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ON INTEGRATION IN BANACH SPACES, IV

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INTRODUCTION

In this part of our theory of integration of vector valued functions with respect to operator valued measures we prove a general convergence theorem (Theorem 1), which we call the diagonal convergence theorem, and we give some applications of it. These applications are concerned with integrals of unconditionally convergent series of functions and measures (§ 3), and with products and double integrals of sequences and series of measures (§ 4). In § 2 we collect miscellaneous results related to the material of the other sections.

We shall use the notations and concepts of the previous parts I, II and III, see [5], [6] and [7]. Particularly, $\mathscr{P}$ and $\mathscr{Q}$ are δ-rings of subsets of non empty sets T and S, respectively. $\mathfrak{S}(\mathscr{P})$ denotes the smallest σ-ring which contains $\mathscr{P}$. $\boldsymbol{X}, \boldsymbol{Y}$, and $\boldsymbol{Z}$ are Banach spaces over the same scalars. By $B(T, \boldsymbol{X})$ we denote the Banach space of all bounded functions $\boldsymbol{f}: T \to \boldsymbol{X}$ with the supremum norm. If it is not specified otherwise, I and J stand for non empty sets of indices. By Φ_1 we denote the collection of all finite subsets of $\omega = \{1, 2, \ldots\}$, and by Φ_2 the collection of all finite subsets of $\omega \times \omega$.

1. DIAGONAL CONVERGENCE THEOREM

In this section we shall use the following three assumptions:

(a_1): *Let $\boldsymbol{f}_k : T \to \boldsymbol{X}$, $k = 1, 2, \ldots$, be a sequence of $\mathscr{P}$-measurable functions, and let $\boldsymbol{f}_k(t) \to \boldsymbol{f}(t) \in \boldsymbol{X}$ for each $t \in T$.*

Since a pointwise limit of a sequence of $\mathscr{P}$-measurable functions is $\mathscr{P}$-measurable, see Section 1.2 in Part I and Lemma 1.2 in [14], the function $\boldsymbol{f}: T \to \boldsymbol{X}$ in (a_1) is $\mathscr{P}$-measurable.

(a_2): *Let $\boldsymbol{m}_n : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$, $n = 1, 2, \ldots$, be operator valued measures countably* *additive in the strong operator topology, let* $\boldsymbol{m}_n(E)\,\boldsymbol{x} \to \boldsymbol{m}(E)\,\boldsymbol{x} \in Y$ *for each* $E \in \mathscr{P}$ *and each* $\boldsymbol{x} \in X$, *and let* $\sup_n \boldsymbol{m}^\wedge{}_n(E) < +\infty$ *for each* $E \in \mathscr{P}$.

By the Vitali-Hahn-Saks theorem, see Theorem III.7.2 in [11], [3], and Theorem 2 in § 2 in [1], and the Uniform Boundedness Principle, see [11, II.1.11 and II.3.21], $\boldsymbol{m} : \mathscr{P} \to L(X, Y)$ in (a_2) is an operator valued measure countably additive in the strong operator topology. Clearly $\boldsymbol{m}^\wedge(E) \leqq \sup_n \boldsymbol{m}^\wedge{}_n(E) < +\infty$ for each $E \in \mathscr{P}$, see Lemma 3 in [8]. Note that if the semivariations $\boldsymbol{m}^\wedge{}_n$, $n = 1, 2, \ldots$, are uniformly continuous on $\mathscr{P}$ and if $\boldsymbol{m}_n(E)\,\boldsymbol{x} \to \boldsymbol{m}(E)\,\boldsymbol{x} \in Y$ for each $E \in \mathscr{P}$ and each $\boldsymbol{x} \in X$, then $\sup_n \boldsymbol{m}^\wedge{}_n(E) < +\infty$ for each $E \in \mathscr{P}$, see Corollary of Theorem 5 below.

(a_2u): (a_2) *and for each* $\boldsymbol{x} \in X$, $\boldsymbol{m}_n(E)\,\boldsymbol{x} \to \boldsymbol{m}(E)\,\boldsymbol{x}$ *uniformly with respect to* $E \in \mathscr{P}$, *i.e.*, $\boldsymbol{m}_n(.)\,\boldsymbol{x} \to \boldsymbol{m}(.)\,\boldsymbol{x}$ *in* $B(\mathscr{P}, Y)$ *for each* $\boldsymbol{x} \in X$.

Theorem 1. (Diagonal convergence theorem.) *Suppose* (a_1) *and* (a_2), *and let* $\boldsymbol{f}_n$ *be integrable with respect to* $\boldsymbol{m}_n$ *for each* $n = 1, 2, \ldots$. *Then the following conditions are equivalent:*

a): *the vector measures* $E \to \int_E \boldsymbol{f}_n \,\mathrm{d}\boldsymbol{m}_n$, $E \in \mathfrak{S}(\mathscr{P})$, $n = 1, 2, \ldots$, *are uniformly countably additive on* $\mathfrak{S}(\mathscr{P})$, *and*

b): *for each* $E \in \mathfrak{S}(\mathscr{P})$ *the sequence* $\int_E \boldsymbol{f}_n \,\mathrm{d}\boldsymbol{m}_n$, $n = 1, 2, \ldots$, *is convergent in* $\boldsymbol{Y}$.

If they hold, then $\boldsymbol{f}$ *is integrable with respect to* $\boldsymbol{m}$, *and*

$$\lim_{n \to \infty} \int_E \boldsymbol{f}_n \,\mathrm{d}\boldsymbol{m}_n = \int_E \boldsymbol{f} \mathrm{d}\boldsymbol{m} \quad \textit{for each} \quad E \in \mathfrak{S}(\mathscr{P}) .$$

If (a_2u) *holds, then this limit is uniform with respect to* $E \in \mathfrak{S}(\mathscr{P})$.

Proof. a) $\Rightarrow$ b). Since $\boldsymbol{f} : T \to X$ is $\mathscr{P}$-measurable, there is, by definition, a sequence of $\mathscr{P}$-simple functions $\boldsymbol{g}_i : T \to X$, $i = 1, 2, \ldots$, such that $\boldsymbol{g}_i(t) \to \boldsymbol{f}(t)$ for each $t \in T$. For $n = 1, 2, \ldots$ let $\mu_n : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ and $\lambda_n : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ be control measures for the countably additive vector measures $E \to \int_E \boldsymbol{f}_n \,\mathrm{d}\boldsymbol{m}_n$, $E \in \mathfrak{S}(\mathscr{P})$, and $E \to \int_E \boldsymbol{g}_n \,\mathrm{d}\boldsymbol{m}$, $E \in \mathfrak{S}(\mathscr{P})$, respectively, see [11, IV.10.5], [12, Theorems 3.2 and 3.10], and also Remark 3 after Corollary of Theorem 6 below. For $E \in \mathfrak{S}(\mathscr{P})$ put

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\mu_n(E) + \lambda_n(E)}{1 + \mu_n(T) + \lambda_n(T)} .$$

Then $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, 1)$ is a countably additive measure, and $N \in \mathfrak{S}(\mathscr{P})$, $\mu(N) = 0$ implies $\left|\int_N \boldsymbol{f}_n \,\mathrm{d}\boldsymbol{m}_n\right| + \left|\int_N \boldsymbol{g}_n \,\mathrm{d}\boldsymbol{m}\right| = 0$ for each $n = 1, 2, \ldots$.

Put $F = \bigcup_{n=1}^{\infty} \{t;\, t \in T,\, |\boldsymbol{f}_n(t)| + |\boldsymbol{g}_n(t)| > 0\}$. Then $F \in \mathfrak{S}(\mathscr{P})$, and by the Egoroff-Lusin theorem, see Section 1.4 in Part I, there are $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$, and $N \in \mathfrak{S}(\mathscr{P})$ such that $F_k \nearrow F - N$, $\mu(N) = 0$, and on each set F_k, $k = 1, 2, \ldots$, the sequence $\{\boldsymbol{f}_1, \boldsymbol{g}_1, \ldots, \boldsymbol{f}_n, \boldsymbol{g}_n, \ldots\}$ converges uniformly to $\boldsymbol{f}$.

Let $\varepsilon > 0$. Then by a) there is a k_0 such that $\|\int f_n \, \mathrm{d}\boldsymbol{m}_n\| \, ((F - N) - F_{k_0}) < \frac{1}{6}\varepsilon$ for each $n = 1, 2, \ldots$. Hence

$$\left| \int_E f_n \, \mathrm{d}\boldsymbol{m}_n - \int_E f_j \, \mathrm{d}\boldsymbol{m}_j \right| < \frac{\varepsilon}{3} + \left| \int_{E \cap F_{k_0}} (f_n - g_i) \, \mathrm{d}\boldsymbol{m}_n \right| +$$

$$+ \left| \int_{E \cap F_{k_0}} g_i \, \mathrm{d}\boldsymbol{m}_n - \int_{E \cap F_{k_0}} g_i \, \mathrm{d}\boldsymbol{m}_j \right| + \left| \int_{E \cap F_{k_0}} (g_i - f_j) \, \mathrm{d}\boldsymbol{m}_j \right|$$

for each $E \in \mathfrak{S}(\mathscr{P})$ and each $i, j, n = 1, 2, \ldots$.

Since on F_{k_0} the sequence $\{f_1, g_1, \ldots, f_n, g_n, \ldots\}$ converges uniformly to f, and since $\sup_n \boldsymbol{m}\hat{\ }_n(F_{k_0}) < +\infty$ by (a_2), there is an i_0 such that $\|f_n - g_{i_0}\|_{F_{k_0}} \cdot$ $. \sup_n \boldsymbol{m}\hat{\ }_n(F_{k_0}) < \frac{1}{6}\varepsilon$ for each $n \geqq i_0$. Hence, by Theorem 14 in Part I,

$$\left| \int_{E \cap F_{k_0}} (f_n - g_{i_0}) \, \mathrm{d}\boldsymbol{m}_n \right| \leqq \|f_n - g_{i_0}\|_{F_{k_0}} \cdot \sup_n \boldsymbol{m}\hat{\ }_n(F_{k_0}) < \frac{\varepsilon}{6}$$

for each $E \in \mathfrak{S}(\mathscr{P})$ and each $n \geqq i_0$.

Let $E \in \mathfrak{S}(\mathscr{P})$. Since $g_{i_0} \cdot \chi_{E \cap F_{k_0}}$ is a $\mathscr{P}$-simple function, according to (a_2) there is an $n_0 \geqq i_0$ such that

$$\left| \int_{E \cap F_{k_0}} g_{i_0} \, \mathrm{d}\boldsymbol{m}_n - \int_{E \cap F_{k_0}} g_{i_0} \, \mathrm{d}\boldsymbol{m}_j \right| < \frac{\varepsilon}{3} \quad \text{for each} \quad n, j \geqq n_0 \, .$$

If we assume $(a_2 u)$, then such n_0 can be taken independently of $E \in \mathfrak{S}(\mathscr{P})$. Thus b) follows from a).

b) $\Rightarrow$ a) by the Vitali-Hahn-Saks theorem, see [11, III.7.2], [3] and [1. Theorem 2 in § 2].

Suppose now a) and b), and let us keep the notations from above. For $E \in \mathfrak{S}(\mathscr{P})$ put $\boldsymbol{v}(E) = \lim_{n \to \infty} \int_E f_n \, \mathrm{d}\boldsymbol{m}_n$. Then $\boldsymbol{v} : \mathfrak{S}(\mathscr{P}) \to Y$ is a countably additive vector measure by a), and $M \in \mathfrak{S}(\mathscr{P})$, $\mu(M) = 0$ implies $\boldsymbol{v}(M) = 0$. Hence $\boldsymbol{v}(E) = \boldsymbol{v}(E \cap F) =$ $= \lim_{k \to \infty} \boldsymbol{v}(E \cap F_k)$ for each $E \in \mathfrak{S}(\mathscr{P}) \, (F_k \nearrow F - N$, and $\|\boldsymbol{v}\| \, (N) = 0)$.

