Ülo Lumiste

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CONNECTIONS IN ASSOCIATED FIBRE BUNDLES

ÜLO LUMISTE*, Tartu

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The purpose of this paper is to give effective conditions for a maximal transversal distribution on an associated fibre bundle to be the horizontal distribution of a connection. In the particular case when the typical fibre is a homogeneous space, such "horizontality conditions" were given in [9]. Here the generalization will be done for the case when the typical fibre is an arbitrary G-space (Theorem 2). The consequence is a splitting of the curvature form into a torsion form and curvature forms of orders $s = 1, \ldots, p$, where p is the order of the higher isotropy of the typical fibre.

The method here differs from the method in [9], where we used more specified frame bundles and where semi-holonomic prolongations were involved. Now in the general situation all prolongations will be holonomic, which makes the form of our conditions simpler.

Our work on this paper was inspired by the great attention, which in the recent years has been paid to the theory of connections by the mathematical physicists, especially in the gauge theory (see, for example, [14] and [15]).

1. THE CARTAN-LAPTEV THEOREM AND THE CONNECTION OBJECT

Let $P(M_n, G_r)$ be a principal fibre bundle and $L_0(P)$ a subbundle of the tangential frame bundle of P formed by all frames $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+r}\}$, where $e_{n+1}, \ldots$ $\ldots, e_{n+r}$ belong to r fixed fundamental vector fields on P. For the canonical form ω on P we have

$$\omega = e_i \omega^i + e_\alpha \omega^\alpha ,$$

where ω^i and ω^α are semi-basic 1-forms on $L_0(P)$. The following structure equations hold, given for the first time by G. F. Laptev [7], [8]:

$$(1) \qquad d\omega^i = \omega^j \wedge \omega^i_j ,$$

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(2) $$d\omega^\alpha = -\tfrac{1}{2}C^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma + \omega^i \wedge \omega^\alpha_i\,,$$

where $C^\alpha_{\beta\gamma}$ are the structure constants of G_r.

A distribution π on P is called *transversal* if $\pi_z \cap T_zG_z = \{0\}$ at every point $z \in P$. A *maximal transversal* distribution π is characterized by $T_zP = \pi_z + T_zG_z$ and can be given as the annihilator of a system of 1-forms

(3) $$\theta^\alpha = \omega^\alpha - \pi^\alpha_i\omega^i\,.$$

Here $\theta^\alpha(X)$, for an arbitrary vector field X on P, are the components of $\mathrm{pr}_2 X$, where pr_2 is the second projector for $\pi_z + T_zG_z$. Therefore θ^α are 1-forms on P and consequently $d\theta^\alpha$ is expressed only in terms of ω^i and ω^α, i.e.

(4) $$d\theta^\alpha = -\tfrac{1}{2}C^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma + R^\alpha_{ij}\omega^i \wedge \omega^j + S^\alpha_{i\beta}\omega^i \wedge \omega^\beta\,.$$

Introducing the g-valued 1-form $\theta = \theta^\alpha A^*_\alpha$, $A_\alpha = e_\alpha$, and calling a 2-form Θ semi-basic provided $\Theta(X, Y) = 0$ for vertical X, or Y, we can state:

Theorem 1. (Cartan-Laptev theorem). *Let $P(M_n, G_r)$ be a principal fibre bundle with a connected structure group G_r. A maximal transversal distribution π on P, which annihilates θ, is G-invariant (i.e. π is the horizontal distribution of a connection) iff the g-valued 2-form*

$$\Theta = d\theta + [\theta, \theta]$$

is semi-basic.

Proof uses the following well-known result (for the necessity see [16], and for the sufficiency [9]): π is G-invariant iff for any vector field Y on P satisfying $Y_z \in \pi_z$ $(z \in P)$ and for any fundamental vector field Z we have $[Y, Z]_z \in \pi_z$.

