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Jiří Jarník

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MULTIVALUED MAPPINGS AND FILIPPOV'S OPERATION

JIŘÍ JARNÍK, Praha

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1. INTRODUCTION

When studying differential equations with discontinuous right hand sides, Filippov [1] introduced a concept of solution in terms of a certain differential relation. Namely, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable, he defined a mapping $\mathscr{F}f$ by

$$(1) \qquad \mathscr{F} f(x) = \bigcap_{\delta > 0} \; \bigcap_{N \subset \mathbb{R}^n, m_n(N) = 0} \overline{\operatorname{conv}}\, f(B(x, \delta) \setminus N)$$

for $x \in \mathbb{R}^n$. Here $m_n(N)$ stands for the n-dimensional Lebesgue measure of the set N, $\overline{\operatorname{conv}}$ denotes the closed convex hull of a set in $\mathbb{R}^n$ and $B(x, \delta) \subset \mathbb{R}^n$ is the open ball with a center $x \in \mathbb{R}^n$ and radius δ. If f satisfies a certain boundedness condition, then $\mathscr{F}f : \mathbb{R}^n \to \mathscr{K}^n$, where $\mathscr{K}^n$ is the family of all nonempty compact convex subsets of $\mathbb{R}^n$. Thus any differential equation

$$\dot{x} = f(x)$$

is associated with a differential relation

$$\dot{x} \in \mathscr{F} f(x)$$

and we can define that x is a solution of the former if it is a solution of the latter in the usual sense.

A natural question to be asked is the following: Given a map $F : \mathbb{R}^n \to \mathscr{K}^n$, is it possible to find a measurable function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $F = \mathscr{F}f$? The aim of this paper is to show that this is indeed possible under some rather natural assumptions. Our result will cover even the nonautonomous case.

It is not difficult to show that $\mathscr{F}f$ has the following properties:

(2) $\mathscr{F}f$ is upper semicontinuous;

(3) $f(x) \in \mathscr{F} f(x)$ for almost all x;

(4) $\mathscr{F}f$ is minimal in the following sense: if $H : \mathbb{R}^n \to \mathscr{K}^n$ satisfies (2), (3) (with the obvious change of notation) then $\mathscr{F} f(x) \subset H(x)$ for all $x \in \mathbb{R}^n$.

On the other hand, (4) implies that these conditions determine the mapping $\mathscr{F}f$ uniquely. Thus, (2)–(4) provide a descriptive definition of $\mathscr{F}f$. (Cf. [2], Chap. 18.)

The definition (1) can be modified to cover a more general case. Let $F : \mathbb{R}^n \to \mathscr{K}^n$ and define

$$(5)\qquad \mathscr{F}\, F(x) = \bigcap_{\delta>0}\ \bigcap_{N\subset \mathbb{R}^n, m_n(N)=0} \overline{\operatorname{conv}} \bigcup_{y\in B(x,\delta)\setminus N} F(y)\,.$$

It is not difficult to show that

$$(6)\qquad \mathscr{F}\, F(x) \subset F(x)$$

for all x provided F is upper semicontinuous. On the other hand, the converse inclusion $F(x) \subset \mathscr{F}\, F(x)$ need not generally hold for all x. Indeed, let e.g. $F(x) = \{0\}$ for $x \neq 0$, $F(0) = [0, 1]$. Then evidently $\mathscr{F}\, F(x) = \{0\}$ for all $x \in \mathbb{R}$. (Nevertheless, it can be shown that the inclusion $F(x) \subset \mathscr{F}\, F(x)$ holds for almost all x.)

The formula (5) gives us the possibility of iterating the operation $\mathscr{F}$ from (1). By (2) and (6) we have $\mathscr{F}\mathscr{F} f(x) \subset \mathscr{F} f(x)$ for all x. Further, it can be proved that $\mathscr{F}\mathscr{F}f$ satisfies conditions (2), (3) with $\mathscr{F}\mathscr{F}f$ instead of $\mathscr{F}f$. Thus the minimality condition (4) implies the converse inclusion and hence

$$(7)\qquad \mathscr{F}\mathscr{F} f(x) = \mathscr{F} f(x) \quad \text{for all} \quad x\,.$$

Actually, Filippov dealt with functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ assuming that f is measurable (as a function of $n + 1$ variables). Nonetheless, the generalization of the above considerations to the nonautonomous case is straightforward: we put $\mathscr{F} f(t, x) = \mathscr{F} f_t(x)$, where $f_t(\cdot) = f(t, \cdot)$. The descriptive definition by (2)–(4) can be used again, with the following obvious modifications:

(2′) for every t, $\mathscr{F}f_t$ is upper semicontinuous;

(3′) for every t, $f(t, x) \in \mathscr{F} f_t(x)$ for almost all x;

(4′) if $H : \mathbb{R}^{n+1} \to \mathscr{K}^n$ satisfies (2′), (3′) then $\mathscr{F} f_t(x) \subset H(t, x)$ for all (t, x).