For each $k = 1, 2, \ldots$ take $n_k \geqq k \vee n_{k-1}$ $(n_0 = 0)$ so that $\|f_n - g_{n_k}\| \, F_k \cdot$ $. \sup_n \boldsymbol{m}\hat{\ }_n(F_k) < 1/k$ for $n \geqq n_k$, and put $h_k = g_{n_k} \cdot \chi_{F_k \cup N}$, $k = 1, 2, \ldots$. Then $h_k : T \to X$, $k = 1, 2, \ldots$, is a sequence of $\mathscr{P}$-simple functions, $h_k(t) \to f(t)$ for each $t \in T$, and

$$\left| \int_E f_n \, \mathrm{d}\boldsymbol{m}_n - \int_E h_k \, \mathrm{d}\boldsymbol{m} \right| \leqq \left\| \int f_n \, \mathrm{d}\boldsymbol{m}_n \right\| ((F - N) - F_k) +$$

$$+ \left| \int_{E \cap F_k} (f_n - h_k) \, \mathrm{d}\boldsymbol{m}_n \right| + \left| \int_{E \cap F_k} h_k \, \mathrm{d}\boldsymbol{m}_n - \int_{E \cap F_k} h_k \, \mathrm{d}\boldsymbol{m} \right|$$

for each $E \in \mathfrak{S}(\mathscr{P})$ and each $n, k = 1, 2, \ldots$.

Let $\varepsilon > 0$. By a) take k_1 so that $1/k_1 < \frac{1}{2}\varepsilon$ and $\|\int f_n \, \mathrm{d}\boldsymbol{m}_n\| \left((F - N) - F_k\right) < \frac{1}{2}\varepsilon$ for each $k \geqq k_1$ and each $n = 1, 2, \ldots$. Then we obtain for $k \geqq k_1$ and for $n \geqq n_k$, in virtue of Theorem 14 from Part I, the inequalities $\left|\int_{E \cap F_k} (\boldsymbol{f}_n - \boldsymbol{h}_k) \, \mathrm{d}\boldsymbol{m}_n\right| \leqq$ $\leqq \|\boldsymbol{f}_n - \boldsymbol{h}_k\|_{F_k} \cdot \sup_n \boldsymbol{m}^{\wedge}{}_n(F_k) < 1/k_1 < \frac{1}{2}\varepsilon$, hence $\left|\int_E \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}_n - \int_E \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m}\right| \leqq \varepsilon +$ $+ \left|\int_{E \cap F_k} \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m}_n - \int_{E \cap F_k} \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m}\right|$ for each $h \in \mathfrak{S}(\mathscr{P})$.

Let $E \in \mathfrak{S}(\mathscr{P})$ and $k \geqq k_1$ be fixed. Since $\boldsymbol{h}_k \cdot \chi_{E \cap F_k}$ is a $\mathscr{P}$-simple function, $\left|\int_{E \cap F_k} \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m}_n - \int_{E \cap F_k} \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m}\right| \to 0$ as $n \to \infty$ by (a_2). Since $\lim_{n \to \infty} \int_E \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}_n = \boldsymbol{v}(E) \in \boldsymbol{Y}$ exists according to b), the last inequality implies for $n \to \infty$ that $\left|\boldsymbol{v}(E) - \int_E \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m}\right| \leqq$ $\leqq \varepsilon$. Thus $\int_E \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m} \to \boldsymbol{v}(E)$ for each $E \in \mathfrak{S}(\mathscr{P})$, hence $\boldsymbol{f}$ is integrable with respect to $\boldsymbol{m}$, and $\int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} = \lim_{k \to \infty} \int_E \boldsymbol{h}_k \, \mathrm{d}\boldsymbol{m} = \boldsymbol{v}(E) = \lim_{n \to \infty} \int_E \boldsymbol{f}_n \, \mathrm{d}\boldsymbol{m}_n$ for each $E \in \mathfrak{S}(\mathscr{P})$ by Theorem 7 in Part I. If $(\mathrm{a}_2\mathrm{u})$ holds, then, as was shown above in a) $\Rightarrow$ b), the last limit is uniform with respect to $E \in \mathfrak{S}(\mathscr{P})$. The theorem is proved.

Obviously, in the special case when $\boldsymbol{m}_n = \boldsymbol{m}$ for all $n = 1, 2, \ldots$, Theorem 1 reduces to Theorems 15 and 16 in Part I. Another important special case occurs when $\boldsymbol{f}_n = \boldsymbol{f}$ for all $n = 1, 2, \ldots$. The author has not succeeded in finding the classical scalar analog of Theorem 1 in literature. He found only a partial result in this direction in [15, Corollary 2 of Theorem 1].

The next corollary is immediate.

Corollary 1. *Suppose* (a_1) *and* (a_2), *and let* $\boldsymbol{f}_k$ *be integrable with respect to* $\boldsymbol{m}_n$ *for all* $k, n = 1, 2, \ldots$. *Then*:

1) *If the iterated limit* $\lim_{n \to \infty} \lim_{k \to \infty} \int_E \boldsymbol{f}_k \, \mathrm{d}\boldsymbol{m}_n$ *exists in* $\boldsymbol{Y}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$, *then* $\boldsymbol{f}$ *is integrable with respect to* $\boldsymbol{m}$ *and each* $\boldsymbol{m}_n$, $n = 1, 2, \ldots$, *and*

$$\lim_{n \to \infty} \lim_{k \to \infty} \int_E \boldsymbol{f}_k \, \mathrm{d}\boldsymbol{m}_n = \lim_{n \to \infty} \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}_n = \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$$

for each $E \in \mathfrak{S}(\mathscr{P})$.

2) *If the iterated limit* $\lim_{k \to \infty} \lim_{n \to \infty} \int_E \boldsymbol{f}_k \, \mathrm{d}\boldsymbol{m}_n$ *exists in* $\boldsymbol{Y}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$, *then* $\boldsymbol{f}$ *and each* $\boldsymbol{f}_k$, $k = 1, 2$, *are integrable with respect to* $\boldsymbol{m}$, *and*

$$\lim_{k \to \infty} \lim_{n \to \infty} \int_E \boldsymbol{f}_k \, \mathrm{d}\boldsymbol{m}_n = \lim_{k \to \infty} \int_E \boldsymbol{f}_k \, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m}$$

for each $E \in \mathfrak{S}(\mathscr{P})$.

Hence, if both the iterated limits above exist in $\boldsymbol{Y}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$, *then they are equal.*

The name diagonal convergence theorem was suggested by the following

Corollary 2. *Suppose* (a_1) *and* (a_2), *and let* $\boldsymbol{f}_k$ *be integrable with respect to* $\boldsymbol{m}_n$

for each $k, n = 1, 2, \ldots$. *Then the following conditions are equivalent*:

a) *the vector measures* $E \to \int_E f_k \, \mathrm{d}m_n$, $E \in \mathfrak{S}(\mathscr{P})$, $k, n = 1, 2, \ldots$ *are uniformly countably additive, and*

b) *for each* $E \in \mathfrak{S}(\mathscr{P})$, $\lim_{k\to\infty} \int_E f_k \, \mathrm{d}m_n$ *exists in* Y *for each* $n = 1, 2, \ldots$, $\lim_{n\to\infty} \int_E f_k \, \mathrm{d}m_n$ *exists in* Y *for each* $k = 1, 2, \ldots$, *and* $\lim_{i\to\infty} \int_E f_{k_i} \, \mathrm{d}m_{n_i}$ *exists in* Y *for all subsequences* $k_i, n_i \to \infty$.

If they hold, then f *is integrable with respect to* m *and each* m_n, $n = 1, 2, \ldots$, *each* f_k, $k = 1, 2, \ldots$, *is integrable with respect to* m, *and*

$$\lim_{k,n\to\infty} \int_E f_k \, \mathrm{d}m_n = \lim_{n\to\infty} \lim_{k\to\infty} \int_E f_k \, \mathrm{d}m_n = \lim_{n\to\infty} \int_E f \mathrm{d}m_n = \int_E f \mathrm{d}m =$$

$$= \lim_{k\to\infty} \int_E f_k \, \mathrm{d}m = \lim_{k\to\infty} \lim_{n\to\infty} \int_E f_k \, \mathrm{d}m_n$$

for each $E \in \mathfrak{S}(\mathscr{P})$. *If* $(a_2 u)$ *holds, then* $\lim_{k,n\to\infty} \int_E f_k \, \mathrm{d}m_n = \int_E f \mathrm{d}m$ *uniformly with respect to* $E \in \mathfrak{S}(\mathscr{P})$.

Proof. a) $\Rightarrow$ b) by Theorem 1, while b) $\Rightarrow$ a) by the Vitali-Hahn-Saks theorem, see [11, III.7.2], [3] and [1, Theorem 2 in § 2]. The other assertions of the corollary follow immediately from Theorem 1, since a) implies the uniform countable additivity of the vector measures $E \to \int_E f \mathrm{d}m_n$, $E \to \int_E f_k \, \mathrm{d}m$, $E \in \mathfrak{S}(\mathscr{P})$, $n, k = 1, 2, \ldots$.

Some special cases. 1. Let $\mathscr{P} = \mathfrak{S}(\mathscr{P})$, let $f_k : T \to X$, $k = 1, 2, \ldots$. be a bounded sequence in $B(T, X)$ of $\mathscr{P}$-measurable functions, and let $f_k(t) \to f(t) \in X$ for each $t \in T$. Let further $m_n : \mathscr{P} \to L(X, Y)$, $n = 1, 2, \ldots$, be operator valued measures countably additive in the strong operator topology, let $m_n(E)\, x \to m(E)\, x \in Y$ for each $E \in \mathscr{P}$ and each $x \in X$, and let the semivariations $\hat{m}_n$, $n = 1, 2, \ldots$, be uniformly continuous on $\mathscr{P}$, see also Theorem 2 below. Then $\hat{m}(T) \leqq \sup_n \hat{m}_n(T) < +\infty$ by Lemma 3 in [8] and Corollary of Theorem 5 below, f_k are integrable with respect to m_n for each $k, n = 1, 2, \ldots$ by Theorem 5 in Part I, and $|\int_E f_k \, \mathrm{d}m_n| \leqq \hat{m}_n(f_k, E) \leqq \|f_k\|_E . \ \hat{m}_n(E) \leqq \sup_k \|f_k\|_T . \sup_n \hat{m}_n(T) < +\infty$ for each $E \in \mathscr{P}$ by Theorems 2 and 1 in Part II. Hence the assumptions and condition a) of Corollary 2 of Theorem 1 are fulfilled. Note that in Corollary of Theorem 6 below we obtain in a certain sense a stronger result than the assertions of Corollary 2 of Theorem 1, namely that $\lim_{k\to\infty} \sup_n \hat{m}_n(f_k - f, T) = 0$.

2. Let $X \otimes_\varepsilon Y$ denote the completion of the tensor product $X \otimes Y$ in the inductive, or weak, crossnorm, see [16, § 20.5], and let $\mu : \mathscr{P} \to Y$ be a countably additive vector measure. (Note that if X is the space of scalars of Y, then $X \otimes_\varepsilon Y = Y$ and $x \otimes y = x . y$.) Then clearly μ may be viewed as an operator valued measure $\mu : \mathscr{P} \to$

$\to L(X, X \otimes_\varepsilon Y)$, $\mu(E)\, x = x \otimes \mu(E)$, $E \in \mathscr{P}$, $x \in X$, countably additive in the uniform operator topology. It easily follows from the properties of the weak crossnorm that $\hat{\mu}(E) = \|\mu\|\,(E)$ for each $E \in \mathscr{P}$. Hence $\hat{\mu}$ is finite and continuous on $\mathscr{P}$ by countable additivity of $\mu : \mathscr{P} \to Y$. If $f : T \to X$ is integrable with respect to $\mu : \mathscr{P} \to L(X, X \otimes_\varepsilon \otimes_\varepsilon Y)$, then we put $\int_E f \otimes_\varepsilon \mathrm{d}\mu = \int_E f \,\mathrm{d}\mu$, $E \in \mathfrak{S}(\mathscr{P})$.