Now, if we evaluate both sides of (4), calculating $d\theta^\alpha(Y, Z)$, then due to $\theta^\alpha(Y) = 0$, $\omega^i(Z) = 0$ and $\theta^\alpha(Z) = Z^\alpha = \text{const}$, we have

$$-2\theta^\alpha[Y, Z]) = S^\alpha_{i\beta}\,\omega^i(Y)\,\omega^\beta(Z)$$

and the last condition is equivalent to $S^\alpha_{i\beta} = 0$, which verifies the theorem.

It follows from (2) and (3) that

(5) $$d\theta^\alpha = \tfrac{1}{2}C^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma - \tfrac{1}{2}C^\alpha_{\beta\gamma}\pi^\beta_i\pi^\gamma_j\omega^i \wedge \omega^j - \Delta\pi^\alpha_i \wedge \omega^i\,,$$

where

(6) $$\Delta\pi^\alpha_i = d\pi^\alpha_i - \pi^\alpha_j\omega^j_i + \pi^\beta_i\omega^\alpha_\beta + \omega^\alpha_i\,,$$

(7) $$\omega^\alpha_\beta = C^\alpha_{\beta\gamma}\omega^\gamma\,.$$

Thus, by virtue of Theorem 1, the annihilator π of the system of 1-forms (3) is G-invariant iff

(8) $$\Delta\pi^\alpha_i = \pi^\alpha_{ij}\omega^j\,.$$

These equations (8) are called horizontality conditions of the maximal transversal distribution π. (They show that the functions π_i^α on $L_0(P)$ define a geometrical object called the connection object [7]). From (5) it follows that

$$\Theta^\alpha = R^\alpha_{ij}\omega^i \wedge \omega^j ,$$

where

$$R^\alpha_{ij} = \pi^\alpha_{[ij]} - \tfrac{1}{2}C^\alpha_{\beta\gamma}\pi^\beta_i\pi^\gamma_j ,$$

so that

$$\mathrm{d}\theta^\alpha = -\tfrac{1}{2}C^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma + \Theta^\alpha . \tag{9}$$

2. MAXIMAL TRANSVERSAL DISTRIBUTION ON AN ASSOCIATED FIBRE BUNDLE

Let $E(P, F)$ be a fibre bundle with a typical fibre F, associated to $P(M_n, G_r)$, and let G_r be connected and act on F effectively to the left. Here E is the coset manifold of $P \times F$ with respect to the action of G_r given by $(z, x) \circ g = (z \circ g, g^{-1} \circ x)$. The orbits are maximal integral submanifolds in $P \times F$ of the involutive distribution, given, for every coordinate neighbourhood U of F, as the annihilator of the system of forms

$$\omega^i, \Omega^a = \mathrm{d}x^a + \xi^a_\alpha(x)\,\omega^\alpha ; \quad a = 1, \ldots, m \quad (= \dim F) ; \tag{10}$$

where x^a are the local coordinate functions on U and

$$\xi^b_\alpha(\partial_b\xi^a_\beta) - \xi^b_\beta(\partial_b\xi^a_\alpha) = -C^\gamma_{\alpha\beta}\xi^a_\gamma \tag{11}$$

(see [4]).

From (2) and (11) it now follows that

$$\mathrm{d}\Omega^a = \Omega^b \wedge \Omega^a_b + \omega^i \wedge \Omega^a_i , \tag{12}$$

where

$$\Omega^a_b = (\partial_b\xi^a_\alpha)\,\omega^\alpha , \tag{13}$$

$$\Omega^a_i = \xi^a_\alpha\omega^\alpha_i . \tag{14}$$

If we differentiate (11) s-times, we obtain

$$2\sum_{t=0}^{s}\binom{s}{t}\{(\partial_{(b_1\ldots b_t}\xi^b_{|\alpha|})\,(\partial_{b_{t+1}\ldots b_s)b}\xi^a_\beta)\}_{[\alpha,\beta]} = -C^\gamma_{\alpha\beta}(\partial_{b_1\ldots b_s}\xi^a_\gamma) , \tag{15}$$

where $[\alpha, \beta]$ means alternation with respect to α and β.

