

Cony M. Lau; Thomas L. Markham

LU factorizations

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 4, 546–550

Persistent URL: <http://dml.cz/dmlcz/101635>

Terms of use:

© Institute of Mathematics AS CR, 1979

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

LU FACTORIZATIONS

CONY M. LAU, THOMAS L. MARKHAM, Columbia

(Received June 26, 1977)

I. INTRODUCTION

Suppose A is an $n \times n$ matrix over the complex field. The problem of factoring A into a product LU , where L is a lower-triangular matrix, and U is an upper-triangular matrix with specified diagonal, is of importance in solving systems of linear equations and in the construction of compact schemes for matrix inversion [1]. In fact, Gaussian elimination is concerned with effecting an LU factorization.

We shall concern ourselves with the following problem. Suppose A is an $n \times n$ matrix. What are necessary and sufficient conditions that A can be factored as LU , where U has a diagonal consisting entirely of ones?

II. LU FACTORIZATIONS

Let α and β be increasing sequences on $\{1, \dots, n\}$. We shall use the following notation. $A(\alpha | \beta)$ denotes the minor of A with rows indexed by α and columns indexed by β . $A[\alpha | \beta]$ is the submatrix of A contained in rows α and columns β . A principal minor is written $A(\alpha)$, and a principal submatrix $A[\alpha]$. $\hat{\alpha}$ is the complement of α . Finally, $R(\cdot)$ denotes the range space.

Theorem 1. *An $n \times n$ matrix A has an LU -factorization with $u_{ii} = 1$ iff $A[n^\wedge]$ has an $\tilde{L}\tilde{U}$ -factorization with $\tilde{u}_{ii} = 1$ and $A[n^\wedge | n] \in R(\tilde{L}) = R(A[n^\wedge])$.*

Proof. Assume A has an LU -factorization. Partition L , U , and A conformally so that

$$LU = \begin{pmatrix} L_{11} & Z^T \\ L_{21} & l \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ Z & 1 \end{pmatrix} = A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix}$$

where Z is a $1 \times (n-1)$ zero vector and L_{11}, U_{11}, A_{11} are $(n-1) \times (n-1)$ in dimension. Let $\tilde{L} = L_{11}$ and $\tilde{U} = U_{11}$, then $A[n^\wedge] = A_{11} = \tilde{L}\tilde{U}$ and $\tilde{L}U_{12} = A_{12}$.

i.e. $A[n^\wedge | n] \in \mathcal{R}(\tilde{L})$. Note that since $A[n^\wedge] = \tilde{L}\tilde{U}$, $\tilde{L} = A[n^\wedge]\tilde{U}^{-1}$ and $\mathcal{R}(\tilde{L}) = \mathcal{R}(A[n^\wedge])$. On the other hand assume $A[n^\wedge]$ has an $\tilde{L}\tilde{U}$ -factorization with $\tilde{u}_{ii} = 1$ and partition A as above. Since $A_{12} = A[n^\wedge | n] \in \mathcal{R}(\tilde{L})$, there exists V such that $\tilde{L}V = A_{12}$. Since \tilde{U} is nonsingular, there exists an $1 \times (n - 1)$ vector Y such that $Y\tilde{U} = A_{21}$. Choose $l = a_{nn} - YV$. Now let

$$L = \begin{pmatrix} \tilde{L} & Z^T \\ Y & l \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \tilde{U} & V \\ Z & l \end{pmatrix}.$$

Then L is lower triangular and U is upper triangular with $u_{ii} = 1$, and

$$LU = \begin{pmatrix} \tilde{L} & Z^T \\ Y & l \end{pmatrix} \begin{pmatrix} \tilde{U} & V \\ Z & l \end{pmatrix} = \begin{pmatrix} \tilde{L}\tilde{U} & \tilde{L}V \\ Y\tilde{U} & YV + l \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} = A.$$

