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PERIODIC SOLUTIONS OF WEAKLY NONLINEAR WAVE EQUATIONS IN UNBOUNDED INTERVALS

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INTRODUCTION AND NOTATION

1. Introduction. We shall investigate the existence of time periodic solutions of wave equations in one spatial variable. This problem has been studied by a number of authors in the case when the spatial variable runs through a bounded interval and the solutions satisfy various types of boundary conditions, see e.g. J. HALE [1], W. S. HALL [2], P. H. RABINOWITZ [3], O. VEJVODA [4], [5] and numerous papers mentioned there.

Here we shall deal with the problem when the spatial variable runs through an unbounded interval. Transformations make it possible to treat only two cases of unbounded intervals: $[0, +\infty)$ and $(-\infty, +\infty)$. The case of the interval $[0, +\infty)$, in which solutions satisfy a boundary condition at 0 and a growth condition at $+\infty$, seems to be more interesting and more instructive than that of the interval $(-\infty, +\infty)$ at the ends of which solutions are to satisfy only two growth conditions both similar to that imposed at $+\infty$ in the preceding case. Thus we shall deal with the case of the interval $[0, +\infty)$.

Now we shall describe the goal of this paper. We shall look for a Banach space U, a subspace of $C^2_\omega(R \times R^+)$ and a neighbourhood of 0 given in the form $[-\hat{\varepsilon}, \hat{\varepsilon}]$, $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$ there exists a function $u \in U$ satisfying

$$(0.1) \qquad u_{tt}(t, x) - u_{xx}(t, x) = \varepsilon F(u, \varepsilon)(t, x) \quad \text{for} \quad (t, x) \in R \times R^+ ,$$

$$(0.2) \qquad u(t, 0) = a(t) \quad \text{for} \quad t \in R .$$

Denoting this u by $u(\varepsilon)$, we shall deal with only such solutions for which the correspondence $\varepsilon \to u(\varepsilon)$ is a continuous mapping of $[-\hat{\varepsilon}, \hat{\varepsilon}]$ into U. In the described problem the function $a \in C^2_\omega(R)$ is given and F is a Nemyckij operator of the form

$$F(u, \varepsilon)(t, x) = f(t, x, u(t, x), u_t(t, x), u_x(t, x), \varepsilon) .$$

Similarly, we shall deal with the problem described by (0.1) and

$$u_x(t, 0) + \sigma(t)\, u(t, 0) = b(t) \quad \text{for} \quad t \in R \tag{0.3}$$

where the functions σ and b are elements of $C^1_\omega(R)$.

Varying assumptions on the functions a, σ, b and the operator F in the separate cases, we shall show that the described problems have solutions which for fixed ε are elements of either a space of functions which grow at most linearly as x tends to $+\infty$ or a space of bounded functions. Both these spaces consist of functions whose derivatives of orderes 1 and 2 are continuous and bounded.

The restrictions on the growth of functions and their derivatives when x tends to $+\infty$ can be considered only as a remote analogue of the boundary conditions of the problems in which the spatial variable runs through a bounded interval. We shall prove that there is a continuous branch of solutions depending on ε, but the local unicity of this branch, which occurs sometimes in the case of a bounded interval, cannot be proved. On the contrary, there are many such branches (cf. Remark 2.2).

This paper is divided into two sections. In Section 1 the problem is investigated for a linear equation. In Section 2 the results of Section 1 are used to prove the existence of periodic solutions to the equation (0.1) under some general assumptions. In two examples at the end of this section some of these assumptions are shown to be satisfied for certain functions.

The authors express their gratitude to O. Vejvoda, who proposed the present problem.

2. Notation. Throughout the paper we shall use the following notation. We denote by R the set of all real numbers and by Z^+ the set of all nonnegative integers. Further, we put

$$R^+ = [0, +\infty) \quad \text{and} \quad Q = \{(t, x);\ t \in R,\ x \in R^+\}.$$

By $C^2(Q)$ we denote the set of all real functions u defined on Q which have continuous derivatives of the type $D_t^k D_x^l u$, $k, l \in Z^+$, $k + l \leqq 2$ on Q. By $C^{1,0}(Q)$ we denote the set of all real functions u defined on Q which have continuous derivatives of the type $D_t^k u$, $k \in Z^+$, $k \leqq 1$ on Q. Finally, by $C^l(R)$ we denote the set of all functions defined on R which have continuous derivatives up to order l.

By $C^2_\omega(Q)$, $C^{1,0}_\omega(Q)$ and $C^l_\omega(R)$ we denote the subspaces of the corresponding spaces formed by the functions which are ω-periodic in the variable t.