Therefore, for 1-forms (13) and

$$\Omega^a_{b_1\ldots b_s} = (\partial_{b_1\ldots b_s}\xi^a_\alpha)\,\omega^\alpha , \tag{16}$$

where $s = 2, \ldots, p,$ we get

$$(17) \qquad d\Omega^a_b = \Omega^c_b \wedge \Omega^a_c + \Omega^c \wedge \Omega^a_{bc} + \omega^i \wedge \Omega^a_{bi} ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(18) \quad d\Omega^a_{b_1 \ldots b_p} = \sum_{t=1}^{p} \binom{p}{t} \Omega^c_{(b_1 \ldots b_t} \wedge \Omega^a_{b_{t+1} \ldots b_p)c} + \Omega^c \wedge \Omega^a_{b_1 \ldots b_p c} + \omega^i \wedge \Omega^a_{b_1 \ldots b_p i} .$$

Let a maximal transversal distribution Γ on E be given as the annihilator of the system of forms

$$\psi^a = \Omega^a - \Gamma^a_i \omega^i .$$

For every two commuting vector fields W and W' on the frame subbundle over E, where W is vertical (i.e. $\omega^i(W) = \Omega^a(W) = 0$), we have $W\psi^a(W') = -\psi^b(W')\,\Omega^a_b(W) - [W\Gamma^a_i - \Gamma^a_j\,\omega^j_i(W) + \Gamma^b_i\,\Omega^a_b(W) + \Omega^a_i(W)]\,\omega^i(W')$. Since Γ_w is invariant, the expression in the brackets must be zero and therefore

$$(19) \qquad \nabla\Gamma^a_i = \Gamma^a_{bi}\Omega^b + \Gamma^a_{ij}\omega^j ,$$

where

$$(20) \qquad \nabla\Gamma^a_i = d\Gamma^a_i - \Gamma^a_j\omega^j_i + \Gamma^b_i\Omega^a_b + \Omega^a_i .$$

Using the exterior differentiation, (17) and the formulas

$$d\omega^i_j = \omega^k_j \wedge \omega^i_k + \omega^k \wedge \omega^i_{jk} ,$$

$$d\Omega^a_i = \omega^j_i \wedge \Omega^a_j + \Omega^b_i \wedge \Omega^a_b + \Omega^b \wedge \Omega^a_{bi} + \omega^i \wedge \Omega^a_{ij} ,$$

which follow from (1) and (14), we obtain

$$\nabla\Gamma^a_{bi} \wedge \Omega^b + \nabla\Gamma^a_{ij} \wedge \omega^j = 0 ,$$

where, in particular,

$$(21) \qquad \nabla\Gamma^a_{bi} = d\Gamma^a_{bi} - \Gamma^a_{bj}\omega^j_i - \Gamma^a_{ci}\Omega^c_b + \Gamma^c_{bi}\Omega^a_c + \Gamma^c_i\Omega^a_{bc} + \Omega^a_{bi} .$$

Consequently,

$$(22) \qquad \nabla\Gamma^a_{bi} = \Gamma^a_{bci}\Omega^c + \Gamma^a_{bij}\omega^j ,$$

where $\Gamma^a_{[bc]i} = 0$. This process of partial differential prolongation of the functions Γ^a_i can be repeated. As the result we get a sequence

$$(23) \qquad \Gamma^a_i, \Gamma^a_{bi}, \Gamma^a_{b_1 b_2 i}, \ldots, \Gamma^a_{b_1 \ldots b_p i}, \ldots$$

of systems of functions, symmetric in lower indices, which are iteratively generated by

$$(24) \qquad \nabla\Gamma^a_{b_1 \ldots b_s i} = \Gamma^a_{b_1 \ldots b_s b i}\Omega^b + \Gamma^a_{b_1 \ldots b_s i j}\omega^j ,$$

where

$$\nabla\Gamma^a_{b_1\ldots b_s i} = \mathrm{d}\Gamma^a_{b_1\ldots b_s i} - \Gamma^a_{b_1\ldots b_s j}\omega^j_i - \sum_{t=0}^{s}\binom{s}{t}\,\Omega^c_{(b_1\ldots b_t}\Gamma^a_{b_{t+1}\ldots b_s)c} + \tag{25}$$

$$+ \sum_{t=0}^{s}\binom{s}{t}\,\Gamma^c_{(b_1\ldots b_t}\Omega^a_{b_{t+1}\ldots b_s)c} + \Omega^a_{b_1\ldots b_s i}\,.$$