Let now $\mathscr{P} = \mathfrak{S}(\mathscr{P})$, let $f_k : T \to X$, $k = 1, 2, \ldots$, be a bounded sequence in $B(T, X)$ of $\mathscr{P}$-measurable functions, and let $f_k(t) \to f(t) \in X$ for each $t \in T$. Let further $\mu_n : \mathscr{P} \to \to Y$, $n = 1, 2, \ldots$, be countably additive vector measures, and let $\mu_n(E) \to \mu(E) \in Y$ for each $E \in \mathscr{P}$. Then the semivariations $\hat{\mu}_n = \|\mu_n\|$, $n = 1, 2, \ldots$, are uniformly continuous on $\mathscr{P}$ by the Vitali-Hahn-Saks theorem, see [11, III.7.2], [3] and [1, Theorem 2 in § 2]. Hence we have the situation of case 1 described above.

3. Let $\mathscr{P} = \mathfrak{S}(\mathscr{P})$, let $f_k \in \bar{\mathfrak{J}}_s$ (= the closure of the space $\mathfrak{J}_s$ of all $\mathscr{P}$-simple functions $f : T \to X$ in $B(T, X)$), $k = 1, 2, \ldots$, and let $f_k(t) \to f(t) \in X$ uniformly with respect to $t \in T$, i.e., $\|f_k - f\|_T \to 0$ (hence $f \in \bar{\mathfrak{J}}_s$ as well). Suppose further (a_2). Then, according to Theorem 9 in Part I, f_k are integrable with respect to m_n for each $k, n = 0, 1, \ldots$, where $f_0 = f$ and $m_0 = m$. Let $\varepsilon > 0$ and take k_0 and a $\mathscr{P}$-simple function $g : T \to X$ so that $(\|f_k - g\|_T + \|f - g\|_T) \,.\, \sup_n m\hat{}_n(T) < \frac{1}{2}\varepsilon$ for $k \geqq k_0$. Then $\left|\int_E f_k \,\mathrm{d}m_n - \int_E f \,\mathrm{d}m\right| \leqq \left|\int_E (f_k - g)\,\mathrm{d}m_n\right| + \left|\int_E g \,\mathrm{d}m_n - \int_E g \,\mathrm{d}m\right| + {}$ $+ \left|\int_E (g - f)\,\mathrm{d}m\right| \leqq \|f_k - g\|_T \,.\, \sup_n m\hat{}_n(T) + \left|\int_E g \,\mathrm{d}m_n - \int_E g \,\mathrm{d}m\right| + {}$ $+ \|g - f\|_T \,.\, \sup_n m\hat{}_n(T) \leqq \frac{1}{2}\varepsilon + \left|\int_E g \,\mathrm{d}m_n - \int_E g \,\mathrm{d}m\right|$ for each $k \geqq k_0$ and each $E \in \mathscr{P}$. Hence $\lim_{k,n\to\infty} \int_E f_k \,\mathrm{d}m_n = \int_E f \,\mathrm{d}m$ for each $E \in \mathscr{P}$ by (a_2), and this limit is uniform with respect to $E \in \mathscr{P}$, provided $(a_2 u)$ holds. The equalities

$$\lim_{n\to\infty} \lim_{k\to\infty} \int_E f_k \,\mathrm{d}m_n = \lim_{n\to\infty} \int_E f \,\mathrm{d}m_n = \int_E f \,\mathrm{d}m = \lim_{k\to\infty} \int_E f_k \,\mathrm{d}m =$$
$$= \lim_{k\to\infty} \lim_{n\to\infty} \int_E f_k \,\mathrm{d}m_n\,, \quad E \in \mathscr{P}\,,$$

may be proved similarly.

Let us note that neither Theorem 1 nor its corollaries have been used. Finally, recall that $C_0(T, X) < \bar{\mathfrak{J}}_s$ if T is a locally compact Hausdorff topological space and if $\mathscr{P} = \mathscr{B}_0$ (= the δ-ring of all relatively compact Baire subsets of T), see Theorem 8 Part I and [8].

2. MISCELLANEOUS RESULTS

The next theorem is a generalization of the $*$-Theorem from Section 1.1 in Part I, and may be proved in the same way.

Theorem 2. (Extended $*$-Theorem.) *Let Y contain no subspace isomorphic to the space c_0, for example, let Y be a weakly complete Banach space,* (see pp. 160 and 161 in [2]), *let* $m_j : \mathscr{P} \to L(X, Y)$, $j \in J$, *be operator valued measures countably additive in the strong operator topology, and let* $\gamma(E) = \sup \{|\sum_{i=1}^{r} m_{j_i}(E_i)\, x_i|,\ j_i \in J,\ E_i \in \mathscr{P},\ E_i \cap E_k = \emptyset \text{ for } i \neq k,\ E = \bigcup_{i=1}^{r} E_i,\ x_i \in X,\ |x_i| \leqq 1,\ i, k = 1, \ldots, r,\ r = 1, 2, \ldots\} < +\infty$ *for each* $E \in \mathscr{P}$. *Then the semivariations* $m^\wedge{}_j$, $j \in J$, *are uniformly continuous on* $\mathscr{P}$.

Since we shall deal with unconditionally convergent series in Banach spaces, let us recall the basic facts. We say that a sequence $y_n \in Y$, $n = 1, 2, \ldots$, is summable to $y \in Y$, if for every $\varepsilon > 0$ there is a finite set $I \subset \omega = \{1, 2, \ldots\}$ such that $|\sum_{j \in J} y_j - y| < \varepsilon$ for each finite $J \subset \omega$, $J \supset I$. It is well-known, see Section IV.1 in [4], that a sequence $y_n \in Y$, $n = 1, 2, \ldots$, is summable to $y \in Y$ if and only if the series $\sum_{n=1}^{\infty} y_n$ is unconditionally convergent to y. By Orlicz-Pettis theorems this is equivalent to the weak unconditional convergence of the series $\sum_{n=1}^{\infty} y_n$ to y, and if Y contains no subspace isomorphic to c_0, then this is equivalent to the condition $\sum_{n=1}^{\infty} |y^* y_n| < +\infty$ for each $y^* \in Y^*$, see [2]. If $y(.)$, $y_n(.) : T \to Y$, $n = 1, 2, \ldots$, then in addition to the pointwise summability of $y_n(t)$, $n = 1, 2, \ldots$, to $y(t)$ we have also the natural concept of uniform summability of $y_n(t)$, $n = 1, 2, \ldots$, to $y(t)$ with respect to $t \in T$, i.e., the summability of $y_n(.)$, $n = 1, 2, \ldots$, to $y(.)$ in $B(T, X)$.

Lemma 1. *Let* $v_n : \mathscr{P} \to Y$, $n = 1, 2, \ldots$, *be a sequence of countably additive vector measures, and for* $I \in \Phi_1$ *and* $E \in \mathscr{P}$ *put* $v_I(E) = \sum_{i \in I} v_i(E)$. *Then the following conditions are equivalent*:

a) *the set function* γ, $\gamma(E) = \sup_{I \in \Phi_1} \|v_I\|(E)$, $E \in \mathscr{P}$, *is continuous on* $\mathscr{P}$,

b) *the vector measures* v_I, $I \in \Phi_1$, *are uniformly countably additive on* $\mathscr{P}$, *and*

c) *for any sequence of pairwise disjoint sets* $I_n \in \Phi_1$, $n = 1, 2, \ldots$, *the vector measures* v_{I_n}, $n = 1, 2, \ldots$, *are uniformly countably additive on* $\mathscr{P}$.

Proof. Clearly a) $\Rightarrow$ b) $\Rightarrow$ c).

c) $\Rightarrow$ a). Suppose c) and non a). Then there is an $\varepsilon > 0$, a sequence $E_k \in \mathscr{P}$, $k = 1, 2, \ldots$, $E_k \searrow \emptyset$, and a sequence $J_k \in \Phi_1$, $k = 1, 2, \ldots$, such that $\|v_{J_k}\|(E_k) > \varepsilon$ for all $k = 1, 2, \ldots$. Put $I_1 = J_1$ and $k_1 = 1$. Since I_1 is a finite set, and since each v_n, $n = 1, 2, \ldots$, is countably additive on $\mathscr{P}$, there is a $k_2 > k_1$ such that $\|v_{I_1}\|(E_{k_2}) < \frac{1}{2}\varepsilon$. Put $I_2 = J_{k_2} - I_1$. Then $I_1 \cap I_2 = \emptyset$ and $\|v_{I_2}\|(E_{k_2}) > \frac{1}{2}\varepsilon$. Similarly, since $I_1 \cup I_2$ is a finite set, there is a $k_3 > k_2$ such that $\|v_{I_1 \cup I_2}\|(E_{k_3}) < \frac{1}{2}\varepsilon$. Put $I_3 = J_{k_3} - (I_1 \cup I_2)$. Then $I_3 \cap (I_1 \cup I_2) = \emptyset$ and $\|v_{I_3}\|(E_{k_3}) > \frac{1}{2}\varepsilon$. Continuing in this way we obtain a contradiction with c). The lemma is proved.

Using this lemma we have the following simple consequence of the Vitali-Hahn-Saks theorem, for the latter see [11, III. 7.2], [3] and [1, Theorem 2 in § 2].

Theorem 3. *Let* $\mathbf{v}_n : \mathscr{P} \to Y$, $n = 1, 2, \ldots$, *be countably additive vector measures, and let for each* $E \in \mathscr{P}$ *the series* $\sum_{n=1}^{\infty} \mathbf{v}_n(E)$ *be unconditionally convergent in* Y. *Then the vector measures* $\mathbf{v}_I$, $I \in \Phi_1$, *are uniformly countably additive on* $\mathscr{P}$.

Proof. Let $I_n \in \Phi_1$, $n = 1, 2, \ldots$, be a sequence of pairwise disjoint sets. According to Lemma 1 it is enough to show that the vector measures $\mathbf{v}_{I_n}$, $n = 1, 2, \ldots$, are uniformly countably additive on $\mathscr{P}$. But this follows immediately from the Vitali-Hahn-Saks theorem, since $\mathbf{v}_{I_n}(E) \to 0$ for each $E \in \mathscr{P}$ by the unconditional convergence of the series $\sum_{n=1}^{\infty} \mathbf{v}_n(E)$ in Y, $E \in \mathscr{P}$.

A partial converse to this theorem is the following

Theorem 4. *Let T be a locally compact Hausdorff topological space and let* $\mathbf{v}_n : \mathfrak{S}(\mathscr{B}) \to Y$, $n = 1, 2, \ldots$, *be countably additive regular Borel vector measures. Further,*

(i) *let the vector measures* $\mathbf{v}_I = \sum_{i \in I} \mathbf{v}_i$, $I \in \Phi_1$, *be uniformly countably additive on* $\mathfrak{S}(\mathscr{B})$,

(ii) *let Y contain no subspace isomorphic to* c_0, (see pp. 160 and 161 in [2]), *and*

(iii) *let* $\sup_{I \in \Phi_1} |\mathbf{v}_I(\{t\})| < +\infty$ *for each* $t \in T$.

Then the series $\sum_{n=1}^{\infty} \mathbf{v}_n(E)$ *is unconditionally convergent in Y for each* $E \in \mathfrak{S}(\mathscr{B})$.