A function $f \in C^0(R)$ is said to be $\frac{1}{2}\omega$-antiperiodic if it satisfies

$$f\left(t + \frac{\omega}{2}\right) = -f(t) \quad \text{for} \quad t \in R.$$

Given a Banach space X, $a \in X$ and $\varrho > 0$, we denote by $B(a, \varrho; X)$ the set $\{x;\ x \in X,\ \|x - a\| \leqq \varrho\}$.

We begin with the following lemma.

Lemma 1.1. *Let* $u \in C^2_\omega(Q)$ *satisfy*

$$\Box\, u(t, x) \equiv u_{tt}(t, x) - u_{xx}(t, x) = 0\,, \quad (t, x) \in Q\,.$$

Then there exist $p, q \in C^2_\omega(R)$ *and* $c \in R$ *such that*

$$u(t, x) = p(t + x) + q(t - x) + 2cx\,, \quad (t, x) \in Q\,.$$

The proof is obvious.

Given $u \in C^2_\omega(Q)$, let us denote

$$\alpha(u) = \sup\{|u(t, x)|\,(1 + x)^{-1};\ (t, x) \in Q\}\,,$$

$$\beta(u) = \sup\{|u(t, x)|;\ (t, x) \in Q\}\,,$$

$$\gamma(u) = \sup\{|D^k_t D^l_x\, u(t, x)|;\ (t, x) \in Q,\ k, l \in Z^+,\ 1 \leqq k + l \leqq 2\}\,,$$

$$\|u\|_{U_1} = \max(\alpha(u), \gamma(u))\,,$$

$$\|u\|_{U_2} = \max(\beta(u), \gamma(u))\,.$$

Finally, let us put

$$U_1 = \{u \in C^2_\omega(Q);\ \|u\|_{U_1} < +\infty\}$$

and

$$U_2 = \left\{u \in C^2(Q);\ \|u\|_{U_2} < +\infty,\ u\left(t + \frac{\omega}{2}, x\right) = -u(t, x),\ (t, x) \in Q\right\}.$$

The spaces U_1 and U_2 endowed with the norms $\|\cdot\|_{U_1}$ and $\|\cdot\|_{U_2}$, respectively, are Banach spaces.

Let $\delta > 0$ (δ is supposed to be fixed throughout this paper). Given $g \in C^{1,0}_\omega(Q)$, let us denote

$$\|g\|_{G_1} = \|g\|_{G_2} = \sup\{(|g(t, x)| + |g_t(t, x)|)(1 + x)^{1+\delta};\ (t, x) \in Q\}\,.$$

Eventually, let us put

$$G_1 = \{g \in C^{1,0}_\omega(Q);\ \|g\|_{G_1} < +\infty\}\,,$$

$$G_2 = \left\{g \in G_1;\ g\left(t + \frac{\omega}{2}, x\right) = -g(t, x),\ (t, x) \in Q\right\}.$$

The spaces G_1 and G_2 equipped with the norms $\|\cdot\|_{G_1}$ and $\|\cdot\|_{G_2}$, respectively, are Banach spaces.

In the sequel we make use of the following operator H. Given $g \in C^{1,0}(Q)$, an element Hg is assigned to g according to the formula

$$(1.1) \qquad (Hg)(t,x) = -\frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} g(\tau,\xi)\,d\tau\,d\xi\,, \quad (t,x) \in Q\,.$$

A straightforward calculation shows that the function Hg satisfies

$$(1.2) \qquad \square\, Hg = g \quad \text{on} \quad Q\,,$$

$$(1.3) \qquad (Hg)(t,0) = (Hg)_x(t,0) = 0\,, \quad t \in R\,,$$

(1.4) if, moreover, $g \in C^{1,0}_{\omega}(Q)$, then $(Hg)(t+\omega,x) = (Hg)(t,x)$, $(t,x) \in Q$.

Lemma 1.2. *The operator H is a linear continuous mapping from G_1 into U_1.*

Proof. Let $g \in G_1$. We shall show that

$$(1.5) \qquad \alpha(Hg) \leqq c_{\alpha}\|g\|_{G_1}$$

and

$$(1.6) \qquad \gamma(Hg) \leqq c_{\gamma}\|g\|_{G_1}$$

where c_α and c_γ are independent of g. Indeed, denoting

$$(\Xi g)(x) = \sup\{|g(t,x)|;\ t \in R\}\,,$$

we have

$$|(Hg)(t,x)|\,(1+x)^{-1} \leqq (2(1+x))^{-1}\int_0^x \int_{t-x+\xi}^{t+x-\xi} |g(\tau,\xi)|\,d\tau\,d\xi \leqq$$

$$\leqq (1+x)^{-1}\int_0^x (x-\xi)(\Xi g)(\xi)d\xi \leqq \|g\|_{G_1}\int_0^\infty (1+\xi)^{-(1+\delta)}\,d\xi\,.$$