The sequence (23) characterizes the given maximal transversal distribution Γ on E. The main purpose of the paper is to get effective conditions on (23) for that Γ to be the horizontal distribution for a connection. To deduce it we need some results about the higher isotropy of the typical fibre F.

3. HIGHER ISOTROPY OF A G-SPACE

An arbitrary fibre in E, isomorphic to F, is given as an integral submanifold of the involutive system $\omega^i = 0$. Denoting $\tilde{\omega}^\alpha = \omega^\alpha \pmod{\omega^i}$, $\tilde{\xi}^a_\alpha = \xi^a_\alpha \pmod{\omega^i}$, $\tilde{\Omega}^a_{b_1\ldots b_s} = \Omega^a_{b_1\ldots b_s} \pmod{\omega^i}$ we get from (10), (13) and (16)

$$\tilde{\Omega}^a = \mathrm{d}x^a + \tilde{\xi}^a_\alpha(x)\,\tilde{\omega}^\alpha\,, \tag{26}$$

$$\tilde{\Omega}^a_b = (\partial_b\tilde{\xi}^a_\alpha)\,\tilde{\omega}^\alpha\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\tilde{\Omega}^a_{b_1\ldots b_s} = (\partial^a_{b_1\ldots b_s}\tilde{\xi}^a_\alpha)\,\tilde{\omega}^\alpha \tag{27}$$

and now (12), (17) … (18) give

$$\mathrm{d}\tilde{\Omega}^a = \tilde{\Omega}^b \wedge \tilde{\Omega}^a_b\,, \tag{28}$$

$$\mathrm{d}\tilde{\Omega}^a_b = \tilde{\Omega}^c_b \wedge \tilde{\Omega}^a_c + \tilde{\Omega}^c \wedge \tilde{\Omega}^a_{bc}\,, \tag{29}$$

$$\mathrm{d}\tilde{\Omega}^a_{b_1\ldots b_p} = \sum\binom{p}{t}\,\tilde{\Omega}^c_{(b_1\ldots b_t} \wedge \tilde{\Omega}^a_{b_{t+1}\ldots b_p)c} + \tilde{\Omega}^c \wedge \tilde{\Omega}^a_{b_1\ldots b_p c}\,. \tag{30}$$

The isomorphism between a fixed fibre and F makes it possible to use these formulas also for F. (Of course, they can be derived for F separately as well.) We shall use this remark in the following calculations.

If we fix a point $x_0 \in F$, then $\tilde{\Omega}^a = 0$, $\mathrm{d}x^a = 0$ and the system of equations $\tilde{\xi}^a_\alpha(x_0)\,\tilde{\omega}^\alpha = 0$, which is involutive on G due to (11), gives the isotropy group H_{x_0} of x_0 as its maximal integral submanifold through $e \in G$.

The formulas (28) … (30) with $\tilde{\Omega}^a = 0$ have the same form as the structure equations of the holonomic differential group L^p_m of order p ($m = \dim F$) (see, for example, [7], [11]). The group L^p_m (denoted also D^p_n) is the structure group of the principal bundle of tangent frames of the order p on a C^∞-manifold and was introduced by V. V. Wagner [18]; obviously $L^1_m = G\,L(n)$.