The proof is found in Kurzweil [2]. Moreover, it is proved there that for f measurable, the mapping $\mathscr{F}f$ is Scorza-Dragonian, i. e. it fulfils the condition

(8) for every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset \mathbb{R}$, $m_1(\mathbb{R} \setminus A_\varepsilon) < \varepsilon$, such that the restriction $\mathscr{F}f$ onto $A_\varepsilon \times \mathbb{R}^n$ is upper semicontinuous (with respect to the couple (t, x)).

Let us notice that (8) implies (2′) but not vice versa.

For a detailed study of condition (8), see [3]. A more recent result extending the above to classes of sets different from $\mathscr{K}^n$ is found in Vrkoč [4, 5].

2. MAIN RESULT

Theorem. *Let* $F : Q^{1+p} = [0, 1] \times [0, 1]^p \to \mathscr{K}^n$, *where* p, n *are positive integers, satisfy the following assumptions*:

(9) *for every* $\varepsilon > 0$ *there exists a measurable set* $A_\varepsilon \subset [0, 1]$ *such that* $m([0, 1] \setminus \setminus A_\varepsilon) < \varepsilon$ *and* $F|_{A_\varepsilon \times [0,1]^p}$ *is upper semicontinuous*;

(10) $$F(t, x) \subset [-1, 1]^n \quad for \quad (t, x) \in Q^{1+p} ;$$

(11) $$F(t, x) = \mathscr{F} F(t, x) \quad for \quad (t, x) \in Q^{1+p}$$

(*cf.* (5); $\mathscr{F} F(t, x) = \mathscr{F} F_t(x)$, *where* $F_t = F(t, \cdot)$).

Then there exists a set $T \subset [0, 1]$, $m_1([0, 1] \setminus T) = 0$, *and a measurable function* $f : Q^{1+p} \to [-1, 1]^n$ *such that*

(12) $$F(t, x) = \mathscr{F} f(t, x)$$

for $(t, x) \in T \times [0, 1]^p$.

Remark. The assumption (9) accords with the property (8) of $\mathscr{F}f$, mentioned in Introduction. The identity (12) together with (7) implies necessity of the assumption (11). On the other hand, the assumption (10) is technical and may be weakened.

3. AUXILIARY RESULTS

Definition. Let $M_0 \subset M \subset \mathbb{R}^r$ be measurable sets. We say that M_0 is *metrically dense in* M if it satisfies the following condition:
(13) If $V \subset \mathbb{R}^r$ is open and $m_r(M \cap V) > 0$, then

$$m_r(M_0 \cap V) > 0 .$$

For $M \subset \mathbb{R}^r$, $\xi \in \mathbb{R}^q$ denote

(14) $$M(\xi, \cdot) = \{x \in \mathbb{R}^{r-q};\ (\xi, x) \in M\} .$$

Lemma 1. *Let* r *be a positive integer,* $A \subset [0, 1]^r$ *a measurable set,* $m_r(A) > 0$. *Let* $0 < \varkappa < 1$.

Then there exist measurable sets D, E *such that* $D \cap E = \emptyset$, $D \cup E = A$ *and*

(15) $$0 < m_r(D) < \varkappa\, m_r(A) , \quad 0 < m_r(E) ;$$

(16) $$both\ D, E\ are\ metrically\ dense\ in\ A .$$

Moreover, if $r > 1$ *and* $t \in \mathbb{R}$ *then*

(17) $$0 < m_{r-1}(D(t, \cdot)) < \varkappa\, m_{r-1}(A(t, \cdot)) , \quad 0 < m_{r-1}(E(t, \cdot))$$

provided $m_{r-1}(A(t, \cdot)) > 0$, *and*

(18) *both* $D(t, \cdot)$, $E(t, \cdot)$ *are metrically dense in* $A(t, \cdot)$.

Remark. Lemma 1 actually claims that every measurable set in $\mathbb{R}^r$ can be split into two disjoint parts, each of them metrically dense and one of an "arbitrarily small" measure, and that this remains valid for the cuts of the set in $\mathbb{R}^{r-1}$.

Proof. First we shall prove the lemma provided $r = 1$, $A = [0, 1]$, $0 < \varkappa < 1$. We shall use the well-known Cantor's sets of positive measure (for construction, see e.g. [6, Chap. 8, Ex. 4]), and construct the set D from the lemma as a union of countably many sets of this type.

Let C_1 be Cantor's set on $[0, 1]$ with $m_1(C_1) = \frac{1}{2}\varkappa$; its complement in $[0, 1]$ has a measure $m_1([0, 1] \setminus C_1) = 1 - \frac{1}{2}\varkappa$ and consists of countably many disjoint intervals, say $\mathcal{I}_{i2}$, $i = 1, 2, \ldots$. On each interval $\mathcal{I}_{i2}$ we construct Cantor's set C_{i2} with $m_1(C_{i2}) = \frac{1}{4}\varkappa\, m_1(\mathcal{I}_{i2})$. Denoting $C_2 = \bigcup_i C_{i2}$, we have $m_1(C_2) = \frac{1}{4}\varkappa \sum_i m_1(\mathcal{I}_{i2}) = \frac{1}{4}\varkappa\, m_1([0, 1] \setminus C_1) = \frac{1}{4}\varkappa(1 - \frac{1}{2}\varkappa)$ and $m_1([0, 1] \setminus (C_1 \cup C_2)) = 1 - \frac{1}{2}\varkappa - \frac{1}{4}\varkappa(1 - \frac{1}{2}\varkappa) = (1 - \frac{1}{2}\varkappa)(1 - \frac{1}{4}\varkappa)$. Proceeding by induction, we construct after n steps a set C_n which consists of countably many Cantor's sets and

$$m_1(C_n) = \frac{1}{2^n}\varkappa(1 - \tfrac{1}{2}\varkappa)\ldots\left(1 - \frac{1}{2^{n-1}}\varkappa\right),$$

$$m_1([0, 1] \setminus \bigcup_{j=1}^{n} C_j) = (1 - \tfrac{1}{2}\varkappa)\ldots\left(1 - \frac{1}{2^n}\varkappa\right).$$