Proof. For $E \in \mathfrak{S}(\mathscr{B})$ put $\mu(E) = \sup_{I \in \Phi_1} \|\mathbf{v}_I\| (E)$. Then clearly $\mu : \mathfrak{S}(\mathscr{B}) \to \langle 0, +\infty \rangle$ is monotone, subadditive, and continuous by (i). Hence μ is a submeasure in the sense of Definition 1 in [9]. Since we suppose (ii), according to Theorem 5 in [2] it is enough to show that $\mu(E) < +\infty$ for each $E \in \mathfrak{S}(\mathscr{B})$. By Theorem 4 in [9] there is a set $Q \in \mathfrak{S}(\mathscr{B})$ such that $\mu(E) = \mu(E \cap Q)$ and $\mu(E - Q) = 0$ for each $E \in \mathfrak{S}(\mathscr{B})$. Owing to the Saks decomposition of Q with respect to μ, see Theorem 8 in [9], there is a finite number r of pairwise disjoint elements $A_0, A_1, \ldots, A_r$ of $\mathfrak{S}(\mathscr{B})$ such that $Q = \bigcup_{i=0}^{r} A_i$, and each A_i, $i = 0, 1, \ldots, r$, is either an atom of μ (see Definition 2 in [9]) with $\mu(A_i) > 1$, or $\mu(A_i) \leqq 1$. According to Theorem 12 in [9] each atom of μ is concentrated at a point $t \in T$. Thus using (iii) and the subadditivity of μ we have $\mu(E) = \mu(E \cap Q) \leqq \mu(Q) < +\infty$ for each $E \in \mathfrak{S}(\mathscr{B})$, which was to be shown. The theorem is proved.

Remark 1. From this proof it is clear that Theorem 4 remains valid for general countably additive vector measures $\mathbf{v}_n : \mathfrak{S}(\mathscr{P}) \to Y$, $n = 1, 2, \ldots$, if the condition (iii) is replaced by

(iii)′ let $\sup_{I \in \Phi_1} \|\mathbf{v}_I\| (E) < +\infty$ for each $E \in \mathfrak{S}(\mathscr{P})$ which is an atom of each $\mathbf{v}_I$, $I \in \Phi_1$.

The properties of submeasures used in the proof of Theorem 4 yield the following result. (For the definition of the L_1-pseudonorm $\boldsymbol{m}^\wedge(.,.)$ and its properties see Part II and the paragraph before Theorem 3 in Part III.)

Theorem 5. *Let $f_i : T \to X$ or $f_i : T \to \langle 0, +\infty)$, $i \in I$, be $\mathscr{P}$-measurable functions, and let $\boldsymbol{m}_j : \mathscr{P} \to L(X, Y)$, $j \in J$, be operator valued measures countably additive in the strong operator topology. Let further the L_1-pseudonorms $\boldsymbol{m}^\wedge_j(f_i, .)$, $j \in J$, $i \in I$, be uniformly continuous on $\mathfrak{S}(\mathscr{P})$, and let $\sup_{i,j} \left|\int_A |f_i| \,\mathrm{d}(\boldsymbol{m}_j(.)\,\boldsymbol{x})\right| < +\infty$ for each $\boldsymbol{x} \in X$ and each $A \in \mathscr{P}$ which is an atom of each $\boldsymbol{m}_j$, $j \in J$. Then $\sup_{i,j} \boldsymbol{m}^\wedge_j(f_i, T) < +\infty$.*

Remark. We do not suppose that the semivariations $\boldsymbol{m}^\wedge_j$, $j \in J$, are finite on $\mathscr{P}$. Since $\widehat{\boldsymbol{m}_j(.)\,\boldsymbol{x}}\,(f_i, E) = \|\boldsymbol{m}_j(.)\,\boldsymbol{x}\|\,(f_i, E) \leqq \|\boldsymbol{m}_j\|\,(f_i, E) \leqq \boldsymbol{m}^\wedge_j(f_i, E)$ for each $\boldsymbol{x} \in X$, $E \in \mathfrak{S}(\mathscr{P})$, $i \in I$ and $j \in J$, the continuity of $\boldsymbol{m}^\wedge_j(f_i, .)$, $j \in J$, $i \in I$, by Lemma 1 in Part II implies that $|f_i|$ is integrable with respect to $\boldsymbol{m}_j(.)\,\boldsymbol{x}$, $j \in J$, $\boldsymbol{x} \in X$, and $\boldsymbol{m}_j$, $j \in J$.

Proof. By the Uniform Boundedness Principle, see [11, II.1.11 and II.3.21], the last condition in the theorem implies that $\sup_{i,j} \sup_{|\boldsymbol{x}| \leqq 1} \left|\int_A |f_i| \,\mathrm{d}(\boldsymbol{m}_j(.)\,\boldsymbol{x})\right| < +\infty$ for each $A \in \mathscr{P}$ which is an atom of each $\boldsymbol{m}_j$, $j \in J$.

For $E \in \mathfrak{S}(\mathscr{P})$ put $\mu(E) = \sup_{i,j} \boldsymbol{m}^\wedge_j(f_i, E)$. Then clearly $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty\rangle$ is a submeasure in the sense of Definition 1 in [9]. If we show that $\mu(A) < +\infty$ for each atom A of μ (see Definition 2 in [9]), then similarly as in the proof of Theorem 4, the subadditivity of μ will imply the desired result by Saks decomposition. Let $A \in \mathfrak{S}(\mathscr{P})$ be an atom of μ. Since μ is continuous on $\mathfrak{S}(\mathscr{P})$, we may suppose that $A \in \mathscr{P}$. Let us have a fixed couple $(i, j) \in I \times J$. Then obviously either $\boldsymbol{m}^\wedge_j(f_i, A) = 0$, or A is an atom of $\boldsymbol{m}^\wedge_j(f_i, .)$. Suppose A is an atom of $\boldsymbol{m}^\wedge_j(f_i, .)$. Since $\boldsymbol{m}^\wedge_j(f_i, A) = \lim_{n\to\infty} \boldsymbol{m}^\wedge_j(f_i, \{t : t \in A,\ 1/n \leqq |f_i(t)| \leqq n\})$, there is an n_0 such that $\boldsymbol{m}^\wedge_j(f_i, A - \{t : t \in A, 1/n_0 \leqq |f_i(t)| \leqq n_0\}) = 0$. Partitioning the interval $\langle 1/n_0, n_0\rangle$ into smaller and smaller parts, by monotone continuity of $\boldsymbol{m}^\wedge_j(f_i, .)$ on $\mathfrak{S}(\mathscr{P})$, see Theorem 1 in [9], we obtain a number a, $1/n_0 \leqq a \leqq n_0$ such that $\boldsymbol{m}^\wedge_j(f_i, A - \{t : t \in A, |f_i(t)| = a\}) = 0$. Hence $\boldsymbol{m}^\wedge_j(f_i, A) = a \,.\, \boldsymbol{m}^\wedge_j(\{t : t \in A, |f_i(t)| = a\}) = \sup_{|\boldsymbol{x}| \leqq 1} \left|\int_A |f_i| \,\mathrm{d}(\boldsymbol{m}_j(\cdot)\,\boldsymbol{x})\right|$. But then $\mu(A) = \sup_{i,j} \boldsymbol{m}^\wedge_j(f_i, A) = \sup_{i,j} \sup_{|\boldsymbol{x}| \leqq 1} \left|\int_A |f_i| \,.\, \mathrm{d}(\boldsymbol{m}_j(.)\,\boldsymbol{x}\right| < +\infty$, which proves the theorem.

From this theorem we immediately obtain the following

Corollary. *Let $\boldsymbol{m}_j : \mathfrak{S}(\mathscr{P}) \to L(X, Y)$, $j \in J$, be operator valued measures countably additive in the strong operator topology, let the semivariations $\boldsymbol{m}^\wedge_j$, $j \in J$, be uni-*

formly continuous on $\mathfrak{S}(\mathscr{P})$, and let $\sup_{j\in J} |m_j(A)\,x| < +\infty$ *for each* $x \in X$ *and each* $A \in \mathscr{P}$ *which is an atom of each* m_j, $j \in J$. *Then* $\sup_{j\in J} m^\wedge{}_j(T) < \infty$.

Remark 2. If T is a locally compact Hausdorff topological space, and if $m_j : \mathscr{B} \to L(X, Y)$, $j \in J$, are Borel measures regular and countably additive in the strong operator topology, then the last condition in Theorem 5 may be replaced by the following: $\sup_{i,j} |f_i(t)| \cdot |m_j(\{t\})\,x| < +\infty$ for each $t \in T$ and each $x \in X$. Similarly, in this case the last condition in the corollary above may be replaced by the following $\sup_{j\in J} |m_j(\{t\})\,x| < +\infty$ for each $t \in T$ and each $x \in X$.

The next theorem is a generalization of the Lebesgue dominated convergence theorem in $\mathscr{L}_1(m)$, see Theorem 17 in Part II.

Theorem 6. (Extended Lebesgue dominated convergence theorem.) *Let* $f_k : T \to X$ *or* $f_k : T \to \langle 0, +\infty)$, $k = 1, 2, \ldots$, *be* $\mathscr{P}$*-measurable functions and let* $f_k(t) \to 0$ *for each* $t \in T$. *Let further* $m_n : \mathscr{P} \to L(X, Y)$, $n = 1, 2, \ldots$, *be operator valued measures countably additive in the strong operator topology, let the* L_1*-pseudo-norms* $m^\wedge{}_n(f_k, .)$, $n, k = 1, 2, \ldots$, *be uniformly continuous on* $\mathfrak{S}(\mathscr{P})$, *and let* $\sup_n m^\wedge{}_n(E) < +\infty$ *for each* $E \in \mathscr{P}$. *Then* $\lim_{k\to\infty} \sup_n m^\wedge{}_n(f_k, T) = 0$.

Proof. For $E \in \mathfrak{S}(\mathscr{P})$ put

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{m^\wedge{}_n(f_k, E)}{1 + m^\wedge{}_n(f_k, T)}.$$

(Since $m^\wedge{}_n(f_k, .)$ is continuous on $\mathfrak{S}(\mathscr{P})$ by assumption, $m^\wedge{}_n(f_k, T) < +\infty$ by Corollary of Theorem 5 in Part II.) Then $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, 1)$ is monotone, subadditive and continuous, and $N \in \mathfrak{S}(\mathscr{P})$, $\mu(N) = 0$ implies $m^\wedge{}_n(f_k, N) = 0$ for all $k, n = 1, 2, \ldots$. Put $F = \bigcup_{k=1}^{\infty} \{t : t \in T, f_k(t) \neq 0\} \in \mathfrak{S}(\mathscr{P})$. Since $f_k(t) \to 0$ for each $t \in T$, the Egoroff-Lusin theorem, see Section 1.4 in Part I (which remains valid for such μ), implies that there are $F_i \in \mathscr{P}$, $i = 1, 2, \ldots$, and $N \in \mathfrak{S}(\mathscr{P})$ such that $N \subset F$, $\mu(N) = 0$, $F_i \nearrow F - N$, and on each F_i, $i = 1, 2, \ldots$, the sequence f_k, $k = 1, 2, \ldots$, converges uniformly to 0. Since $m^\wedge{}_n(f_k, T) = m^\wedge{}_n(f_k, F) \leqq m^\wedge{}_n(f_k, F_i) + m^\wedge{}_n(f_k, (F - N) - F_i) \leqq \|f_k\|_{F_i} \cdot \sup_n m^\wedge{}_n(F_i) + m^\wedge{}_n(f_k, (F - N) - F_i)$ for all $i, k, n = 1, 2, \ldots$, the assertion of the theorem is now evident.