It follows that (26) ... (27) at a fixed x_0 give a homomorphism X^p of the Lie algebra h_{x_0} into the Lie algebra l^p_n. Here $\operatorname{Im} X^p$ is called the *l^p_m-isotropy algebra at the point* x_0 (linear isotropy algebra for $p = 1$) and $h^p_{x_0} = \operatorname{Ker} X^p$ is called *the isotropy kernel of the order p at* x_0. An element $X \in g$ belongs to $h^p_{x_0}$ iff for $\omega^\alpha(X) = X^\alpha$

$$\zeta^a_\alpha(x_0)\, X^\alpha = 0\,, \tag{31}$$

$$(\partial_b \zeta^a_\alpha)_{x_0} X^\alpha = 0\,, \tag{32}$$

$$(\partial_{b_1 \ldots b_p} \zeta^a_\alpha)_{x_0} X^\alpha = 0\,. \tag{33}$$

It is clear that

$$h_{x_0} \supset h^1_{x_0} \supset \ldots \supset h^p_{x_0} \supset \ldots\,; \tag{34}$$

moreover, for $h^p_{x_0}$ we have

$$[h^p_{x_0}, g] \subset h^{p-1}_{x_0}\,, \tag{35}$$

which gives a recurrent definition of kernels (34). In fact, if we contract both sides of (15) with X^α and Y^β, where $X \in h^{p-1}_{x_0}$ and $Y \in g$, then for $s = 1, \ldots, p$ we get due to (31)–(33)

$$\xi^a_\gamma(x_0)\, [X, Y]^\gamma = 0\,,$$

$$(\partial_{b_1 \ldots b_{p-1}} \xi^a_\gamma)_{x_0} [X, Y]^\gamma = 0\,,$$

$$[\zeta^b_\beta(x_0)\, Y^\beta]\, [(\partial_{b_1 \ldots b_p b} \xi^a_\alpha)_{x_0} X^\alpha] = (\partial_{b_1 \ldots b_p} \zeta^a_\gamma)_{x_0} [X, Y]^\gamma\,,$$

which verifies (35).

Now (35) implies that, if x_0 is the point of a maximal dimensional orbit of G in F, then $h^p_{x_0} = h^{p-1}_{x_0}$ is possible only for $h^p_{x_0} = \{0\}$, because otherwise $h^p_{x_0} \neq \{0\}$ is a nontrivial ideal of g in h and G acts on F noneffectively. Thus $\dim h^s_{x_0} > \dim h^{s+1}_{x_0}$ if $\dim h^s_{x_0} \neq 0$. Consequently, there exists a least p with $\dim h^p_{x_0} = 0$. This integer p is called *the holonomic isotropy order of the G-space F*.

The concept of the isotropy order of a homogeneous space F was introduced in the semi-holonomic case in [9] and in the holonomic case in [5] (see also [13]).

In terms of (31)–(33), the holonomic isotropy order p is the first order p, for which the system of linear equations (31)–(33) has only the trivial solution $X^\alpha = 0$, i.e., the matrix of the system has the full rank.

4. GENERAL HORIZONTALITY CONDITIONS

The maximal transversal distribution Γ on a fibre bundle $E(P, F)$, associated to the principal fibre bundle $P(M, G)$, is a horizontal distribution of a connection iff there exists a connection in P with the horizontal distribution π, such that if Γ (for some

$U \subset F$) and π are annihilators of the system of forms $\psi^a = \Omega^a - \Gamma^a_i \omega^i$ and $\theta^\alpha = \omega^\alpha - \pi^\alpha_i \omega^i$, respectively, then

$$\psi^a = \mathrm{d}x^a + \xi^a_\alpha(x)\, \theta^\alpha \tag{36}$$

(see [6], Proposition 3). Consequently,

$$\xi^a_\alpha \pi^\alpha_i = \Gamma^a_i \,. \tag{37}$$

Substitution in (19) gives, due to (7) and (11),

$$(\partial \xi^a_\alpha)\, \pi^\alpha_i = \Gamma^a_{bi} \tag{38}$$

and another substitution in (22) gives

$$(\partial_{bc} \xi^a_\alpha)\, \pi^\alpha_i = \Gamma^a_{bci} \,. \tag{39}$$

This process can be repeated using (24) and (15) and leads to

$$(\partial_{b_1 \dots b_s} \xi^a_\alpha)\, \pi^\alpha_i = \Gamma^a_{b_1 \dots b_s i} \,. \tag{40}$$