Denote $D_1 = \bigcup_{n=1}^{\infty} C_n$, $E_1 = [0, 1] \setminus D_1$ so that $D_1 \cap E_1 = \emptyset$, $D_1 \cup E_1 = [0, 1]$. Further, $0 < m_1(C_n) < (1/2^n)\varkappa$, hence $0 < m_1(D_1) < \varkappa$, $m_1(E_1) > 1 - \varkappa > 0$. Finally, let V be an open subset of $[0, 1]$. It is easily seen that there exist positive integers j, k such that V contains the interval $\mathcal{I}_{jk}$. Cantor's set C_{jk} constructed on $\mathcal{I}_{jk}$ in the way described above has a measure $m_1(C_{jk}) = (1/2^k)\varkappa\, m_1(\mathcal{I}_{jk})$; due to the similarity of construction of Cantor's sets on $\mathcal{I}_{jk}$ and on the whole interval $[0, 1]$ we conclude that the union of all Cantor's sets C_{in} which fulfil the inclusion $C_{in} \subset \mathcal{I}_{jk}$ has a measure less than $m_1(\mathcal{I}_{jk})$ but certainly positive. This yields immediately $m_1(D_1 \cap \mathcal{I}_{jk}) > 0$, $m_1(E_1 \cap \mathcal{I}_{jk}) > 0$ and hence also $m_1(D_1 \cap V) > 0$, $m_1(E_1 \cap V) > 0$ which completes the proof of the lemma in the case $r = 1$, $A = [0, 1]$.

Now let $r = 1$, $A \subset [0, 1]$, $0 < \varkappa < 1$, A measurable with $m_1(A) > 0$. Denote by χ_A the charactetistic function of the set A and define

$$X(s) = \frac{1}{m_1(A)} \int_0^s \chi_A(\sigma)\, d\sigma \quad \text{for} \quad s \in [0, 1],$$

$$D = \{t \in A\,;\, X(t) \in D_1\}, \quad E = \{t \in A\,;\, X(t) \in E_1\},$$

where D_1, E_1 are the sets constructed in the first part of the proof. The sets D, E are evidently measurable, disjoint and $D \cup E = A$. Moreover,

$$\int_M \mathrm{d}\xi = \int_{X_{-1}(M)} \mathrm{d}X(\sigma) = \frac{1}{m_1(A)} \int_{X_{-1}(M)} \chi_A(\sigma)\,\mathrm{d}\sigma \tag{19}$$

provided $M \subset [0, 1]$ is measurable. (See e.g. [7], Chap. IV, Sec. 9.43, Theorem 1, or [8].) Substituting $M = D_1$, we obtain (notice that $D \subset A$)

$$\int_{D_1} \mathrm{d}\xi = \frac{1}{m_1(A)} \int_{X_{-1}(D_1)} \chi_A(\sigma)\,\mathrm{d}\sigma = \frac{1}{m_1(A)} \int_D \mathrm{d}\sigma\,,$$

i.e. $m_1(D_1)\, m_1(A) = m_1(D)$, and similarly $M = E_1$ yields $m_1(E_1)\, m_1(A) = m_1(E)$. Hence (15) holds (with $r = 1$) in virtue of the properties of the sets D_1, E_1.

Let $(\alpha, \beta) \subset [0, 1]$, $\alpha < \beta$ and $m_1(A \cap (\alpha, \beta)) > 0$. Then obviously $0 \leqq X(\alpha) < X(\beta) \leqq 1$ and the substitution $M = D_1 \cap (X(\alpha), X(\beta))$ or $M = E_1 \cap (X(\alpha), X(\beta))$ into (19) yields analogously as above

$$\begin{aligned} m_1(D_1 \cap (X(\alpha), X(\beta)))\, m_1(A) &= m_1(D \cap (\alpha, \beta))\,, \\ m_1(E_1 \cap (X(\alpha), X(\beta)))\, m_1(A) &= m_1(E \cap (\alpha, \beta))\,. \end{aligned} \tag{20}$$

This immediately implies the assertion (16) of the lemma.

Let us now pass to the general case of the lemma. If $A \subset [0, 1]^r$, $r > 1$, $m_r(A) > 0$, let us write the elements of the set A in the form (x, s), $x \in [0, 1]^{r-1}$, $s \in [0, 1]$. Let again $\chi_A(x, s)$ be the characteristic function of the set A and define

$$X(x, s) = \frac{1}{m_1(A(x,\cdot))} \int_0^s \chi_A(x, \sigma)\,\mathrm{d}\sigma$$

(cf. (14)).