Theorem 2. *An $n \times n$ matrix A has an LU -factorization with $u_{ii} = 1$ iff $A[1, 2, \dots, k | k + 1] \in \mathcal{R}(A[1, 2, \dots, k])$ for $k = 1, 2, \dots, n - 1$.*

Proof. This statement is the result of iteration of the necessary and sufficient condition of Theorem 1. Since A has an LU -factorization, $A[1, \dots, n - 1 | n] \in \mathcal{R}(A[1, \dots, n - 1])$ and $A[1, \dots, n - 1]$ has an LU -factorization. Now the same argument applies to $A[1, \dots, n - 1]$, obtaining $A[1, \dots, n - 2 | n - 1] \in \mathcal{R}(A[1, \dots, n - 2])$ and the process continuous until finally we have $A[1 | 2] \in \mathcal{R}(A[1])$. Conversely $A[1]$ has an LU -factorization trivially and $A[1 | 2] \in \mathcal{R}(A[1])$ implies that $A[1, 2]$ has an LU -factorization, which together with $A[1, 2 | 3] \in \mathcal{R}(A[1, 2])$ implies, in turn, that $A[1, 2, 3]$ has an LU -factorization. The argument repeats until we obtain that A has an LU -factorization.

Corollary 1. *If an $n \times n$ matrix A has an LU -factorization with $u_{ii} = 1$, then $A(1, 2, \dots, k | 1, 2, \dots, k - 1, j) = 0$ whenever $A(1, 2, \dots, k) = 0$ for $1 \leq k \leq n - 1$ and $k < j \leq n$. In particular, if $a_{11} = 0$, then the first row of A must be zero.*

Proof. If $A(1, 2, \dots, k) = 0$, then there is a nontrivial linear relation between the columns of $A[1, \dots, k]$, say $\sum_{i=1}^k C_i A[1, 2, \dots, k | i] = 0$. Suppose $C_k = 0$. Then the first $k - 1$ columns in $A[1, 2, \dots, k]$ are linearly dependent and $A(1, \dots, k | 1, \dots, k - 1, j) = 0$. If $C_k \neq 0$, then the k -th column in $A[1, 2, \dots, k]$ depends on the first $k - 1$ columns. Since it follows from the theorem that $A[1, 2, \dots, k | j] \in \mathcal{R}(A[1, 2, \dots, k])$ for $k < j \leq n$, $A[1, 2, \dots, k | j]$ also depends on these $k - 1$ columns and $A(1, 2, \dots, k | 1, 2, \dots, k - 1, j) = 0$.

The converse of the last corollary is not true. The matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

satisfies the condition of the corollary but not that of Theorem 2 and hence cannot have an LU -factorization.

Theorem 3. *An $n \times n$ matrix A has an LU -factorization with $u_{ii} = 1$ iff $A[1, 2, \dots, k \mid k + 1] \in R(A[1, 2, \dots, k])$ for $m \leq k \leq n - 1$ where m is the smallest positive integer such that $A(1, 2, \dots, m) = 0$.*

Proof. Note that if $A(1, 2, \dots, k) \neq 0$, the columns of $A[1, 2, \dots, k]$ form a basis of k -space and $A[1, 2, \dots, k \mid k + 1] \in R(A[1, 2, \dots, k])$ is automatically satisfied. Thus it follows from the Theorem 2 that A has an LU -factorization with $u_{ii} = 1$ iff $A[1, 2, \dots, k \mid k + 1] \in R(A[1, 2, \dots, k])$ whenever $A(1, 2, \dots, k) = 0$. However, when A has an LU -factorization, $A(1, 2, \dots, k) = L(1, 2, \dots, k) U(1, 2, \dots, k) = \prod_{i=1}^k l_{ii}$. So if $A(1, 2, \dots, m) = 0$ where m is the smallest such integer, necessarily $l_{mm} = 0$ and $A(1, 2, \dots, k) = 0$ for $m \leq k \leq n - 1$. Hence it is required that $A[1, 2, \dots, k \mid k + 1] \in R(A[1, 2, \dots, k])$ for $m \leq k \leq n - 1$.