Theorem 2. *Let $E(P, F)$ be an associated fibre bundle with a connected Lie group G, acting on the typical fibre F. A maximal transversal distribution Γ on E, given as the annihilator of the system of forms $\psi^a = \Omega^a - \Gamma^b_i \omega^i$, is a horizontal distribution of a connection iff the following system of linear equations with unknowns π^α_i is compatible*:

$$\xi^a_\alpha \pi^\alpha_i = \Gamma^a_i \,, \tag{41}$$

$$(\partial_b \xi^a_\alpha)\, \pi^\alpha_i = \Gamma^a_{bi} \,, \tag{42}$$

$$(\partial_{b_1 \dots b_{p+1}} \xi^a_\alpha)\, \pi^\alpha_i = \Gamma^a_{b_1 \dots b_{p+1} i} \,, \tag{43}$$

where the right hand sides are the systems of functions from (23) and p is the holonomic isotropy order of the typical fibre F.

Proof. Necessity follows from (37)–(40). Sufficiency let us prove first for $p = 1$. The matrix of the system (41), (42) has the full rank and the system has a unique solution π^α_i, which fulfils also the relation

$$(\partial_{bc} \xi^a_\alpha)\, \pi^\alpha_i = \Gamma^a_{bci} \,.$$

Differentiating now (41) and (42) using (10), (11) and (15) we get

$$\xi^a_\alpha (\Delta \pi^\alpha_i) = \Gamma^a_{ij} \omega^j \,,$$

$$(\partial_b \xi^a_\alpha)(\Delta \pi^\alpha_i) = \Gamma^a_{bij} \omega^j \,,$$

where $\Delta \pi^\alpha_i$ is given by (6), and therefore (8) holds, i.e., on P there exists a unique horizontal distribution π that gives Γ on E.

The general case $p > 1$ can be treated analogously. The system (41)–(43) has a unique solution π_i^α. After differentiation we get

$$\xi_\alpha^a(\Delta\pi_i^\alpha) = \Gamma_{ij}^a\omega^j\,,$$
$$(\partial_b\xi_\alpha^a)(\Delta\pi_i^\alpha) = \Gamma_{bij}^a\omega^j\,,$$
$$(\partial_{b_1\dots b_p}\xi_\alpha^a)(\Delta\pi_i^\alpha) = \Gamma_{b_1\dots b_p ij}^a\omega^j$$

and therefore (8) holds. Theorem 2 is proved.

The conditions that we have put on the distribution Γ in Theorem 2 are called the *horizontality conditions*. If they hold, the object, defined by Γ_i^a, is called *a connection object*.

Analogous horizontality conditions have been obtained in [9], but provided F is a homogeneous G-space and instead of holonomic differential groups L_m^s the semi-holonomic differential groups $\bar{L}_m^s$ are involved (for them the isotropy order $\bar{p}$ of F is $\leqq p$).

Let us remark that there is a class of the so called *v-reductive homogeneous spaces* (see [2], [12]), which covers all "classical" cases, for which the semiholonomic point of view reduces to the holonomic one (as follows from a result of [2]).

5. SPLITTING OF THE CURVATURE

Analogously as in § 2, (36) due to (11) and (9) implies

$$\text{(44)}\qquad \mathrm{d}\psi^a = \psi^b \wedge \psi_b^a + \Psi^a\,,$$

where $\psi_b^a = (\partial_b\xi_\alpha^a)\,\theta^\alpha$ and $\Psi^a = \xi_\alpha^a\Theta^\alpha$.