The function $X(x, \cdot)$ is defined for $x \in Y = \{y \in [0, 1]^{r-1}\,;\, m_1(A(y, \cdot)) > 0\}$. As $m_1(A(x, \cdot)) = 0$ for $x \in [0, 1]^{r-1} \setminus Y$, we find that the function X is defined for all $(x, s) \in A \setminus N$, where $m_r(N) = 0$. We put

$$D = \{(x, s) \in A \setminus N\,;\, X(x, s) \in D_1\}\,, \quad E = \{(x, s) \in A \setminus N\,;\, X(x, s) \in E_1\}\,.$$

The sets D, E are evidently measurable, disjoint and $D \cup E = A \setminus N$. Quite analogously as in the preceding case we obtain the inequalities

$$\begin{aligned} &0 < m_1(D(x, \cdot)) < \varkappa\, m_1(A(x, \cdot))\,, \\ &0 < m_1(E(x, \cdot)) \end{aligned} \tag{21}$$

provided $m_1(A(x, \cdot)) > 0$. As $m_r(A) > 0$ by assumption, we immediately have the assertion (15) of the lemma.

Let V be an open set, $V \subset [0, 1]^r$, $m_r(A \cap V) > 0$. As we are concerned with the metrical density only, we may assume without loss of generality that V is a p-dimensional open interval. The set $V(x, \cdot)$ is a one-dimensional interval. As above, we establish identities analogous to (20):

$$m_1(D_1 \cap X(V(x, \cdot)))\, m_1(A(x, \cdot)) = m_1(D(x, \cdot) \cap V(x, \cdot)),$$

$$m_1(E_1 \cap X(V(x, \cdot)))\, m_1(A(x, \cdot)) = m_1(E(x, \cdot) \cap V(x, \cdot)).$$

If $m_1(A(x, \cdot)) > 0$, we have

$$m_1(D(x, \cdot) \cap V(x, \cdot)) > 0, \quad m_1(E(x, \cdot) \cap V(x, \cdot)) > 0 \tag{22}$$

and hence also

$$m_1(D_1 \cap X(V(x, \cdot))) > 0, \quad m_1(E_1 \cap X(V(x, \cdot))) > 0.$$

However, the set of x such that $m_1(A(x, \cdot)) > 0$ has a positive $(r - 1)$-dimensional measure in virtue of the assumption $m_r(A) > 0$. Hence the assertion (16) of the lemma follows.

Moreover, if $r = 2$ then the inequalities (21), (22) coincide (after unessential changes of notation) with the assertions (17), (18) of the lemma.

If $r > 2$, we can write the elements of A in the form (t, y, s), $t \in [0, 1]$, $y \in [0, 1]^{r-2}$, $s \in [0, 1]$ to obtain (21), (22) as above (with $x = (t, y)$). Now the assumption $m_{r-1}(A(t, \cdot, \cdot)) > 0$ implies (17), (18) in the same way as $m_r(A) > 0$ implies (15), (16). The proof of the lemma is complete.

4. FUNDAMENTAL LEMMA

Lemma. *Let $p \geqq 0$ be an integer, $A_j \subset [0, 1]^{1+p}$, $j = 1, 2, \ldots$. Let A_j be measurable, $\bigcup_{j=1}^{\infty} A_j = [0, 1]^{1+p}$. Then there exist measurable pairwise disjoint sets C_j, $C_j \subset A_j$ for $j = 1, 2, \ldots$ such that $\bigcup_{j=1}^{\infty} C_j = [0, 1]^{1+p}$ and C_j is metrically dense in A_j, $j = 1, 2, \ldots$.*

Moreover, if $t \in [0, 1]$, then $C_j(t, \cdot)$ is metrically dense in $A_j(t, \cdot)$, $j = 1, 2, \ldots$.

Proof. We shall describe a step-by-step construction which eventually yields the sets C_j from Fundamental Lemma.

1st step. Denote $A_1 = C_1^1$.

kth step, $k \geqq 2$. We start with disjoint measurable sets $C_1^{k-1}, C_2^{k-1}, \ldots, C_{k-1}^{k-1}$ and the set A_k. We introduce the following family of cubes:

Let $r = (r_1, r_2, \ldots, r_{p+1})$ be a $(p + 1)$-tuple of integers, $0 \leqq r_j \leqq 2^k - 1$ for $j = 1, 2, \ldots, p + 1$ and denote

$$(22a) \qquad K_r^k = (r_1 2^{-k}, (r_1 + 1) 2^{-k}) \times \ldots \times (r_{p+1} 2^{-k}, (r_{p+1} + 1) 2^{-k}) .$$

According to Lemma 1 we find for each set $C_i^{k-1} \cap A_k \cap K_r^k$ (which is obviously measurable) disjoint measurable sets ${}^r D_i^{k-1}, {}^r E_i^{k-1}$ such that

$$C_i^{k-1} \cap A_k \cap K_r^k = {}^r D_i^{k-1} \cup {}^r E_i^{k-1} ,$$

which satisfy the assertion of Lemma 1 with $\varkappa = \varkappa_k = 1/2^k$ and with the set $A = C_i^{k-1} \cap A_k \cap K_r^k$.