This proves (1.5). Further,

$$|(Hg)_x(t,x)| \leqq \frac{1}{2}\int_0^x |g(t+x-\xi,\xi)|\,d\xi + \frac{1}{2}\int_0^x |g(t-x+\xi,\xi)|\,d\xi \leqq$$

$$\leqq \|g\|_{G_1}\int_0^\infty (1+\xi)^{-(1+\delta)}\,d\xi$$

and

$$|(Hg)_{xx}(t,x)| \leqq |g(t,x)| + \frac{1}{2}\int_0^x |g_t(t+x-\xi,\xi)|\,d\xi +$$

$$+ \frac{1}{2}\int_0^x |g_t(t-x+\xi,\xi)|\,d\xi \leqq \|g\|_{G_1}\left(1 + \int_0^\infty (1+\xi)^{-(1+\delta)}\,d\xi\right).$$

These estimates and the similar ones which can be easily obtained for $(Hg)_t$, $(Hg)_{tt}$ and $(Hg)_{tx}$ give (1.6). Hence, the proof is complete.

Lemma 1.3. *The operator H is a linear continuous mapping from G_2 into U_2.*

Proof. The estimate

$$\gamma(Hg) \leqq c_\gamma \|g\|_{G_2}$$

can be obtained similarly as in the proof of Lemma 1.2. Given $g \in G_2$, the relation

$$(Hg)\left(t + \frac{\omega}{2}, x\right) = -(Hg)(t, x) \quad \text{for all} \quad (t, x) \in Q$$

can be proved easily. As soon as we have derived the estimate

$$\beta(u) \leqq c_\beta \|g\|_{G_2} \tag{1.7}$$

with c_β independent of g, the lemma will be proved. To simplify the forthcoming formulae we put

$$(J\sigma)(t) = \int_0^t \sigma(\eta)\,\mathrm{d}\eta - \frac{1}{2}\int_0^{\omega/2} \sigma(\eta)\,\mathrm{d}\eta\,, \quad t \in R$$

for an $\frac{1}{2}\omega$ – antiperiodic and continuous function σ. The function $J\sigma$ is also $\frac{1}{2}\omega$ – antiperiodic and satisfies

$$|(J\sigma)(t)| \leqq c_J \|\sigma\|\,, \quad t \in R\,,$$

where

$$\|\sigma\| = \sup\{|\sigma(t)|;\ t \in R\}$$

and c_J is a constant independent of σ. We now have

$$|(Hg)(t, x)| = \frac{1}{2}\int_0^x \left|\int_{t-x+\xi}^{t+x-\xi} g(\tau, \xi)\,\mathrm{d}\tau\right| \mathrm{d}\xi \leqq$$

$$\leqq \frac{1}{2}\int_0^x \{|(Jg(\cdot, \xi))(t + x - \xi)| + |(Jg(\cdot, \xi))(t - x + \xi)|\}\,\mathrm{d}\xi \leqq$$

$$\leqq c_J \int_0^x (\Xi g)(\xi)\,\mathrm{d}\xi \leqq c_J \|g\|_{G_2} \int_0^\infty (1 + \xi)^{-(1+\delta)}\,\mathrm{d}\xi$$

where $(\Xi g)(\xi)$ is defined in the proof of Lemma 1.2. Thus, the proof is complete

Lemma 1.4. *Let $a \in C^2_\omega(R)$.*

a) *Then the set of all functions $u \in U_1$ satisfying*

$$\Box u = 0 \quad on \quad Q\,, \tag{1.8}$$

$$u(t, 0) = a(t)\,, \quad t \in R \tag{1.9}$$

coincides with the set of functions of the form

$$u(t, x) = \int_{t-x}^{t+x} s(\tau)\,d\tau + a(t - x)\,, \quad (t, x) \in Q \tag{1.10}$$

where $s \in C^1_\omega(R)$ *is arbitrary.*

b) *If, moreover,* a *is* $\frac{1}{2}\omega$ *– antiperiodic, then the set of all functions* $u \in U_2$ *satisfying* (1.8), (1.9) *coincides with the set of all functions of the form* (1.10) *where* s *is an arbitrary* $\frac{1}{2}\omega$ *– antiperiodic function from* $C^1_\omega(R)$.

The proof is similar to that of the following lemma and is omitted.