For ψ_b^a we get

$$\text{(45)}\qquad \mathrm{d}\psi_b^a = \psi_b^c \wedge \psi_c^a + \psi^c \wedge \psi_{bc}^a + \Psi_b^a\,,$$

where $\psi_{bc}^a = (\partial_{bc}\xi_\alpha^a)\,\theta^\alpha$ and $\Psi_b^a = (\partial_b\xi_\alpha^a)\,\Theta^\alpha$.

In general by a recurrent algorithm we get 1-forms $\psi_{b_1\dots b_s}^a = (\partial_{b_1\dots b_s}\xi_\alpha^a)\,\theta^\alpha$, $s = 1, 2, \dots$ and the formulas

$$\text{(46)}\qquad \mathrm{d}\psi_{b_1\dots b_s}^a = \sum_{t=1}^{s}\binom{s}{t}\psi_{(b_1\dots b_t}^c \wedge \psi_{b_{t+1}\dots b_s)c}^a + \psi^c \wedge \psi_{b_1\dots b_s c}^a + \psi_{b_1\dots b_s}^a\,,$$

where

$$\text{(47)}\qquad \Psi_{b_1\dots b_s}^a = (\partial_{b_1\dots b_s}\xi_\alpha^a)\,\Theta^\alpha\,.$$

It is remarkable that the 2-forms $\Psi^a, \Psi_b^a, \dots, \Psi_{b_1\dots b_s}^a$ can be expressed only in terms of (23):

$$\text{(48)}\qquad \Psi^a = R_{ij}^a\omega^i \wedge \omega^j\,,$$

$$R^a_{ij} = \Gamma^a_{[ij]} + \Gamma^a_{b[i}\Gamma^b_{j]}\,,$$

(49) $$\Psi^a_b = R^a_{bij}\omega^i \wedge \omega^j\,,$$

$$R^a_{bij} = \Gamma^a_{b[ij]} + \Gamma^a_{bc[i}\Gamma^c_{j]} + \Gamma^a_{c[i}\Gamma^c_{|b|j]}\,,$$

(50) $$\Psi^a_{b_1\dots b_s} = R^a_{b_1\dots b_s ij}\omega^i \wedge \omega^j\,,$$

$$R^a_{b_1\dots b_s} = \Gamma^a_{b_1\dots b_s[ij]} + \Gamma^a_{b_1\dots b_s c[i}\Gamma^c_{j]} +$$

$$+ \Big\{\sum_{t=1}^{s}\binom{s}{t}\Gamma^a_{(b_1\dots b_t|ci|}\Gamma^c_{b_{t+1}\dots b_s)j}\Big\}_{[i,j]} + \Gamma^a_{c[i}\Gamma^c_{|b_1\dots b_s|j]}\,,$$

where $\Gamma^a_{b_1\dots b_s ij}$ are determined by (24).

If we consider (47) for $s = 1, \dots, p$ (where p is the holonomic isotropy order of F) we see that the system of 2-forms $\Psi^a, \Psi^a_b, \dots, \Psi^a_{b_1\dots b_p}$ is equivalent to the system of curvature 2-forms Θ^α of the connection π in the principal fibre bundle $P(M, G)$. The 2-forms Ψ^a are called *torsion forms* and the 2-forms $\Psi^a_{b_1,\dots,b_s}$ *curvature forms of the order s for a connection Γ in the associated fibre bundle $E(P, F)$*. They satisfy the following Bianchi identities, obtained by the exterior differentation from (44)–(46):

(51) $$\mathrm{d}\Psi^a = \psi^b \wedge \Psi^a_b - \psi^b \wedge \Psi^a_b\,,$$

(52) $$\mathrm{d}\Psi^a_b = \psi^c_b \wedge \Psi^a_c - \Psi^c_b \wedge \psi^a_c + \psi^c \wedge \Psi^a_{bc} - \Psi^c \wedge \Psi^a_{bc}\,,$$