Denote $D_i^{k-1} = \bigcup_r {}^r D_i^{k-1}$, $E_i^{k-1} = \bigcup_r {}^r E_i^{k-1}$, the unions being taken over all multi-indices r described above. It is evident that these sets D_i^{k-1}, E_i^{k-1} are again measurable and disjoint. We may assume that

$$(23) \qquad C_i^{k-1} \cap A_k = D_i^{k-1} \cup E_i^{k-1} \ *)$$

and the sets D_i^{k-1}, E_i^{k-1} satisfy the assertion of Lemma 1 with $\varkappa = \varkappa_k = 1/2^k$ and with the set $C_i^{k-1} \cap A_k$ instead of A. We denote

$$(24) \qquad C_i^k = C_i^{k-1} \setminus D_i^{k-1} \quad \text{for} \quad i = 1, 2, \ldots, k - 1 ,$$
$$C_k^k = A_k \setminus \bigcup_{i=1}^{k-1} E_i^{k-1} .$$

Let us denote

$$(25) \qquad C_j = \bigcap_{k=j}^{\infty} C_j^k .$$

We assert that these sets meet the requirements of Fundamental Lemma.

Indeed, we have

$$C_j^k = A_j \setminus [\bigcup_{i=1}^{j-1} E_i^{j-1} \cup \bigcup_{i=j}^{k-1} D_j^i]$$

provided $k \geqq j$, hence $C_j \subset A_j$. Further, it is clear from the construction that $C_{j_1}^k \cap C_{j_2}^k = \emptyset$ provided $j_1 \neq j_2$. Hence evidently $C_{j_1} \cap C_{j_2} = \emptyset$ under the same assumption according to (25). The measurability of C_j is evident. We shall prove that

$$(26) \qquad m_{p+1}(\bigcup_{j=1}^{\infty} C_j) = 1 .$$

To this aim we shall first prove that

$$(27) \qquad \bigcup_{j=1}^{k} C_j^k = \bigcup_{j=1}^{k} A_j .$$

*) Actually, the set $(C_i^{k-1} \cap A_k) \setminus (D_i^{k-1} \cup E_i^{k-1})$ is not void but only of measure zero. Nonetheless, this does not affect our further considerations.

Indeed, for $k = 1$ the identity is immediately verified by the first step of the construction. Further, (24) implies

$$\bigcup_{j=1}^{k} C_j^k = \bigcup_{j=1}^{k-1} (C_j^{k-1} \setminus D_j^{k-1}) \cup (A_k \setminus \bigcup_{j=1}^{k-1} E_j^{k-1}) . \tag{28}$$

The sets D_j^{k-1}, E_j^{k-1} being disjoint, we have by (23)

$$E_j^{k-1} \subset C_j^{k-1} \setminus D_j^{k-1} , \quad D_j^{k-1} \subset A_k \setminus E_j^{k-1}$$

and, since $C_1^{k-1}, \ldots, C_{k-1}^{k-1}$ are pairwise disjoint, we have even

$$D_j^{k-1} \subset A_k \setminus \bigcup_{i=1}^{k-1} E_i^{k-1}$$

(here always $i, j \leqq k - 1$). Hence (28) yields

$$\bigcup_{j=1}^{k} C_j^k = \bigcup_{j=1}^{k-1} C_j^{k-1} \cup A_k$$

and, by the induction hypothesis, we conclude that (27) holds.

Now assume that (26) is false, i.e.

$$m_{p+1}(\bigcup_{j=1}^{\infty} C_j) = \alpha < 1 . \tag{29}$$

The first formula in (24) implies $C_j = C_j^k \setminus \bigcup_{l=k}^{\infty} D_j^l$ $(k \geqq j)$ and hence

$$m_{p+1}(C_j^k) \leqq m_{p+1}(C_j) + \sum_{l=k}^{\infty} m_{p+1}(D_j^l) \leqq m_{p+1}(C_j) + 2^{-k+1}$$

(cf. (15) in Lemma 1; recall that $\varkappa_k = 2^{-k}$).

Thus

$$m_{p+1}(\bigcup_{j=1}^{k} C_j^k) \leqq \sum_{j=1}^{k} m_{p+1}(C_j^k) \leqq \sum_{j=1}^{k} [m_{p+1}(C_j) + 2^{-k+1}] \leqq \alpha + k\, 2^{-k+1}$$

by (29). On the other hand, (27) together with the last inequality yields $m_{p+1}(\bigcup_{j=1}^{\infty} A_j) < 1$ which contradicts the assumption of Fundamental Lemma. Hence (26) holds and consequently

$$m_{p+1}([0, 1]^{1+p} \setminus \bigcup_{j=1}^{\infty} C_j) = 0 .$$

Since adding sets of measure zero to the sets C_j does not affect their properties involved in Fundamental Lemma, we may assume without loss of generality that $\bigcup_{j=1}^{\infty} C_j = [0. 1]^{1+p}$.

The last assertion of Fundamental Lemma to be proved is that concerning the metrical density of the sets C_j and $C_j(t, \cdot)$ in A_j and $A_j(t, \cdot)$, respectively. We shall prove it for C_j, A_j, the proof for $C_j(t, \cdot)$, $A_j(t, \cdot)$ being quite analogous.