Lemma 1.5. *Let* σ, $b \in C^1_\omega(R)$.

a) *Then the set of all functions* $u \in U_1$ *satisfying*

$$\Box u = 0 \ \text{on}\ Q\,, \tag{1.11}$$

$$u_x(t, 0) + \sigma(t)\,u(t, 0) = b(t)\,, \quad t \in R \tag{1.12}$$

coincides with the set of functions of the form

$$u(t, x) = \frac{1}{2}\left\{s(t + x) + s(t - x) + \int_{t-x}^{t+x} (b(\tau) - \sigma(\tau)\,s(\tau))\,d\tau\right\}, \quad (t, x) \in Q \tag{1.13}$$

where $s \in C^2_\omega(R)$ *is arbitrary.*

b) *If, moreover,* σ *is* $\frac{1}{2}\omega$ *– periodic and* b *is* $\frac{1}{2}\omega$ *– antiperiodic, then the set of all functions* $u \in U_2$ *satisfying* (1.11) *and* (1.12) *coincides with the set of functions of the form* (1.13) *where* s *is an arbitrary* $\frac{1}{2}\omega$ *– antiperiodic function from* $C^2_\omega(R)$.

Proof. a) Let $u \in U_1$ satisfy (1.11) and (1.12). By Lemma 1.1 there exist $p, q \in C^2_\omega(R)$ and $c \in R$ such that

$$u(t, x) = p(t + x) + q(t - x) + 2cx\,, \quad (t, x) \in Q\,. \tag{1.14}$$

Putting this expression into (1.12), we have

$$p'(t) - q'(t) + 2c + \sigma(t)\,(p(t) + q(t)) = b(t)\,, \quad t \in R\,. \tag{1.15}$$

Setting $s = p + q$ and $r = p - q$, we obtain

$$u(t, x) = \frac{1}{2}\left\{s(t + x) + s(t - x) + \int_{t-x}^{t+x} (r'(\tau) + 2c)\,d\tau\right\}$$

from (1.14) and

$$r'(t) = b(t) - \sigma(t)\,s(t) - 2c$$

from (1.15). Hence u is given by (1.13).

Conversely, one easily verifies that u given by (1.13) satisfies (1.11) and (1.12).

b) In this case $u \in U_2$ is given by (1.14) with $c = 0$. The $\frac{1}{2}\omega$ – antiperiodicity of the function u implies that the function s defined in part a) of the proof is $\frac{1}{2}\omega$ – antiperiodic.

This completes the proof.

Remark 1.1. Lemmas 1.2–1.5 show that the problem given by

$$\Box\, u(t, x) = g(t, x), \quad (t, x) \in Q$$

and either (1.9) or (1.12) has a solution $u \in U_1$ provided $a \in C^2_\omega(R)$, σ, $b \in C^1_\omega(R)$ and $g \in G_1$. The solution is not determined uniquely; it depends on an arbitrary function from $C^1_\omega(R)$ or from $C^2_\omega(R)$, respectively. If, moreover, a, b are $\frac{1}{2}\omega$ – antiperiodic, σ is $\frac{1}{2}\omega$ – periodic and $g \in G_2$, then the problem has a solution $u \in U_2$; it depends on an arbitrary $\frac{1}{2}\omega$ – antiperiodic function from $C^1_\omega(R)$ or $C^2_\omega(R)$, respectively.

The problem on the whole space given by

$$\Box\, u(t, x) = g(t, x), \quad (t, x) \in R^2$$

can be solved in the spaces similar to U_i $(i = 1, 2)$ if they are modified in a natural manner together with the spaces G_i. The solution then depends on an arbitrary function from $C^2_\omega(R)$ and an arbitrary function from $C^1_\omega(R)$ (or, which is the same, on two arbitrary functions from $C^2_\omega(R)$ and a constant $c \in R$).

Remark 1.2. The solution of the classical Cauchy problem

$$\Box\, u(t, x) = g(t, x), \quad (t, x) \in R^2,$$

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x), \quad x \in R$$

is given by the well-known formula

$$u(t, x) = \frac{1}{2}\left(\varphi(x + t) + \varphi(x - t) + \int_{x-t}^{x+t} \psi(\xi)\, d\xi\right) + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} g(\tau, \xi)\, d\xi\, d\tau. \tag{1.16}$$

If we interchange the roles of t and x, we easily obtain that the solution of the problem

$$\Box\, u(t, x) = g(t, x), \quad (t, x) \in R^2,$$

$$u(t, 0) = \alpha(t). \quad u_x(t, 0) = \beta(t), \quad t \in R$$

has the form

$$u(t, x) = \frac{1}{2}\left(\alpha(t + x) + \alpha(t - x) + \int_{t-x}^{t+x} \beta(\tau)\, d\tau\right) - \frac{1}{2}\int_0^x \int_{t-x+\xi}^{t+x-\xi} g(\tau, \xi)\, d\tau\, d\xi. \tag{1.17}$$

The formula (1.17) forms the starting point of the present investigation. Looking for an ω-periodic solution we have noticed these facts:

i) the functions α and β are to be ω-periodic;

ii) the operator (1.1) occurring on the right hand side of (1.17) has the excellent property (1.4).