From here and from Corollary of Theorem 5 we immediately have the following

Corollary. (Extended Lebesgue bounded convergence theorem.) *Let* $f_k : T \to X$ *or* $f_k : T \to \langle 0, +\infty)$ *be a bounded sequence in* $B(T, X)$ *or* $B(T, R)$, *and let* $f_k(t) \to f(t) \in X$ *or* R *for each* $t \in T$. *Let further* $m_n : \mathfrak{S}(\mathscr{P}) \to L(X, Y)$, $n = 1, 2, \ldots$, *be operator valued measures countably additive in the strong operator topology,* *let their semivariations* $\boldsymbol{m}^\wedge_n$, $n = 1, 2, \ldots$, *be uniformly continuous on* $\mathfrak{S}(\mathscr{P})$, *and let* $\sup_n |\boldsymbol{m}_n(A)\, x| < +\infty$ *for each* $x \in X$ *and each* $A \in \mathscr{P}$ *which is an atom of each* $\boldsymbol{m}_n$, $n = 1, 2, \ldots$. *Then* $\lim_{k\to\infty} \sup_n \boldsymbol{m}^\wedge_n(f_k - f, T) = 0$.

Remark 3. The basic tool in our theory of integration is the Egoroff-Lusin theorem, see Section 1.4 in Part I. It has been used, until the proof of Theorem 6 above, via the existence of a control measure for a countably additive vector measure or for a continuous L_1-pseudonorm, see the previous parts. Obviously, it remains valid when $\mu : \mathfrak{S}(\mathscr{P}) \Rightarrow \langle 0, +\infty)$ is monotone, subadditive, and continuous (in fact, it clearly remains valid for general submeasures $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ in the sense of Definition 1 in [9]). Therefore instead of control measures we can use either directly the continuous L_1-pseudonorms as in the proof of Theorem 6, or, in the case of a countably additive vector measure $\boldsymbol{\mu} : \mathfrak{S}(\mathscr{P}) \to Y$, the set function $\bar{\boldsymbol{\mu}}(E) = \sup \{|\boldsymbol{\mu}(F)|,\ F \in \mathfrak{S}(\mathscr{P}),\ F \subset E\}$ (which by elementary straightforward arguments, see Theorems 2.6 and 3.5 in [12], is bounded, monotone, subadditive, and continuous). On the other hand, the existence of a control measure for a countably additive vector measure was essential for the proofs of Theorems 14 and 15 (The Fubini theorem) in Part III.

We are now ready to prove (for notations and terminology see Part III)

Theorem 7. *Let* $\boldsymbol{m}_i : \mathscr{P} \to L(X, Y)$, $i \in I$, *be operator valued measures countably additive in the strong operator topology, let the semivariations* $\boldsymbol{m}^\wedge_i$, $i \in I$, *be uniformly continuous on* $\mathscr{P}$, *and let* $\sup_{i\in I} |\boldsymbol{m}_i(A)\, x| < +\infty$ *for each* $x \in X$ *and each* $A \in \mathscr{P}$ *which is an atom of each* $\boldsymbol{m}_i$, $i \in I$. *Let us subject* $\boldsymbol{l}_j : \mathscr{Q} \to L(Y, Z)$, $j \in J$ *to analogous assumptions. Then the product measures* $\boldsymbol{l}_j \otimes \boldsymbol{m}_i : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$, $i \in I$, $j \in J$, *exist*, $\sup_{i,j} (\widehat{\boldsymbol{l}_j \otimes \boldsymbol{m}_i})(E) < +\infty$ *for each* $E \in \mathscr{P} \otimes \mathscr{Q}$, *and the semivariations* $\widehat{\boldsymbol{l}_j \otimes \boldsymbol{m}_i}$, $i \in I$, $j \in J$, *are uniformly continuous on* $\mathscr{P} \otimes \mathscr{Q}$.

Proof. The product measures $\boldsymbol{l}_j \otimes \boldsymbol{m}_i : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$, $i \in I$, $j \in J$, exist by Theorem 3 in Part III.

Let $E \in \mathscr{P} \otimes \mathscr{Q}$ and take $A \in \mathscr{P}$ and $B \in \mathscr{Q}$ so that $E \subset A \times B$. Since $\sup_{i\in I} \boldsymbol{m}^\wedge_i(A) < +\infty$ and $\sup_{j\in J} \boldsymbol{l}^\wedge_j(B) < +\infty$ by Corollary of Theorem 5, we have $\sup_{(i,j)\in I\times J} (\widehat{\boldsymbol{l}_j \otimes \boldsymbol{m}_i})(E) \leqq \sup_{i\in I} \boldsymbol{m}^\wedge_i(A) \, . \sup_{j\in J} \boldsymbol{l}^\wedge_j(B) < +\infty$ by Theorem 2 in Part III.

Concerning the last assertion of the theorem, we immediately see by indirect proof that it is enough to prove it when $I = J = \{1\ 2, \ldots\}$, i.e., when we have a sequence $\boldsymbol{m}_k$, $k = 1, 2, \ldots$, and a sequence $\boldsymbol{l}_n$, $n = 1, 2, \ldots$. Let $E_r \in \mathscr{P} \otimes \mathscr{Q}$, $r = 1, 2, \ldots$, and let $E_r \searrow \emptyset$. For $r = 1, 2, \ldots$ and $s \in S$ put $f_r(s) = \sup_k \boldsymbol{m}^\wedge_k(E_r)$. Since each semivariation $\boldsymbol{m}^\wedge_k$, $k = 1, 2, \ldots$, is continuous on $\mathscr{P}$, each function $s \to$

$\to m^\wedge_k(E^s_r)$, $s \in S$, $k, r = 1, 2, \ldots$, is $\mathcal{Q}$-measurable by Lemma 2 c) in Part III. Hence each f_r, $r = 1, 2, \ldots$, is $\mathcal{Q}$-measurable by Theorem A in § 20 in [13]. Take $A_1 \in \mathcal{P}$ and $B_1 \in \mathcal{Q}$ so that $E_1 \subset A_1 \times B_1$. Then $f_{r+1}(s) \leqq f_r(s) \leqq \sup_k m^\wedge_k(A_1) < +\infty$ for each $r = 1, 2, \ldots$ and each $s \in S$. Hence the functions f_r, $r = 1, 2, \ldots$, are uniformly bounded on B_1. Since we assume that the semivariations $m^\wedge_k$, $k = 1, 2, \ldots$, are uniformly continuous on $\mathcal{P}$, $f_r(s) \to 0$ for each $s \in B_1$. Thus $\lim_{r\to\infty} \sup_n l^\wedge_n(f_r, B_1) = 0$ by Theorem 6. Since $\widehat{(l_n \otimes m_k)}(E_r) \leqq l^\wedge_n(m_k(E_r), B_1) \leqq l^\wedge_n(f_r, B_1)$ for all $n, k, r = 1, 2, \ldots$ by Theorem 2 in Part III, we conclude that $\lim_{r\to\infty} \sup_{n,k} \widehat{(l_n \otimes m_k)}(E_r) = 0$. The theorem is proved.

3. APPLICATIONS TO INTEGRALS OF UNCONDITIONALLY CONVERGENT SERIES OF FUNCTIONS AND MEASURES

The Orlicz-Pettis theorems, see [4, IV. Theorem 1], [11, IV.10.1] and [2], is a powerful tool for establishing unconditional convergence of a given series in a Banach space. Hence it is reasonable to apply our diagonal convergence theorem to integrals of unconditionally convergent series of functions and measures.

We shall use the following three assumptions corresponding to assumptions (a_1), (a_2) and (a_2u) in § 1.

(b_1): *Let $f_k : T \to X$, $k = 1, 2, \ldots$, be $\mathcal{P}$-measurable functions, and let* $\sum_{k=1}^{\infty} f_k(t) = f(t) \in X$ *unconditionally for each $t \in T$.*

(b_2): *Let $m_n : \mathcal{P} \to L(X, Y)$, $n = 1, 2, \ldots$, be operator valued measures countably additive in the strong operator topology, let* $\sum_{n=1}^{\infty} m_n(E)\, x = m(E)\, x \in Y$ *unconditionally for each $E \in \mathcal{P}$ and each $x \in X$, and let* $\sup_{I \in \Phi_1} \widehat{(\sum_{i \in I} m_i)}(E) < +\infty$ *for each $E \in \mathcal{P}$.*

(b_2u) : (b_2) and $\sum_{n=1}^{\infty} m_n(.)\, x = m(.)\, x$ unconditionally in $B(\mathcal{P}, Y)$ for each $x \in X$.

Note that if (b_2) holds, then $m^\wedge(E) \leqq \sup_{I \in \Phi_1} \widehat{(\sum_{i \in I} m_i)}(E) < +\infty$ for each $E \in \mathcal{P}$.

For the next result, Theorem 15 in Part I (a special case of the diagonal convergence theorem) is sufficient.

Theorem 8. *Suppose (b_1). Let further $m : \mathcal{P} \to L(X, Y)$ be an operator valued measure countably additive in the strong operator topology, let $m^\wedge(E) < +\infty$ for each $E \in \mathcal{P}$, and let each f_k, $k = 1, 2, \ldots$, be integrable with respect to m. Then the following conditions are equivalent:*

a) *the series* $\sum_{k=1}^{\infty} \int_E f_k \, dm$ *is unconditionally convergent in* Y *for each* $E \in \mathfrak{S}(\mathscr{P})$, *and*

b) *the vector measures* $E \to \sum_{i \in I} \int_E f_i \, dm$, $E \in \mathfrak{S}(\mathscr{P})$, $I \in \Phi_1$, *are uniformly countably additive.*

If they hold, then f *is integrable with respect to* m, *and* $\sum_{k=1}^{\infty} \dot{\int} f_k \, dm = \dot{\int} f \, dm$ *unconditionally in* $B(\mathfrak{S}(\mathscr{P}), Y)$.

Proof. a) $\Rightarrow$ b) by Theorem 3. We show that b) implies the second assertion of the theorem, hence also a). Suppose b) and let $p(i)$, $i = 1, 2, \ldots$, be any permutation of $\omega = \{1, 2, \ldots\}$. Then $s_k(t) = \sum_{i=1}^{k} f_{p(i)}(t) \to f(t)$ for each $t \in T$ by (b_1), and the vector measures $E \to \int_E s_k \, dm$, $E \in \mathfrak{S}(\mathscr{P})$, $k = 1, 2, \ldots$, are uniformly countably additive by b). Hence f is integrable with respect to m and $\sum_{i=1}^{k} \dot{\int} f_{p(i)} \, dm \to \dot{\int} f \, dm$ in $B(\mathfrak{S}(\mathscr{P}), Y)$ by Theorem 15 in Part I. This yields, since $p(.)$ is any permutation of $\{1, 2, \ldots\}$, that $\sum_{k=1}^{\infty} \dot{\int} f_k \, dm = \dot{\int} f \, dm$ unconditionally in $B(\mathfrak{S}(\mathscr{P}), Y)$, see [4, IV. § 1, (B) $\Leftrightarrow$ (C)]. The theorem is proved.

Similarly, using another special case of Theorem 1, we immediately obtain

Theorem 9. *Suppose* (b_2). *Let further* $f: T \to X$ *be a* $\mathscr{P}$*-measurable function integrable with respect to each* m_n, $n = 1, 2, \ldots$. *Then the following conditions are equivalent:*

a) *the series* $\sum_{n=1}^{\infty} \int_E f \, dm_n$ *is unconditionally convergent in* Y *for each* $E \in \mathfrak{S}(\mathscr{P})$, *and*

b) *the vector measures* $E \to \sum_{i \in I} \int_E f \, dm_i$, $E \in \mathfrak{S}(\mathscr{P})$, $I \in \Phi_1$, *are uniformly countably additive.*

If they hold, then f *is integrable with respect to* m, *and* $\sum_{n=1}^{\infty} \int_E f \, dm_n = \int_E f \, dm$ *unconditionally for each* $E \in \mathfrak{S}(\mathscr{P})$. *If, moreover,* (b_2u) *holds, then* $\sum_{n=1}^{\infty} \dot{\int} f \, dm_n = \dot{\int} f \, dm$ *in* $B(\mathfrak{S}(\mathscr{P}), Y)$.