Thus, let $V \subset \mathbb{R}^{1+p}$ be open,

$$m_{1+p}(A_j \cap V) > 0 . \tag{30}$$

We shall prove by induction that then

$$m_{1+p}(C_j^k \cap V) > 0 , \quad k \geqq j . \tag{31}$$

Firstly, recalling (23) we have

$$\begin{aligned} A_j &= (A_j \cap C_1^{j-1}) \cup (A_j \cap C_2^{j-1}) \cup \ldots \cup (A_j \cap C_{j-1}^{j-1}) \cup \\ &\cup [A_j \setminus (C_1^{j-1} \cup C_2^{j-1} \cup \ldots \cup C_{j-1}^{j-1})] = \\ &= (\bigcup_{i=1}^{j-1} D_i^{j-1}) \cup (\bigcup_{i=1}^{j-1} E_i^{j-1}) \cup (A_j \setminus \bigcup_{i=1}^{j-1} C_i^{j-1}) . \end{aligned}$$

Now (30) implies that either

$$m_{1+p}[(A_j \setminus \bigcup_{i=1}^{j-1} C_i^{j-1}) \cap V] > 0$$

or there exists h, $1 \leqq h \leqq j - 1$, such that

$$m_{1+p}[(A_j \cap C_h^{j-1}) \cap V] > 0 .$$

In both cases we conclude with regard to the second identity in (24) that (31) with $k = j$ holds. (Recall (23) and, in the latter case, also the fact that D_h^{j-1} is metrically dense in $A_j \cap C_h^{j-1}$.)

Secondly, let us assume that (30) implies (31) with $k = 1, 2, \ldots, s - 1$ where $s > j$. We have

$$C_j^s = C_j^{s-1} \setminus D_j^{s-1} = (C_j^{s-1} \setminus A_s) \cup E_j^{s-1}$$

(cf. (24), (23)). By the induction hypothesis it is either

$$m_{1+p}[(C_j^{s-1} \setminus A_s) \cap V] > 0$$

or

$$m_{1+p}(C_j^{s-1} \cap A_s \cap V) > 0 ;$$

in the former case, (31) with $k = s$ follows immediately, in the latter we recall similarly as above that E_j^{s-1} is metrically dense in $C_j^{s-1} \cap A_s$ which completes the proof of implication (30) $\Rightarrow$ (31).

We have (cf. (25), (24))

$$C_j = \bigcap_{k=j}^{\infty} C_j^k = \bigcap_{k=q}^{\infty} C_j^k = C_j^q \setminus \bigcup_{k=q}^{\infty} D_j^k$$

provided $q \geqq j$. Considering an open cube K_r^q (see (22a)) with $q \geqq j$, we can write

$$\mu = m_{1+p}(C_j \cap K_r^q) = m_{1+p}(C_j^q \cap K_r^q) - \sum_{k=q}^{\infty} m_{1+p}(D_j^k \cap K_r^q).$$

As $D_j^k = \bigcup_\varrho {}^\varrho D_j^k$ by construction where ϱ denotes multiindices described at the beginning of the proof, and ${}^\varrho D_j^k \subset K_\varrho^{k+1}$, we may write further

$$\mu = m_{1+p}(C_j^q \cap K_r^q) - \sum_{k=q}^{\infty} \sum_\varrho{}^* m_{1+p}({}^\varrho D_j^k \cap K_\varrho^{k+1}),$$

where the star indicates that the sum is taken over all multiindices ϱ such that ${}^\varrho D_j^k \subset K_\varrho^{k+1} \subset K_r^q$. Thus (cf. (23) and the text below)

$$\mu \geqq m_{1+p}(C_j^q \cap K_r^q) - \sum_{k=q}^{\infty} \varkappa_{k+1} \sum_\varrho{}^* m_{1+p}(C_j^k \cap A_{k+1} \cap K_\varrho^{k+1}) \geqq$$

$$\geqq m_{1+p}(C_j^q \cap K_r^q) - \sum_{k=q}^{\infty} \varkappa_{k+1} m_{1+p}(C_j^q \cap K_r^q) = m_{1+p}(C_j^q \cap K_r^q)(1 - 2^{-q}).$$

Thus, in virtue of the above proved implication (30) $\Rightarrow$ (31), $m_{1+p}(A_j \cap K_r^q) > 0$ implies $m_{1+p}(C_j \cap K_r^q) > 0$. Since the sets K_r^q form a countable basis of open sets in $\mathbb{R}^{1+p}$, this implication yields metrical density of C_j in A_j.

Fundamental Lemma is completely proved.