If the problem is considered for $x \in R$ we could employ also directly the formula (1.16) and look for φ and ψ for which u would be ω-periodic. This approach is rather more complicated since it requires to solve the functional equations

$$(1.18) \qquad m(x + \omega) - m(x) + \frac{1}{2}\int_0^x \int_{-\omega}^0 g(\tau, \xi - \tau)\, d\tau\, d\xi =$$

$$= \text{const} = n(x) - n(x - \omega) +$$

$$+ \frac{1}{2}\int_0^x \int_{-\omega}^0 g(\tau, \xi + \tau)\, d\tau\, d\xi - \frac{1}{2}\int_{-\omega}^0 \int_{\tau}^{-\tau} g(\tau, \xi)\, d\xi\, d\tau\,, \quad x \in R$$

for

$$m(x) = \frac{1}{2}\left(\varphi(x) + \int_0^x \psi(\xi)\, d\xi\right),$$

$$n(x) = \frac{1}{2}\left(\varphi(x) - \int_0^x \psi(\xi)\, d\xi\right).$$

One finds that the general solutions read

$$(1.19) \qquad m(x) = \mu(x) + cx - \frac{1}{2}\int_0^x \int_0^{\xi} g(\tau, \xi - \tau)\, d\tau\, d\xi$$

and

$$(1.20) \qquad n(x) = \nu(x) + cx - \frac{1}{2}\int_0^x \int_{-\xi}^0 g(\tau, \xi + \tau)\, d\tau\, d\xi$$

respectively, where μ, ν are ω-periodic and $c \in R$. Inserting (1.19), (1.20) into (1.16), we obtain the formula (1.17) with

$$\alpha(t) = \mu(t) + \nu(-t)\,, \quad \beta(t) = \mu'(t) + \nu'(-t) + 2c\,, \quad t \in R\,.$$

2. WEAKLY NONLINEAR CASE

In this part we shall use the following lemma the proof of which can be obtained directly.

Lemma 2.1. *Let U be a Banach space and let G be a normed space, $u_0 \in U$, $\varrho > 0$, $\varepsilon_0 > 0$ and $\lambda > 0$. Let $\tilde{u}$ be a continuous mapping from $[-\varepsilon_0, \varepsilon_0]$ into U*

such that $\tilde{u}(0) = u_0$. *Let H be a linear continuous mapping from G into U. Let F be a continuous mapping from* $B(u_0, \varrho; U) \times [-\varepsilon_0, \varepsilon_0]$ *into G satisfying*

$$\|F(u_1, \varepsilon) - F(u_2, \varepsilon)\|_G \leqq \lambda \|u_1 - u_2\|_U$$

for all $u_1, u_2 \in B(u_0, \varrho; U)$ *and* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Then there exists $\hat{\varepsilon} \in (0, \varepsilon_0]$ *such that for every* $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$ *there is a unique* $u \in B(u_0, \varrho; U)$ *satisfying*

$$u = \tilde{u}(\varepsilon) + \varepsilon HF(u, \varepsilon).$$

Moreover, the correspondence $\varepsilon \mapsto u$ *is a continuous mapping from* $[-\hat{\varepsilon}, \hat{\varepsilon}]$ *into* $B(u_0, \varrho; U)$.

Now, we formulate the assumptions which guarantee that the operator F given by

$$F(u, \varepsilon)(t, x) = f(t, x, u(t, x), u_t(t, x), u_x(t, x), \varepsilon) \tag{2.1}$$

possesses the following two properties ($i = 1, 2$):

(2.2)$_i$ F is a continuous mapping from $B(\bar{u}_i, \varrho; U_i) \times [-\varepsilon_0, \varepsilon_0]$ into G_i;

(2.3)$_i$ $\|F(u, \varepsilon) - F(v, \varepsilon)\|_{G_i} \leqq \lambda \|u - v\|_{U_i}$ for all $u, v \in B(\bar{u}_i, \varrho; U_i)$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

where $\bar{u}_i \in U_i$, $\varrho > 0$, $\lambda > 0$, $\varepsilon_0 > 0$ are fixed. We start by introducing three conditions which characterize the properties of a function f.

