$-adjacency graph. It now remains only to show that these four kinds of proper induced subgraph of G are $(2,1)$-adjacency graphs. This is done by displaying for each in Figure 3 below a collection $\{G_i : i \in I\}$ of subgraphs.

Each subgraph in the above collections is indicated by dashed edges drawn parallel to the corresponding edges of G they are meant to represent. It is observed that each collection $\{G_1 : i \in I\}$ satisfies conditions (i), (ii), and (iii) of Corollary 1b. This completes the proof of Theorem 2.

UNSOLVED PROBLEM

In view of the failure of Beineke's theorem to have such a direct generalization along these lines, it may still be asked whether there might not exist a finite number of easily described classes of $(2,1)$-forbidden graphs. This question is perhaps not unreasonable: for example by allowing m to have the value 3 in the description of $G(m)$ in the proof of Theorem 2, one of the graphs in Figure 1b is obtained. Perhaps each graph in Figure 1b is the simplest representative of a (possibly infinite) class of $(2,1)$-forbidden graphs. On the other hand, it is still possible that there are only a finite number of $(2,0)$-forbidden graphs.

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