5. PROOF OF MAIN RESULT

Let j be a positive integer, $r = (r_1, \ldots, r_n)$ a multiindex, r_i integers, $0 \leqq r_i \leqq 2^j - 1$ for $i = 1, 2, \ldots, n$. Let us denote

$$Q_r^j = [r_1 2^{-j}, (r_1 + 1) 2^{-j}] \times \ldots \times [r_n 2^{-j}, (r_n + 1) 2^{-j}]$$

(a closed cube in $\mathbb{R}^n$ with edges of the length 2^{-j}),

$$q_r^j = (r_1 2^{-j}, \ldots, r_n 2^{-j}) \in \mathbb{R}^n.$$

Further, denote

$$A_r^j = \{(t, x); F(t, x) \cap Q_r^j \neq \emptyset\}. \tag{32}$$

The family of sets A_r^j for r, j described above is countable, the sets A_r^j are measurable (cf. (9)) and

$$\bigcup_{r,j} A_r^j = [0, 1]^{1+p}.$$

(Evidently, it is even

$$\bigcup_r A_r^j = [0, 1]^{1+p} \tag{33}$$

for every positive integer j – cf. (10).)

By Fundamental Lemma there exist measurable pairwise disjoint sets C_r^j, $C_r^j \subset A_r^j$ such that

$$\bigcup_{r,j} C_r^j = [0, 1]^{1+p} \tag{34}$$

and the sets C_r^j, $C_r^j(t, \cdot)$ are metrically dense in A_r^j, $A_r^j(t, \cdot)$, respectively, for $t \in [0,1]$. (To avoid misunderstanding, let us point out that the sets C_r^j correspond to those denoted in Fundamental Lemma by C_j.)

We shall construct the function f satisfying (12) as the limit of a sequence of functions f_k.

For $(t, x) \in C_r^j$ let us define $f_1(t, x) = q_r^j$. Then obviously

$$d(f_1(t, x), F(t, x)) \leqq \tfrac{1}{2}$$

by (32). (For the sake of simplicity, we take $d(\xi, \eta) = \max |\xi_i - \eta_i|$ if $\xi = (\xi_1, \ldots$ $\ldots, \xi_n) \in \mathbb{R}^n$, $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, and define the distance of a point from a set as usual.) Due to (34) and to the disjointness of the sets C_r^j the function f_1 is defined uniquely on the whole $[0, 1]^{1+p}$.

Now let us assume that $k \geqq 2$ and the function $f_{k-1} : [0, 1]^{1+p} \to \mathbb{R}^n$ has been defined so that it is measurable, satisfies

$$d(f_{k-1}(t, x), F(t, x)) \leqq 2^{-(k-1)} \tag{35}$$

for $(t, x) \in [0, 1]^{1+p}$, and for $(t, x) \in C_r^j$ its value is $f_{k-1}(t, x) = q_\varrho^l$ where $l \geqq j$, $Q_\varrho^l \subset Q_r^j$. (For $k = 2$, these conditions for $f_{k-1} = f_1$ are immediately verified.)

If $(t, x) \in C_r^j$, then three cases excluding each other may occur:

(i) $j > k - 1$;
(ii) $j \leqq k - 1$, $d(f_{k-1}(t, x), F(t, x)) \leqq 2^{-k}$;
(iii) $j \leqq k - 1$, $d(f_{k-1}(t, x), F(t, x)) > 2^{-k}$.

In the cases (i), (ii) put

$$f_k(t, x) = f_{k-1}(t, x) .$$

To verify the inequality (35) with k instead of $k - 1$, notice that in the case (i) we evidently have $d(z, F(t, x)) \leqq 2^{-j} \leqq 2^{-k}$ for $z \in Q_r^j$, hence in particular $d(q_\varrho^l, F(t, x)) \leqq 2^{-k}$ for $f_{k-1}(t, x) = q_\varrho^l \in Q_\varrho^l \subset Q_r^j$; the case (ii) is trivial.

In the case (iii), notice that the cube Q_ϱ^l may be divided into 2^n parts, namely the cubes Q_σ^{l+1} where $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i = 2\varrho_i$ or $\sigma_i = 2\varrho_i + 1$. We can always choose one of these smaller cubes, say Q_s^{l+1}, such that

$$d(q_s^{l+1}, F(t, x)) \leqq 2^{-k} . \tag{36}$$

(In order to determine this choice uniquely, we have to order the multiindices σ in a fixed way and then to use always the "least" multiindex s satisfying (36).) Now we set

$$f_k(t, x) = q_s^{l+1}$$

which completes the definition of f_k on the whole interval $[0, 1]^{1+p}$. The function f_k evidently satisfies the assumptions imposed on f_{k-1} (with k instead of $k-1$).

Moreover, it is clear that if $(t, x) \in C_r^j$ then $f_1(t, x) = f_2(t, x) = \ldots = f_j(t, x)$. Further,

$$|f_s(t, x) - f_{s-1}(t, x)| \leqq 2^{-s}$$

for $s = j, j+1, \ldots,$ hence

$$|f_j(t, x) - f_{j+i}(t, x)| \leqq 2^{-j}$$

provided i is a positive integer.

Thus the (pointwise) limit

$$f(t, x) = \lim_{k \to \infty} f_k(t, x)$$

exists and (35) implies

$$f(t, x) \in F(t, x)$$

and consequently

(37) $$\mathscr{F} f(t, x) \subset F(t, x)$$

(cf. Introduction, (4)).

It remains to prove the converse inclusion, i.e.