Theorems 8 and 9 immediately imply

Theorem 10. *Suppose* (b_1) *and* (b_2). *Let further* f_k *be integrable with respect to* m_n *for all* $k, n = 1, 2, \ldots$. *Then:*

1) *If* $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_E f_k \, dm_n$ *is iteratively unconditionally convergent in* Y *for each* $E \in \mathfrak{S}(\mathscr{P})$, *then* f *is integrable with respect to* m *and each* m_n, $n = 1, 2, \ldots$, *and*

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_E f_k \, dm_n = \sum_{n=1}^{\infty} \int_E f \, dm_n = \int_E f \, dm$$

for each $E \in \mathfrak{S}(\mathscr{P})$.

2) *If* $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n$ *is iteratively unconditionally convergent in* $\boldsymbol{Y}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$, *then* f *and all* f_k, $k = 1, 2, \ldots$, *are integrable with respect to* $\boldsymbol{m}$ *and*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n = \sum_{k=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m} = \int_E f \, \mathrm{d}\boldsymbol{m}$$

for each $E \in \mathfrak{S}(\mathscr{P})$.

From Theorems 8, 9 and 10 we easily deduce

Theorem 11. *Suppose* (b_1) *and* (b_2). *Let further* f_k *be integrable with respect to* $\boldsymbol{m}_n$ *for all* $k, n = 1, 2, \ldots$, *and let the vector measures* $E \to \sum_{(i,j) \in I} \int_E f_i \, \mathrm{d}\boldsymbol{m}_j$, $E \in \mathfrak{S}(\mathscr{P})$, $I \in \Phi_2$, *be uniformly countably additive. Then* f *is integrable with respect to* $\boldsymbol{m}$ *and both* $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n$ *and* $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n$ *are iteratively unconditionally convergent to* $\int_E f \, \mathrm{d}\boldsymbol{m}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$.

Using Theorem 1 we obtain (see also Theorem 4)

Theorem 12. *Suppose* (b_1) *and* (b_2). *Let further* f_k *be integrable with respect to* $\boldsymbol{m}_n$ *for all* $k, n = 1, 2, \ldots$, *and let the series* $\sum_{k,n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n$ *be unconditionally convergent in* $\boldsymbol{Y}$ *for each* $E \in \mathfrak{S}(\mathscr{P})$. *Then the vector measures* $E \to \sum_{(k,n) \in I} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n$, $E \in \mathfrak{S}(\mathscr{P})$, $I \in \Phi_2$, *are uniformly countably additive,* f *is integrable with respect to* $\boldsymbol{m}$ *and* $\sum_{k,n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n = \int_E f \, \mathrm{d}\boldsymbol{m}$ *unconditionally for each* $E \in \mathfrak{S}(\mathscr{P})$. *If* $(\mathrm{b}_2\mathrm{u})$ *holds, then* $\sum_{k,n=1}^{\infty} \int_{\cdot} f_k \, \mathrm{d}\boldsymbol{m}_n = \int_{\cdot} f \, \mathrm{d}\boldsymbol{m}$ *unconditionally in* $B(\mathfrak{S}(\mathscr{P}), \boldsymbol{Y})$.

Proof. The first assertion follows immediately from Theorem 3. Since $\sum_{k,n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n = \lim_{i \to \infty} \int_E \left(\sum_{k=1}^{i} f_k\right) \mathrm{d}\left(\sum_{n=1}^{i} \boldsymbol{m}_n\right)$ for each $E \in \mathfrak{S}(\mathscr{P})$, f is integrable with respect to $\boldsymbol{m}$, and $\sum_{k,n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n = \int_E f \, \mathrm{d}\boldsymbol{m}$ for each $E \in \mathfrak{S}(\mathscr{P})$ by Theorem 1. By assumption the last convergence is unconditional for each $E \in \mathfrak{S}(\mathscr{P})$.

Let us assume $(\mathrm{b}_2\mathrm{u})$ and suppose that the series $\sum_{k,n=1}^{\infty} \int_{\cdot} f_k \, \mathrm{d}\boldsymbol{m}_n$ is not summable to $\int_{\cdot} f \, \mathrm{d}\boldsymbol{m}$ in $B(\mathfrak{S}(\mathscr{P}), \boldsymbol{Y})$. Then, since $\sum_{k,n=1}^{\infty} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n = \int_E f \, \mathrm{d}\boldsymbol{m}$ unconditionally for each $E \in \mathfrak{S}(\mathscr{P})$, the series $\sum_{k,n=1}^{\infty} \int_{\cdot} f_k \, \mathrm{d}\boldsymbol{m}_n$ is not summable in $B(\mathfrak{S}(\mathscr{P}), \boldsymbol{Y})$. Hence there is an $\varepsilon > 0$, a sequence of pairwise disjoint non empty sets $I_i \in \Phi_2$, $i = 1, 2, \ldots$, and a sequence $E_i \in \mathfrak{S}(\mathscr{P})$, $i = 1, 2, \ldots$, such that $\left|\sum_{(k,n) \in I_i} \int_{E_i} f_k \, \mathrm{d}\boldsymbol{m}_n\right| > \varepsilon$ for each $i = 1, 2, \ldots$. Since $\sum_{(k,n) \in I_i} \int_{E_1} f_k \, \mathrm{d}\boldsymbol{m}_n \to 0$ as $i \to \infty$, the unconditional convergence of the series $\sum_{k,n=1}^{\infty} \int_{E_1} f_k \, \mathrm{d}\boldsymbol{m}_n$ in $\boldsymbol{Y}$ implies that there is an $i_2 > i_1 = 1$ such that $\left| \sum_{(k,n)\in I_{i_2}} \int_{E_{i_2} - E_{i_1}} f_k \, \mathrm{d}\boldsymbol{m}_n \right| > \varepsilon$. Similarly there is an $i_3 > i_2$ such that $\left| \sum_{(k,n)\in I_{i_3}} \int_{E_{i_3} - (E_{i_1} \cup E_{i_2})} f_k \, \mathrm{d}\boldsymbol{m}_n \right| > \varepsilon$. Continuing in this manner we obtain a sequence of pairwise disjoint sets $F_r = E_{i_r} - (E_{i_1} \cup \ldots \cup E_{i_{r-1}}) \in \mathfrak{S}(\mathscr{P})$, $r = 1, 2, \ldots$, such that $\left| \sum_{(k,n)\in I_{i_r}} \int_{F_r} f_k \, \mathrm{d}\boldsymbol{m}_n \right| > \varepsilon$ for all $r = 1, 2, \ldots$. Hence we have obtained a contradiction with the uniform countable additivity of the vector measures $E \to \to \sum_{(k,n)\in I} \int_E f_k \, \mathrm{d}\boldsymbol{m}_n$, $E \in \mathfrak{S}(\mathscr{P})$, $I \in \Phi_2$. The theorem is proved.

4. APPLICATIONS TO PRODUCTS AND DOUBLE INTEGRALS OF SEQUENCES AND SERIES OF OPERATOR VALUED MEASURES

The results obtained above combined with the main theorems of Part III yield interesting results about products and double integrals of sequences and series of operator valued measures. We state and prove only few of them for illustration.

Let us note that the next theorem, by suitable localization, remains valid if $\mathfrak{S}(\mathscr{P})$, $\mathfrak{S}(\mathscr{Q})$ and $\mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ are replaced by $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{P} \otimes \mathscr{Q}$ respectively, but then we have a somewhat weaker result.

Theorem 13. *Let* $\boldsymbol{m}_n : \mathfrak{S}(\mathscr{P}) \to L(X, Y)$ *and* $\boldsymbol{l}_n : \mathfrak{S}(\mathscr{Q}) \to L(Y, Z)$, $n = 1, 2, \ldots$ *be operator valued measures countably additive in strong operator topologies, let* $\boldsymbol{m}_n(A)\, \boldsymbol{x} \to \boldsymbol{m}(A)\, \boldsymbol{x} \in \boldsymbol{Y}$ *for each* $A \in \mathfrak{S}(\mathscr{P})$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$, *and let* $\boldsymbol{l}_n(B)\, \boldsymbol{y} \to \boldsymbol{l}(B)\, \boldsymbol{y} \in \boldsymbol{Z}$ *for each* $B \in \mathscr{Q}$ *and each* $\boldsymbol{y} \in \boldsymbol{Y}$. *Let further* $\sup_n \boldsymbol{l}\hat{}_n(B) < +\infty$ *for each* $B \in \mathscr{Q}$, *and let the product measures* $\boldsymbol{l}_n \otimes \boldsymbol{m}_n : \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \to L(X, Z)$, $n = 1, 2, \ldots$, *exist. Then the following conditions are equivalent*:

a) *the vector measures* $E \to (\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E)\, \boldsymbol{x}$, $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$, $n = 1, 2, \ldots$, *are uniformly countably additive for each* $\boldsymbol{x} \in \boldsymbol{X}$,

b) *the vector measures* $B \to (\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E \cap (T \times B))\, \boldsymbol{x}$, $B \in \mathfrak{S}(\mathscr{Q})$, $n = 1, 2, \ldots$, *are uniformly countably additive for each* $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$, *and*

c) $\lim_{n\to\infty} (\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E)\, \boldsymbol{x} \in \boldsymbol{Z}$ *exists for each* $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$.

If they hold, then the product measure $\boldsymbol{l} \otimes \boldsymbol{m} : \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \to L(X, Z)$ *exists, and* $\lim_{n\to\infty} (\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E)\, \boldsymbol{x} = (\boldsymbol{l} \otimes \boldsymbol{m})(E)\, \boldsymbol{x}$ *for each* $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$.

Finally, if $\boldsymbol{l}_n(B)\, \boldsymbol{y} \to \boldsymbol{l}(B)\, \boldsymbol{y}$ *uniformly with respect to* $B \in \mathscr{Q}$, *then* $\lim_{n\to\infty} (\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E \cap (T \times B))\, \boldsymbol{x} = (\boldsymbol{l} \otimes \boldsymbol{m})(E \cap (T \times B))\, \boldsymbol{x}$ *uniformly with respect to* $B \in \mathfrak{S}(\mathscr{Q})$ *for each* $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$.

Proof. a) $\Rightarrow$ b) trivially, while c) $\Rightarrow$ a) by the Vitali-Hahn-Saks theorem, see [11, III.7.2], [3] and [1, Theorem 2 in § 2].

Suppose b). According to Theorem 1 in Part III we have

$$(1) \qquad (\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E)\,\boldsymbol{x} = \int_S \boldsymbol{m}_n(E^s)\,\boldsymbol{x}\,\mathrm{d}\boldsymbol{l}_n \quad \text{for each} \quad E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}), \quad \boldsymbol{x} \in \boldsymbol{X}$$
$$\text{and} \quad n = 1, 2, \ldots .$$

Let $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$ and $\boldsymbol{x} \in \boldsymbol{X}$ be fixed. Since $\boldsymbol{m}_n(E^s)\,\boldsymbol{x} \to \boldsymbol{m}(E^s)\,\boldsymbol{x}$ for each $s \in S$, the function $s \to \boldsymbol{m}(E^s)\,\boldsymbol{x}$, $s \in S$, is integrable with respect to $\boldsymbol{l}$, and

$$(2) \qquad \int_B \boldsymbol{m}_n(E^s)\,\boldsymbol{x}\,\mathrm{d}\boldsymbol{l}_n \to \int_B \boldsymbol{m}(E^s)\,\boldsymbol{x}\,\mathrm{d}\boldsymbol{l} \quad \text{for each} \quad B \in \mathfrak{S}(\mathscr{Q})$$

by Theorem 1. Hence the product measure $\boldsymbol{l} \otimes \boldsymbol{m} : \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}) \to L(\boldsymbol{X}, \boldsymbol{Z})$ exists by Theorem 1 in Part III. Taking $B \in \mathfrak{S}(\mathscr{Q})$ so that $T \times B \supset E$, we obtain from (1), (2) and Theorem 1 in Part III that $(\boldsymbol{l}_n \otimes \boldsymbol{m}_n)(E)\,\boldsymbol{x} \to \int_S \boldsymbol{m}(E^s)\,\boldsymbol{x}\,\mathrm{d}\boldsymbol{l} = (\boldsymbol{l} \otimes \boldsymbol{m})(E)\,\boldsymbol{x}$. The last assertion of the theorem also follows from (1), (2) and Theorem 1.