A function f is said to satisfy the condition (A_r^1) if it is defined on

$$M_r = \{(t, x, u_0, u_1, u_2, \varepsilon);\ t \in R,\ x \in R^+,\ |u_0| \leqq (1 + x) r,\ |u_1| \leqq r,\ |u_2| \leqq r,\ |\varepsilon| \leqq \varepsilon_0\}$$

and

$$\sup \{|f(t, x, u_0, u_1, u_2, \varepsilon)| (1 + x)^{1+\delta};\ (t, x, u_0, u_1, u_2, \varepsilon) \in M_r\} < +\infty .$$

A function f is said to satisfy the condition (A_r^2) if it is defined on M_r and

$$\sup \{|f(t, x, u_0, u_1, u_2, \varepsilon)| (1 + x)^{2+\delta};\ (t, x, u_0, u_1, u_2, \varepsilon) \in M_r\} < +\infty .$$

A function f is said to satisfy the condition (A_r^3) if it is defined on M_r and

$$\limsup_{\varepsilon \to \bar{\varepsilon}} \{|f(t, x, u_0, u_1, u_2, \varepsilon) - f(t, x, u_0, u_1, u_2, \bar{\varepsilon})| (1 + x)^{1+\delta};\ t \in R,\ x \in R^+,\ |u_0| \leqq (1 + x) r,\ |u_1| \leqq r,\ |u_2| \leqq r\} = 0$$

for all $\bar{\varepsilon} \in [-\varepsilon_0, \varepsilon_0]$.

Lemma 2.2. *Let $r > 0$ and let the following assumptions be satisfied:*

(i) *All the functions*

$$D_t^k D_{u_1}^{l_1} D_{u_2}^{l_2} f, \quad k, l_i \in Z^+ , \quad k + l_1 + l_2 \leqq 2 , \quad k \leqq 1$$

satisfy the condition (A_r^1).

(ii) *All the functions*

$$D_t^k D_{u_0}^{l_0} D_{u_1}^{l_1} D_{u_2}^{l_2} f, \quad k, l_i \in Z^+ , \quad k + l_0 + l_1 + l_2 \leqq 2 , \quad l_0 \geqq 1 , \quad k \leqq 1$$

satisfy the condition (A_r^2).

(iii) *All the functions*

$$D_t^k D_{u_0}^{l_0} D_{u_1}^{l_1} D_{u_2}^{l_2} f, \quad k, l_i \in Z^+ , \quad k + l_0 + l_1 + l_2 \leqq 1$$

satisfy the condition (A_r^3).

(iv) *The function f is ω-periodic in t.*

Then the operator F satisfies $(2.2)_1$ *and* $(2.3)_1$ *provided* $\|\bar{u}_1\|_{U_1} + \varrho < r$.

Proof. The assumptions of the lemma imply that $F(u, \varepsilon) \in G_1$ for every $u \in$ $\in B(\bar{u}_1, \varrho; U_1)$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and

$$(2.4) \qquad \lim_{\varepsilon \to \bar{\varepsilon}} \|F(u, \varepsilon) - F(u, \bar{\varepsilon})\|_{G_1} = 0$$

for every $u \in B(\bar{u}_1, \varrho; U_1)$ and $\bar{\varepsilon} \in [-\varepsilon_0, \varepsilon_0]$.

The inequality $(2.3)_1$ follows immediately from the relation

$$F(u_1, \varepsilon)(t, x) - F(u_2, \varepsilon)(t, x) =$$

$$= (u_1 - u_2) \int_0^1 f_u(t, x, u_2 + \sigma(u_1 - u_2), u_{1t}, u_{1x}, \varepsilon) \, d\sigma +$$

$$+ (u_1 - u_2)_t \int_0^1 f_{u_t}(t, x, u_2, u_{2t} + \sigma(u_1 - u_2)_t, u_{1x}, \varepsilon) \, d\sigma +$$

$$+ (u_1 - u_2)_x \int_0^1 f_{u_x}(t, x, u_2, u_{2t}, u_{2x} + \sigma(u_1 - u_2)_x, \varepsilon) \, d\sigma .$$

Now (2.4) and $(2.3)_1$ yield $(2.2)_1$. This completes the proof.

The following conditions define again some properties of a function f.