Corollary 2. *If the proper leading principal minors of an $n \times n$ matrix A are nonzero, then A has an LU -factorization with $u_{ii} = 1$.*

This corollary is well-known (see, for example, [3]). Also we can obtain the following corollary, which is given in Gantmacher ([2], p. 35).

Corollary 3. *Let A be an $n \times n$ matrix of rank r and $A(1, 2, \dots, k) \neq 0$ for $k = 1, 2, \dots, r$. Then A has an LU -factorization in which the last $n - r$ columns of L are zero and $u_{ii} = 1$.*

Proof. Such a matrix A satisfies the conditions of the theorem. It is sufficient to show that $A[1, 2, \dots, k \mid k + 1] \in R(A[1, 2, \dots, k])$ for $k = r + 1, r + 2, \dots, n - 1$. Since $A(1, 2, \dots, r) \neq 0$, it follows that for $k > r$, the first r columns in $A[1, 2, \dots, k]$ are linearly independent. Suppose $A[1, 2, \dots, k \mid k + 1] \notin R(A[1, 2, \dots, k])$; then A would have at least $r + 1$ linearly independent columns, contradicting that $\text{rank}(A) = r$. Thus A has an LU -factorization with $u_{ii} = 1$.

Note that since U is invertible, $\text{rank}(L) = r$. $A(1, 2, \dots, r) = \prod_{i=1}^r l_{ii} \neq 0$ implies that the first r columns of L are linearly independent. Any of the last $n - r$ columns of L being nonzero would imply that L has more than r linearly independent columns, contradicting that $\text{rank}(L) = r$.

Theorem 4. *If an $n \times n$ matrix A has an LU -factorization with $u_{ii} = 1$, L and U are unique iff all proper leading principal minors of A are nonzero.*

Proof. Note that in the conformal partitioning

$$LU = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A,$$

$L_{11}U_{11} = A_{11}$, i.e., if A has an LU -factorization with $u_{ii} = 1$, every leading principal submatrix of A does also. Since $A[1]$ is factored uniquely as $(a_{11})(1)$, it is sufficient, by mathematical induction, to show that whenever $A_k = A[1, 2, \dots, k]$ has a unique factorization, the factorization in A_{k+1} is also unique. Consider the partitioning

$$A_{k+1} = \begin{pmatrix} A_k & C \\ R & a_{k+1,k+1} \end{pmatrix} = L_{k+1}U_{k+1} = \begin{pmatrix} L_k & 0 \\ Y & l \end{pmatrix} \begin{pmatrix} U_k & V \\ 0 & 1 \end{pmatrix}$$

where L_k, U_k are unique with $L_kU_k = A_k$. Note that for $k = 1, 2, \dots, n-1$, L_k is invertible since both U_k and A_k are invertible. Now $C = L_kV$ and $YU_k = R$ imply Y and V are both unique, and $YV + l = a_{k+1,k+1}$ implies l is unique. Thus L_{k+1}, U_{k+1} are unique.

Conversely, in the above partitioning, $C = L_kV$ and V is unique imply that L_k is invertible. Hence $A_k = L_kU_k$ is invertible and $A(1, 2, \dots, k) \neq 0$. This holds true for $k = 1, 2, \dots, n-1$.