From this theorem and its proof it is evident how to apply Corollaries 1 and 2 of Theorem 1, and Theorems 8–12 to products of sequences and series of operator valued measures.

Theorem 14. *Let* $\boldsymbol{m}_k : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ *and* $\boldsymbol{l}_n : \mathscr{Q} \to L(\boldsymbol{Y}, \boldsymbol{Z})$, $k, n = 1, 2, \ldots$, *be operator valued measures countably additive in strong operator topologies, let* $\boldsymbol{m}_k(A)\,\boldsymbol{x} \to \boldsymbol{m}(A)\,\boldsymbol{x} \in \boldsymbol{Y}$ *for each* $A \in \mathscr{P}$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$, *let* $\boldsymbol{l}_n(B)\,\boldsymbol{y} \to \boldsymbol{l}(B)\,\boldsymbol{y} \in \boldsymbol{Z}$ *for each* $B \in \mathscr{Q}$ *and each* $\boldsymbol{y} \in \boldsymbol{Y}$, *and let the semivariations* $\boldsymbol{l}^\wedge{}_n$, $n = 1, 2, \ldots$, *be uniformly continuous on* $\mathscr{Q}$. *Then* $\boldsymbol{l}^\wedge$ *is continuous on* $\mathscr{Q}$, $\sup_n \boldsymbol{l}^\wedge{}_n(B) < +\infty$ *for each* $B \in \mathscr{Q}$, *all product measures* $\boldsymbol{l}_n \otimes \boldsymbol{m}_k : \mathscr{P} \otimes \mathscr{Q} \to L(\boldsymbol{X}, \boldsymbol{Z})$, $k, n = 0, 1, 2, \ldots$, *where* $\boldsymbol{l}_0 = \boldsymbol{l}$ *and* $\boldsymbol{m}_0 = \boldsymbol{m}$, *exist, and*

$$\lim_{n,k \to \infty} (\boldsymbol{l}_n \otimes \boldsymbol{m}_k)(E)\,\boldsymbol{x} = \lim_{n \to \infty} \lim_{k \to \infty} (\boldsymbol{l}_n \otimes \boldsymbol{m}_k)(E)\,\boldsymbol{x} = \lim_{n \to \infty} (\boldsymbol{l}_n \otimes \boldsymbol{m})(E)\,\boldsymbol{x} =$$
$$= (\boldsymbol{l} \otimes \boldsymbol{m})(E)\,\boldsymbol{x} = \lim_{k \to \infty} (\boldsymbol{l} \otimes \boldsymbol{m}_k)(E)\,\boldsymbol{x} = \lim_{k \to \infty} \lim_{n \to \infty} (\boldsymbol{l}_n \otimes \boldsymbol{m}_k)(E)\,\boldsymbol{x}$$

for each $E \in \mathscr{P} \otimes \mathscr{Q}$ *and each* $\boldsymbol{x} \in \boldsymbol{X}$.

Proof. Since $\boldsymbol{l}^\wedge(B) \leqq \sup_n \boldsymbol{l}^\wedge{}_n(B)$ for each $B \in \mathscr{Q}$, $\boldsymbol{l}^\wedge$ is continuous on $\mathscr{Q}$. $\sup_n \boldsymbol{l}^\wedge{}_n(B) < +\infty$ by Corollary of Theorem 5. According to Theorem 3 in Part III the product measures $\boldsymbol{l}_n \otimes \boldsymbol{m}_k$, $k, n = 0, 1, 2, \ldots$, exist in virtue of continuity of the semivariations $\boldsymbol{l}^\wedge{}_n$, $n = 0, 1, 2, \ldots$, on $\mathscr{Q}$. By Theorem 1 in Part III

$$(1) \qquad (\boldsymbol{l}_n \otimes \boldsymbol{m}_k)(E)\,\boldsymbol{x} = \int_S \boldsymbol{m}_k(E^s)\,\boldsymbol{x}\,\mathrm{d}\boldsymbol{l}_n$$

for each $n, k = 0, 1, \ldots$, each $E \in \mathscr{P} \otimes \mathscr{Q}$, and each $\boldsymbol{x} \in \boldsymbol{X}$.

Let $E \in \mathscr{P} \otimes \mathscr{Q}$ and $x \in X$ be fixed. Take $A \in \mathscr{P}$ and $B \in \mathscr{Q}$ so that $E \subset A \times B$, and put $f_k(s) = m_k(E^s)\, x$, $s \in S$, $k = 0, 1, \ldots$. Then the functions $f_k : S \to Y$, $k = 0, 1, \ldots$, are $\mathscr{Q}$-measurable by Lemma 2.1 in Part III. Since $m_k(C)\, x \to m(C)\, x \in Y$ for each $C \in \mathscr{P}$, we have $f_k(s) \to f_0(s)$ for each $s \in S$, and the semivariations $\|m_k(.)\, x\|$, $k = 1, 2, \ldots$, are uniformly continuous on $\mathscr{P}$ by the Vitali-Hahn-Saks theorem, see [11, III.7.2], [3] and [1, Theorem 2 in § 2]. Thus $K = \sup_k \|m_k(.)\, x\|\,(A) < +\infty$ by Corollary of Theorem 5. (The Nikodym theorem on uniform boundedness of measures, see [11, IV.9.8] and [1, Theorem 1 in § 2], may be used alternatively.) Since clearly $\|f_k\|_S \leqq K$ for all $k = 1, 2, \ldots$, and since we assume that the semivariations $l^\wedge{}_n$, $n = 1, 2, \ldots$, are uniformly continuous on $\mathscr{Q}$, we have the situation of case 1 after Corollary 2 of Theorem 1 ($\mathfrak{S}(B \cap \mathscr{Q}) = B \cap \mathscr{Q}$). Hence (1) and Corollary 2 of Theorem 1 imply the last assertion of the theorem. The theorem is proved.

Theorem 15. *Let the following assumptions be satisfied*:

(i) $\mathscr{P} = \mathfrak{S}(\mathscr{P})$, $\mathscr{Q} = \mathfrak{S}(\mathscr{Q})$, $f_j : T \times S \to X$, $j = 1, 2, \ldots$, *is a bounded sequence in* $B(T \times S, X)$ *of* $\mathscr{P} \otimes \mathscr{Q}$*-measurable functions, and* $f_j(t, s) \to f(t, s) \in X$ *for each* $(t, s) \in T \times S$,

(ii) $m_k : \mathscr{P} \to L(X, Y)$, $k = 1, 2, \ldots$, *are operator valued measures countably additive in the strong operator topology,* $m_k(A)\, x \to m(A)\, x \in Y$ *for each* $A \in \mathscr{P}$ *and each* $x \in X$, *and the semivariations* $m^\wedge{}_k$, $k = 1, 2, \ldots$, *are uniformly continuous on* $\mathscr{P}$, *and*

(iii) $l_n : \mathscr{Q} \to L(Y, Z)$, $n = 1, 2, \ldots$, *are operator valued measures countably additive in the strong operator topology,* $l_n(B)\, y \to l(B)\, y \in Z$ *for each* $B \in \mathscr{Q}$ *and each* $y \in Y$, *and the semivariations* $l^\wedge{}_n$, $n = 1, 2, \ldots$, *are uniformly continuous on* $\mathscr{Q}$.

Then:

1) $m^\wedge$ *is continuous on* $\mathscr{P}$, $l^\wedge$ *is continuous on* $\mathscr{Q}$, $\sup_k m^\wedge{}_k(T) < +\infty$ *and* $\sup_n l^\wedge{}_n(S) < +\infty$,

2) *the product measures* $l_n \otimes m_k : \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$, $k, n = 0, 1, \ldots$, *where* $m_0 = m$ *and* $l_0 = l$, *exist and fulfil*

$$\lim_{k,n\to\infty} (l_n \otimes m_k)(E)\, x = \lim_{n\to\infty} \lim_{k\to\infty} (l_n \otimes m_k)(E)\, x =$$
$$= \lim_{n\to\infty} (l_n \otimes m)(E)\, x = (l \otimes m)(E)\, x = \lim_{k\to\infty} (l \otimes m_k)(E)\, x =$$
$$= \lim_{k\to\infty} \lim_{n\to\infty} (l_n \otimes m_k)(E)\, x \quad \textit{for each} \quad E \in \mathscr{P} \otimes \mathscr{Q} \quad \textit{and each} \quad x \in X,$$

and

$$\widehat{(l \otimes m)}(T \times S) \leqq \sup_{n,k} \widehat{(l_n \otimes m_k)}(T \times S) \leqq \sup_n l^\wedge{}_n(S) \,.\, \sup_k m^\wedge{}_k(T) < +\infty\,,$$

3) *the semivariations* $\widehat{l_n \otimes m_k}$, $n, k = 0, 1, \ldots$, *are uniformly continuous on* $\mathscr{P} \otimes \mathscr{Q} = \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$,

4) *the* L_1*-pseudonorms* $(\widehat{l_n \otimes m_k})(f_j, .)$, $j, k, n = 0, 1, \ldots$, *are uniformly continuous on* $\mathscr{P} \otimes \mathscr{Q}$,

5) $\lim\limits_{j \to \infty} \sup\limits_{n,k} (\widehat{l_n \otimes m_k})(f_j - f, T \times S) = 0$, *and*

6) *all integrals below exist and the following identities hold:*

$$\int_E f_j \, \mathrm{d}(l_n \otimes m_k) = \int_S \int_{E^s} f_j(., s) \, \mathrm{d}m_k \, \mathrm{d}l_n \quad \textit{for each} \quad j, k, n = 0, 1, \ldots,$$

and each $E \in \mathscr{P} \otimes \mathscr{Q}$, *and*

$$\lim_{j,k,n \to \infty} \int_E f_j \, \mathrm{d}(l_n \otimes m_k) = \lim_{\alpha,\beta \to \infty} \lim_{\gamma \to \infty} \int_E f_j \, \mathrm{d}(l_n \otimes m_k) =$$

$$= \lim_{\alpha \to \infty} \lim_{\beta,\gamma \to \infty} \int_E f_j \, \mathrm{d}(l_n \otimes m_k) = \lim_{\alpha \to \infty} \lim_{\beta \to \infty} \lim_{\gamma \to \infty} \int_E f_j \, \mathrm{d}(l_n \otimes m_k) = \int_E f \, \mathrm{d}(l \otimes m)$$

for each $E \in \mathscr{P} \otimes \mathscr{Q}$ *and each permutation* (α, β, γ) *of* (j, k, n).

If $f_j \in \mathfrak{J}_s(\mathscr{P} \otimes \mathscr{Q})$ (= *the closure of all* $\mathscr{P} \otimes \mathscr{Q}$*-simple functions* $f: T \times S \to X$ *in* $B(T \times S, X)$), $j = 1, 2, \ldots$, *if* $f_j(t, s) \to f(t, s)$ *uniformly with respect to* $(t, s) \in T \times S$, *and if* $\sup\limits_k m\hat{}_k(T) < \infty$, *then assertions* 2) *and* 6) *remain valid even if we do not assume the uniform continuity of the semivariations* $m\hat{}_k$, $k = 1, 2, \ldots$, *on* $\mathscr{P}$.