A function f is said to satisfy the condition (B_r^1) if it is defined on

$$N_r = R \times R^+ \times [-r, r]^3 \times [-\varepsilon_0, \varepsilon_0]$$

and

$$\sup \{|f(t, x, u_0, u_1, u_2, \varepsilon)| (1 + x)^{1+\delta};\ (t, x, u_0, u_1, u_2, \varepsilon) \in N_r\} < +\infty .$$

A function f is said to satisfy the condition (B_r^2) if it is defined on N_r and

$$\limsup_{\varepsilon \to \bar{\varepsilon}} \{|f(t, x, u_0, u_1, u_2, \varepsilon) - f(t, x, u_0, u_1, u_2, \bar{\varepsilon})| (1 + x)^{1+\delta} ;$$
$$t \in R,\ x \in R^+,\ |u_i| \leqq r,\ i = 0, 1, 2\} = 0 \quad \text{for all} \quad \bar{\varepsilon} \in [-\varepsilon_0, \varepsilon_0] .$$

Lemma 2.3. *Let $r > 0$ and let the following assumptions be satisfied:*
(i) *All the functions*

$$D_t^k D_{u_0}^{l_0} D_{u_1}^{l_1} D_{u_2}^{l_2} f , \quad k, l_i \in Z^+ , \quad k + l_0 + l_1 + l_2 \leqq 2 , \quad k \leqq 1$$

satisfy the condition (B_r^1).
(ii) *All the functions*

$$D_t^k D_{u_0}^{l_0} D_{u_1}^{l_1} D_{u_2}^{l_2} f , \quad k, l_i \in Z^+ , \quad k + l_0 + l_1 + l_2 \leqq 1$$

satisfy the condition (B_r^2).

(iii) $$f\left(t + \frac{\omega}{2}, x, -u_0, -u_1, -u_2, \varepsilon\right) = -f(t, x, u_0, u_1, u_2, \varepsilon) \quad \text{on} \quad N_r .$$

Then the operator F satisfies $(2.2)_2$ and $(2.3)_2$ provided $\|\bar{u}_2\|_{U_2} + \varrho < r$.
The proof is similar to that of Lemma 2.2.

Theorem 2.1. *Let $r > 0$ and let $\bar{u}_1 \in U_1$ be a function satisfying*

$$\Box \bar{u}_1 = 0 \quad on \quad Q , \quad \bar{u}_1(t, 0) = a(t) , \quad t \in R$$

and $r > \|\bar{u}_1\|_{U_1}$. Let a function f satisfy the assumptions of Lemma 2.2.
Then there exists a continuous mapping $u : [-\hat{\varepsilon}, \hat{\varepsilon}] \to U_1$, $0 < \hat{\varepsilon} \leqq \varepsilon_0$ satisfying

(2.5) $$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \quad on \quad Q$$

and

(2.6) $$u(t, 0) = a(t) , \quad t \in R$$

for every $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$. Moreover,

(2.7) $$u(\varepsilon)|_{\varepsilon=0} = \bar{u}_1 .$$

Proof. The relations (1.2) and (1.3) show that a function $u : [-\hat{\varepsilon}, \hat{\varepsilon}] \to U_1$ satisfying

(2.8) $$u = \bar{u}_1 + \varepsilon H F(u, \varepsilon)$$

is a solution to (2.5) and (2.6). Hence applying Lemma 2.1 to (2.8) we obtain the assertion of the theorem.

Similarly we obtain the next three theorems.

Theorem 2.2. *Let $r > 0$ and let $\bar{u}_1 \in U_1$ be a function satisfying*

$$\Box\, \bar{u}_1 = 0 \quad on \quad Q\,, \qquad \bar{u}_{1x}(t, 0) + \sigma(t)\, \bar{u}_1(t, 0) = b(t)\,, \quad t \in R$$

and $r > \|\bar{u}_1\|_{U_1}$. Let a function f satisfy the assumptions of Lemma 2.2.
Then there exists a continuous mapping $u : [-\hat{\varepsilon}, \hat{\varepsilon}] \to U_1$, $0 < \hat{\varepsilon} \leqq \varepsilon_0$ satisfying

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \quad on \quad Q$$

and

$$u_x(t, 0) + \sigma(t)\, u(t, 0) = b(t)\,, \quad t \in R$$

for every $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$. Moreover,

$$u(\varepsilon)|_{\varepsilon=0} = \bar{u}_1\,.$$

Theorem 2.3. *Let $r > 0$ and let $\bar{u}_2 \in U_2$ be a function satisfying*

$$\Box\, \bar{u}_2 = 0 \quad on \quad Q\,, \qquad \bar{u}_2(t, 0) = a(t)\,, \quad t \in R$$

and $r > \|\bar{u}_2\|_{U_2}$. Let a function f satisfy the assumptions of Lemma 2.3.
Then there exists a continuous mapping $u : [-\hat{\varepsilon}, \hat{\varepsilon}] \to U_2$, $0 < \hat{\varepsilon} \leqq \varepsilon_0$ satisfying