(38) $$F(t, x) \subset \mathscr{F} f(t, x) .$$

Let us denote

$$W = \{(t, x); F(t, x) \setminus \mathscr{F} f(t, x) \neq \emptyset\}$$

and assume

(39) $$m_{1+p}(W) > 0 .$$

Denote

(40) $$B_r^j = \{(t, x); \mathscr{F} f(t, x) \cap Q_r^j = \emptyset\} ,$$

$$H_r^j = A_r^j \cap B_r^j .$$

Then it is easy to see that

(41) $$W = \bigcup_{j,r} H_r^j .$$

Indeed, if $(t, x) \in W$ then there is $z \in F(t, x)$ with $d(z, \mathscr{F} f(t, x)) > 0$; hence there is Q_r^j, $\mathscr{F} f(t, x) \cap Q_r^j = \emptyset$ and simultaneously $z \in Q_r^j$ which together with the inclusion $z \in F(t, x)$ yields $(t, x) \in H_r^j$. On the other hand, if $(t, x) \in H_r^j$ then there is $z \in F(t, x) \cap Q_r^j$ and hence $z \in Q_r^j$; thus necessarily $z \notin \mathscr{F} f(t, x)$ which completes the proof of (41).

The relations (39), (41) imply that there exist ϱ, ι such that

$$m_{1+p}(H^{\iota}_{\varrho}) > 0\,. \tag{42}$$

We shall prove that then also

$$m_{1+p}(C^{\iota}_{\varrho} \cap B^{\iota}_{\varrho}) > 0\,. \tag{43}$$

Indeed, by (2') we know that $\mathscr{F} f(t, \cdot)$ is upper semicontinuous for $t \in [0, 1]$. If $t \in [0, 1]$, $x \in B^{\iota}_{\varrho}(t, \cdot)$ then $\mathscr{F} f(t, x) \cap Q^{\iota}_{\varrho} = \emptyset$ by definition of B^{ι}_{ϱ}, hence there exists $\varepsilon > 0$ such that $\Omega(\mathscr{F} f(t, x), \varepsilon) \cap Q^{\iota}_{\varrho} = \emptyset$. By the upper semicontinuity of $\mathscr{F} f(t, \cdot)$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\mathscr{F} f(t, y) \subset \Omega(\mathscr{F} f(t, x), \varepsilon)$, consequently $\mathscr{F} f(t, y) \cap Q^{\iota}_{\varrho} = \emptyset$. Thus $B^{\iota}_{\varrho}(t. \cdot)$ is open (in $\mathbb{R}^p$) provided $t \in [0, 1]$. (The symbol $\Omega(M, \varepsilon)$ means the ε-neighbourhovp of the set M.)

Now (42) implies that there is a set $A \subset [0, 1]$, $m_1(A) > 0$, such that

$$m_p(H^{\iota}_{\varrho}(t, \cdot)) = m_p(A^{\iota}_{\varrho}(t, \cdot) \cap B^{\iota}_{\varrho}(t, \cdot)) > 0$$

for $t \in A$. Taking into account that the sets C^j_r have been constructed so as to satisfy the assertions of Fundamental Lemma (see (34) and the following text) and the just proved openness of $B^{\iota}_{\varrho}(t, \cdot)$ we conclude that also

$$m_p(C^{\iota}_{\varrho}(t, \cdot) \cap B^{\iota}_{\varrho}(t, \cdot)) > 0$$

for $t \in A$. This evidently yields (43).

Let $(t, x) \in C^{\iota}_{\varrho} \cap B^{\iota}_{\varrho}$. Then, as mentioned above, $f_1(t, x) = \ldots = f_{\iota}(t, x) = q^{\iota}_{\varrho}$, $|f(t, x) - f_{\iota}(t, x)| \leqq 2^{-\iota}$, i.e.

$$|f(t, x) - q^{\iota}_{\varrho}| \leqq 2^{-\iota}\,.$$

Moreover, it follows from the construction that

$$f(t, x) \in Q^{\iota}_{\varrho}\,.$$

Since $f(t, x) \in \mathscr{F} f(t, x)$ for almost all (t, x) (cf. (3')), we have

$$\mathscr{F} f(t, x) \cap Q^{\iota}_{\varrho} \neq \emptyset \tag{44}$$

for $(t, x) \in C^{\iota}_{\varrho} \cap B^{\iota}_{\varrho} \setminus N$, $m_{1+p}(N) = 0$ (this is a nonempty set due to (43)). However, (44) implies $(t, x) \notin H^{\iota}_{\varrho}$ (cf. (40)) which is a contradiction since $C^{\iota}_{\varrho} \cap B^{\iota}_{\varrho} \subset H^{\iota}_{\varrho}$. Thus (39) is impossible and hence

$$m_{1+p}(W) = 0\,.$$

The rest of the proof is easy. There exists a set $N_1 \subset [0, 1]$, $m_1(N_1) = 0$ such that $m_p(W(t, \cdot)) = 0$ provided $t \in [0, 1] \setminus N_1$. This means $F(t, x) \subset \mathscr{F} f(t, x)$ for almost all x ($t \in [0, 1] \setminus N_1$ being arbitrary but fixed) which obviously yields

$$\mathscr{F} F(t, x) \subset \mathscr{F} f(t, x)$$

for all x (t as above) since $\mathscr{F}\mathscr{F} f = \mathscr{F} f$.

Now the assumption (11) yields (38) which completes the proof of **Main Result.**

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Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).