Theorem 5. *An $n \times n$ matrix A with $a_{11} = 0$ has an LU -factorization with $l_{ii} = 0$ and $u_{ii} = 1$ iff the first row of A is zero and $A[1^\wedge | n^\wedge]$ has an $\tilde{L}\tilde{U}$ -factorization with $\tilde{u}_{ii} = 1$ and $A[1^\wedge | n] \in R(\tilde{L}) = R(A[1^\wedge | n^\wedge])$. Furthermore, such a factorization is unique iff all leading principal minors of $A[1^\wedge | n^\wedge]$ are nonzero.*

Proof. First, assume A has an LU -factorization. By Corollary 1 the first row of L is necessarily zero. Partition L, U and A conformally so that

$$LU = \begin{pmatrix} Z & 0 \\ L_{21} & Z^T \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ Z & 1 \end{pmatrix} = A = \begin{pmatrix} Z & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

where Z is an $1 \times (n-1)$ zero vector and L_{21}, U_{11}, A_{21} are all of dimension $(n-1) \times (n-1)$. Then $A[1^\wedge | n^\wedge] = A_{21} = L_{21}U_{11}$. Let $\tilde{L} = L_{21}$ and $\tilde{U} = U_{11}$ implies $A[1^\wedge | n^\wedge]$ has an $\tilde{L}\tilde{U}$ -factorization with $\tilde{u}_{ii} = 1$. Also $A[1^\wedge | n] = A_{22} = L_{21}U_{12}$ implies $A[1^\wedge | n] \in R(L_{21}) = R(\tilde{L})$. Note that since $A_{21} = \tilde{L}\tilde{U}$ and \tilde{U} is nonsingular, $\tilde{L} = A_{21}\tilde{U}^{-1}$ and $R(\tilde{L}) = R(A_{21}) = R(A[1^\wedge | n^\wedge])$. For the converse, partition A into $\begin{pmatrix} Z & 0 \\ A_{21} & A_{22} \end{pmatrix}$ as above. Then $A_{21} = A[1^\wedge | n^\wedge]$ has an $\tilde{L}\tilde{U}$ -factorization with $\tilde{u}_{ii} = 1$. Since $A_{22} = A[1^\wedge | n] \in R(\tilde{L})$, there exists V such that $\tilde{L}V = A_{22}$. Let

$$L = \begin{pmatrix} Z & 0 \\ \tilde{L} & Z^T \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \tilde{U} & V \\ Z & 1 \end{pmatrix}.$$

Then L is lower triangular with $l_{ii} = 0$ and U is upper triangular with $u_{ii} = 1$, and

$$LU = \begin{pmatrix} Z & 0 \\ \tilde{L} & Z^T \end{pmatrix} \begin{pmatrix} \tilde{U} & V \\ Z & 1 \end{pmatrix} = \begin{pmatrix} Z & 0 \\ \tilde{L}\tilde{U} & \tilde{L}V \end{pmatrix} = \begin{pmatrix} Z & 0 \\ A_{21} & A_{22} \end{pmatrix} = A.$$

For the uniqueness statement, consider L , U and A partitioned as in the last equation. If all leading minors of A_{21} are nonzero, it follows from the Theorem 4 that \tilde{L} , \tilde{U} are unique. Also A_{21} is invertible implies \tilde{L} is invertible, so it follows from $\tilde{L}V = A_{22}$ that V is unique. Hence L and U are unique. On the other hand, assume that L , U are unique, then \tilde{L} , \tilde{U} are unique. So by the last theorem all proper leading principal minors of $A_{21} = \tilde{L}\tilde{U}$ are nonzero. Also V is unique and $\tilde{L}V = A_{22}$ implies that \tilde{L} is invertible. Hence A_{21} is invertible and $\det A_{21} \neq 0$. This completes the proof.

Example.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 5 & 4 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

References

- [1] *V. N. Faddeeva*: Computational Methods of Linear Algebra, Dover, New York, 1959.
- [2] *F. R. Gantmacher*: The Theory of Matrices, Vol. 1, Chelsea, New York, 1959.
- [3] *A. S. Householder*: Lectures on Numerical Algebra, Mathematical Association of America, 1972.

Author's address: Department of Mathematics and Computer Science, University of South Carolina, Columbia, South Carolina 29208, U.S.A.