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \quad on \quad Q$$

and

$$u(t, 0) = a(t)\,, \quad t \in R$$

for every $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$. Moreover,

$$u(\varepsilon)|_{\varepsilon=0} = \bar{u}_2\,.$$

Theorem 2.4. *Let $r > 0$ and let $\bar{u}_2 \in U_2$ be a function satisfying*

$$\Box\, \bar{u}_2 = 0 \quad on \quad Q\,, \qquad \bar{u}_{2x}(t, 0) + \sigma(t)\, \bar{u}_2(t, 0) = b(t)\,, \quad t \in R$$

and $r > \|\bar{u}_2\|_{U_2}$. Let a function f satisfy the assumptions of Lemma 2.3.
Then there exists a continuous mapping $u : [-\hat{\varepsilon}, \hat{\varepsilon}] \to U_2$, $0 < \hat{\varepsilon} \leqq \varepsilon_0$ satisfying

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon) \quad on \quad Q$$

and

$$u_x(t, 0) + \sigma(t)\, u(t, 0) = b(t)\,, \quad t \in R$$

for every $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$. Moreover,

$$u(\varepsilon)|_{\varepsilon=0} = \bar{u}_2\,.$$

Remark 2.1. In Lemma 1.4 and Lemma 1.5, assumptions under which the functions $\bar{u}_1$ and $\bar{u}_2$ from Theorems 2.1, 2.2, 2.3, 2.4 exist are formulated.

Remark 2.2. Let $u_1 \in U_1$ be a function which does not equal zero identically and which satisfies

$$\square u_1 = 0 \quad \text{on} \quad Q\,, \qquad u_1(t, 0) = 0\,, \quad t \in R\,.$$

Let $\bar{u}_1$ be the function from Theorem 2.1. Let a function u satisfy

$$u = \bar{u}_1 + \varepsilon u_1 + \varepsilon HF(u, \varepsilon)$$

on an interval $[-\hat{\varepsilon}_1, \hat{\varepsilon}_1]$, $\hat{\varepsilon}_1 > 0$. Then this function satisfies (2.5), (2.6) and (2.7) but does not coincide with that obtained from (2.8). Hence the branch of solutions guaranteed by Theorem 2.1 is not unique.

Example 2.1. Let p be a positive integer. Let

$$f(t, x, u_0, u_1, u_2, \varepsilon) = \sum_{\substack{k,m,n \in Z^+ \\ k+m+n \leq p}} b_{k,m,n}(t, x)\, u_0^k u_1^m u_2^n$$

with $b_{k,m,n} \in C_\omega^{1,0}(Q)$ satisfying

$$\sup \{(|b_{k,m,n}(t, x)| + |D_t^1\, b_{k,m,n}(t, x)|)\, (1 + x)^{k+1+\delta};\ (t, x) \in Q\} < +\infty\,.$$

Then the assumptions of Lemma 2.2 are satisfied for an arbitrary $r > 0$.

Example 2.2. Let $b \in C_\omega^1(R)$, $a_0 \in R$, $\alpha > 0$. Let φ be a twice continuously differentiable function on R^2.

Then the function

$$f(t, x, u_0, u_1, u_2, \varepsilon) = b(t) \exp(-\alpha x + a_0 u_0)\, \varphi(u_1, u_2)$$

satisfies the assumptions of Lemma 2.2 provided

$$|a_0|\, r < \alpha\,.$$

References

[1] *J. K. Hale:* Periodic solutions of a class of hyperbolic equations containing a small parameter. Archive Rat. Mech. Anal *23* (1967), 380—398.

[2] *W. S. Hall:* Periodic solutions of a class of weakly nonlinear evolution equations. Archive Rat. Mech. Anal. *39* (1970), 294—322.

[3] *P. H. Rabinowitz:* Periodic solutions of nonlinear hyperbolic partial differential equations. Comm. Pure Appl. Math. *20* (1967), 145—206.

[4] *O. Vejvoda:* Periodic solutions of a linear and weakly nonlinear wave equation in one dimension, I. Czechoslovak Math. J. *14* (*89*) (1964), 341—382.

[5] *O. Vejvoda:* Periodic solutions of nonlinear partial differential equations of evolution. Equadiff II, Differential Equations and their Applications, Proceedings of the Conference, Bratislava 1966, published in Acta Facultatis Rerum Naturalium Universitatis Comenianae, Mathetica, *17* (1967), 293—300.

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