Proof. 1). Since $m\hat{}(A) \leqq \sup\limits_k m\hat{}_k(A)$ for each $A \in \mathscr{P}$, $m\hat{}$ is continuous on $\mathscr{P}$ by uniform continuity of $m\hat{}_k$, $n = 1, 2, \ldots$, on $\mathscr{P}$. Further, $\sup\limits_k m\hat{}_k(T) < +\infty$ by Corollary of Theorem 5. Similarly $l\hat{}$ is continuous on $\mathscr{Q}$ and $\sup\limits_n l\hat{}_n(S) < +\infty$.

2) follows from Theorem 14.

3) follows from Theorem 7.

4) follows from 3) and the inequalities $(\widehat{l_n \otimes m_k})(f_j, E) \leqq \sup\limits_j \|f_j\|_{T \times S} \, . \sup\limits_{n,k} (\widehat{l_n \otimes m_k})$ . . (E), $j, k, n = 0, 1, \ldots$ and $E \in \mathscr{P} \otimes \mathscr{Q}$, see Theorem 1 in Part II.

5) follows from 4) by Corollary of Theorem 6.

6) f_j is integrable with respect to $l_n \otimes m_k$, $j, k, n = 0, 1, \ldots$, by 3) and Theorem 5 in Part I. Similarly, by Theorem 5 in Part I the function $f_j(., s)$ is integrable with respect to m_k for $j, k = 0, 1, \ldots$, and each $s \in S$. By Theorem 9 in Part III the function $s \to \int_{E^s} f_j(., s) \, \mathrm{d}m_k$, $s \in S$, is $\mathscr{Q}$-measurable, and by Theorem 5 in Part I it is integrable with respect to l_n for $j, k, n = 0, 1, \ldots$ and each $E \in \mathscr{P} \otimes \mathscr{Q}$. Finally, $\int_E f_j \, \mathrm{d}(l_n \otimes m_k) = \int_S \int_{E^s} f_j(., s) \, \mathrm{d}m_k \, \mathrm{d}l_n$ for $j, k, n = 0, 1, \ldots$ and each

$E \in \mathscr{P} \otimes \mathscr{Q}$ by Theorem 16 in Part III. Now the identities concerning limits follow from Corollary 2 of Theorem 1, see case 1 after the same Corollary.

The last assertion of the theorem follows from Theorem 14 and Corollary 2 of Theorem 1 by virtue of the considerations of case 3 at the end of § 1.

Let us recall the notations of case 2 at the end of § 1 and, for countably additive vector measures $\mu : \mathscr{P} \to Y$ and $\nu : \mathscr{Q} \to Z$, let $\mu \otimes_\varepsilon \nu : \mathscr{P} \otimes \mathscr{Q} \to Y \otimes_\varepsilon Z$ be the inductive tensor product measure, which exists by Theorem 3 in Part III, i.e., it is a countably additive vector measure.

Corollary. *Let $f_j : T \times S \to X, j = 1, 2, \ldots,$ be a bounded sequence in $B(T \times S, X)$ of $\mathscr{P} \otimes \mathscr{Q}$-measurable functions and let $f_j(t, s) \to f(t, s) \in X$ for each $(t, s) \in T \times S$. Let further $\mu_k : \mathfrak{S}(\mathscr{P}) \to Y$ and $\nu_n : \mathfrak{S}(\mathscr{Q}) \to Z$, $k, n = 1, 2, \ldots,$ be countably additive vector measures, and let $\mu_k(A) \to \mu(A) \in Y$ and $\nu_n(B) \to \nu(B) \in Z$ for each $A \in \mathfrak{S}(\mathscr{P})$ and each $B \in \mathfrak{S}(\mathscr{Q})$. Then:*

1) $$\lim_{k,n\to\infty} (\mu_k \otimes_\varepsilon \nu_n)(E) = \lim_{n\to\infty} \lim_{k\to\infty} (\mu_k \otimes_\varepsilon \nu_n)(E) = \lim_{n\to\infty} (\mu \otimes_\varepsilon \nu_n)(E) = (\mu \otimes_\varepsilon \nu)(E) =$$
$$= \lim_{k\to\infty} (\mu_k \otimes_\varepsilon \nu)(E) = \lim_{k\to\infty} \lim_{n\to\infty} (\mu_k \otimes_\varepsilon \nu_n)(E) \quad \textit{for each} \quad E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q}),$$

2) $$\limsup_{j\to\infty} \sup_{k,n} \left\| \int (f_j - f) \otimes_\varepsilon \mathrm{d}(\nu_n \otimes_\varepsilon \mu_k) \right\| (T \times S) = 0,$$

and

3) *analogs of the assertions 6) of the theorem hold.*

If, moreover, $\mu_k(A) \to \mu(A)$ uniformly with respect to $A \in \mathscr{P}$ and if $\nu_n(B) \to \nu(B)$ uniformly with respect to $B \in \mathscr{Q}$, then these limits are uniform with respect to $A \in \mathfrak{S}(\mathscr{P})$ and $B \in \mathfrak{S}(\mathscr{Q})$ respectively, $\lim_{k,n\to\infty} (\mu_k \otimes_\varepsilon \nu_n)(E) = (\mu \otimes_\varepsilon \nu)(E)$ uniformly with respect to $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$, and $\lim_{j,k,n\to\infty} \int_E f_j \otimes_\varepsilon \mathrm{d}(\mu_k \otimes_\varepsilon \nu_n) = \int_E f \otimes_\varepsilon \mathrm{d}(\mu \otimes_\varepsilon \nu)$ uniformly with respect to $E \in \mathfrak{S}(\mathscr{P} \otimes \mathscr{Q})$.

Proof. Only the assertion concerning uniform limits is perhaps not immediate. Suppose $\mu_k(A) \to \mu(A)$ and $\nu_n(B) \to \nu(B)$ uniformly with respect to $A \in \mathscr{P}$ and $B \in \mathscr{Q}$ respectively. Since, by the Vitali-Hahn-Saks theorem, see [11, III.7.2], [3] and [1, Theorem 2 in § 2], the vector measures $\mu_k : \mathfrak{S}(\mathscr{P}) \to Y$, $k = 0, 1, \ldots$, $\mu_0 = \mu$, and $\nu_n : \mathfrak{S}(\mathscr{Q}) \to Z$, $n = 0, 1, \ldots$, $\nu_0 = \nu$, are uniformly countably additive, and since to each $A \in \mathfrak{S}(\mathscr{P})$ and $B \in \mathfrak{S}(\mathscr{Q})$ there are $A_i \in \mathscr{P}$, $B_i \in \mathscr{Q}$, $i = 1, 2, \ldots,$ such that $A_i \nearrow A$, and $B_i \nearrow B$, it is clear that $\mu_k(A) \to \mu(A)$ and $\nu_n(B) \to \nu(B)$ uniformly with respect to $A \in \mathfrak{S}(\mathscr{P})$ and $B \in \mathfrak{S}(\mathscr{Q})$ respectively. But then,

$$\left| \int_S \mu_k(E^s) \otimes_\varepsilon \mathrm{d}\nu_n - \int_S \mu(E^s) \otimes_\varepsilon \mathrm{d}\nu \right| \leqq \left| \int_S \mu_k(E^s) \otimes_\varepsilon \mathrm{d}\nu_n - \int_S \mu_n(E^s) \otimes_\varepsilon \mathrm{d}\nu_n \right| +$$
$$+ \left| \int_S \mu_n(E^s) \otimes_\varepsilon \mathrm{d}\nu_n - \int_S \mu(E^s) \otimes_\varepsilon \mathrm{d}\nu_n \right| + \left| \int_S \mu(E^s) \otimes_\varepsilon \mathrm{d}\nu_n - \int_S \mu(E^s) \otimes_\varepsilon \mathrm{d}\nu \right| \leqq$$

$$\leq \sup_{A \in \mathfrak{S}(\mathcal{P})} |\mu_k(A) - \mu_n(A)| \,.\, \sup_n \|\nu_n\| (S) +$$

$$+ \sup_{A \in \mathfrak{S}(\mathcal{P})} |\mu_n(A) - \mu(A)| \,.\, \sup_n \|\nu_n\| (S) + \|\mu\| (T) \cdot \|\nu_n - \nu\| (S),$$

and $\|\nu_n - \nu\| (S) \leq 4 \sup_{B \in \mathfrak{S}(\mathcal{Q})} |\nu_n(B) - \nu(B)| \to 0$, see [11, IV.10.4], yields $\lim_{k,n \to \infty} (\mu_k \otimes_\varepsilon \nu_n)(E) = (\mu \otimes_\varepsilon \nu)(E)$ uniformly with respect to $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$. Hence

$$\lim_{j,k,n \to \infty} \int_E f_j \otimes_\varepsilon \mathrm{d}(\mu_k \otimes_\varepsilon \nu_n) = \int_E f \otimes_\varepsilon \mathrm{d}(\mu \otimes_\varepsilon \nu)$$

uniformly with respect to $E \in \mathfrak{S}(\mathcal{P} \otimes \mathcal{Q})$ by Corollary 2 of Theorem 1.

From Theorem 2 in Part III, using inequalities similar to those in the proof of Corollary of Theorem 15 above, we immediately obtain

Theorem 16. *Let $\boldsymbol{m}_k : \mathcal{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ and $\boldsymbol{l}_n : \mathcal{Q} \to L(\boldsymbol{Y}, \boldsymbol{Z})$, $k, n = 0, 1, \ldots,$ be operator valued measures countably additive in strong operator topologies, let the product measures $\boldsymbol{l}_n \otimes \boldsymbol{m}_k : \mathcal{P} \otimes \mathcal{Q} \to L(\boldsymbol{X}, \boldsymbol{Z})$, $k, n = 0, 1, \ldots,$ exist, let $\boldsymbol{l}^\wedge{}_n(S) < +\infty$ for all $n = 1, 2, \ldots,$ and let $\widehat{(\boldsymbol{l}_n - \boldsymbol{l}_0)}(S) \to 0$. Put $\boldsymbol{m} = \boldsymbol{m}_0$ and $\boldsymbol{l} = \boldsymbol{l}_0$. Then:*

1) *if $\boldsymbol{m}_k(A)\, \boldsymbol{x} \to \boldsymbol{m}(A)\, \boldsymbol{x}$ uniformly with respect to $A \in \mathcal{P}$ for each $\boldsymbol{x} \in \boldsymbol{X}$, then $(\boldsymbol{l}_n \otimes \boldsymbol{m}_k)(E)\, \boldsymbol{x} \to (\boldsymbol{l} \otimes \boldsymbol{m})(E)\, \boldsymbol{x}$ as $k, n \to \infty$ uniformly with respect to $E \in \mathcal{P} \otimes \otimes \mathcal{Q}$ for each $\boldsymbol{x} \in \boldsymbol{X}$,*
2) *if $\boldsymbol{m}_k(A) \to \boldsymbol{m}(A)$ uniformly with respect to $A \in \mathcal{P}$, then $(\boldsymbol{l}_n \otimes \boldsymbol{m}_k)(E) \to \to (\boldsymbol{l} \otimes \boldsymbol{m})(E)$ as $k, n \to \infty$ uniformly with respect to $E \in \mathcal{P} \otimes \mathcal{Q}$, and*
3) *if $\boldsymbol{m}^\wedge(T) < +\infty$ and $\widehat{(\boldsymbol{m}_k - \boldsymbol{m})}(T) \to 0$, then $\widehat{(\boldsymbol{l}_n \otimes \boldsymbol{m}_k - \boldsymbol{l} \otimes \boldsymbol{m})}(T \times S) \to 0$ as $k, n \to \infty$.*

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