(53) $$\mathrm{d}\Psi^a_{b_1\dots b_s} = \sum_{t=1}^{s}\binom{s}{t}\big[\psi^c_{(b_1\dots b_t} \wedge \Psi^a_{b_{t+1}\dots b_s)c} -$$

$$- \Psi^c_{(b_1\dots b_t} \wedge \psi^a_{b_{t+1}\dots b_s)c}\big] + \psi^c \wedge \Psi^a_{b_1\dots b_s c} - \Psi^c \wedge \psi^a_{b_1\dots b_s c}\,.$$

Theorem 3. *The horizontal distribution Γ on E is involutive iff the torsion forms Ψ^a vanish identically on E. As a consequence, all curvature forms of orders $s = 1, \dots, p$ vanish, too, and the horizontal distribution π on P, associated to Γ on E, is involutive. The converse is also true.*

The proof follows immediately from (44), the Bianchi identities (51)–(53) and (47) for $s = 1, \dots, p$.

In the particular case of a homogeneous F, Theorem 3 was proved in [9] by involving semi-holonomic curvature forms of order $s = 1, \dots, p$. An analogous holonomic splitting of the curvature was given in [5].

6. EXAMPLE: PROJECTIVE CONNECTION

To demonstrate the efficiency (effectiveness) of our horizontality conditions in a non-trivial case, we consider a connection in a projective fibre bundle, in which F is a projective space P_m and G is the Lie group $GP(m)$ of its collineations.

If a point $x \in P_m$ (a 1-dimensional subspace of a vector space L_{m+1}) is given by a vector $X = X^0 e_0 + X^a e_a$, then in the open domain, where $X^0 \neq 0$, we can introduce coordinates of x as $x^a = X^a/X^0$. In virtue of the wellknown formulas

$$\mathrm{d}e_0 = \omega_0^0 e_0 + \omega_0^a e_a , \quad \mathrm{d}e_a = \omega_a^0 e_0 + \omega_a^b e_b ,$$

the condition for fixing the point x is that the vector

$$\mathrm{d}(e_0 + x^a e_a) = (\omega_0^0 + x^a \omega_a^0)\, e_0 + (\mathrm{d}x^a + x^b \omega_b^a + \omega_0^a)\, e_a$$

is collinear to $e_0 + x^a e_a$, i.e.,

$$\mathrm{d}x^a + x^b \omega_b^a + \omega_0^a = x^a(\omega_0^0 + x^b \omega_b^0) .$$

Consequently,

$$\Omega^a = \mathrm{d}x^a + \omega_0^a + x^b \hat{\omega}_b^a - x^a x^b \omega_b^0 ,$$

where ω_0^a, $\hat{\omega}_b^a = \omega_b^a - \delta_b^a \omega_0^0$ and ω_b^0 play the role of ω^α. Therefore

$$\Omega_b^a = \hat{\omega}_b^a - 2x^c \delta_{(b}^a \omega_{c)}^0 , \quad \Omega_{bc}^a = -2\delta_{(b}^a \omega_{c)}^0 .$$

The isotropy order of P_m is $p = 2$.

From (41)–(43) it follows that the horizontality conditions are now the conditions of compatibility of the following system:

$$\pi_{0i}^a + x^b \pi_{bi}^a - x^a x^b \pi_{bi}^0 = \Gamma_i^a , \tag{54}$$

$$\pi_{bi}^a - 2x^c \delta_{(b}^a \pi_{c)i}^0 = \Gamma_{bi}^a , \tag{55}$$

$$-2\delta_{(b}^a \pi_{c)i}^0 = \Gamma_{bci}^a , \tag{56}$$

$$0 = \Gamma_{bcdi}^a \tag{57}$$

and they reduce to

$$\Gamma_{bci}^a = -2\delta_{(b}^a \Gamma_{c)i} , \quad \Gamma_{bcdi}^a = 0 . \tag{58}$$

The connection object is characterized by the system

$$\mathrm{d}\Gamma_i^a - \Gamma_j^a \omega_i^j + \Gamma_i^b \Omega_b^a + \Omega_i^a = \Gamma_{bi}^a \Omega^b + \Gamma_{ij}^a \omega^j , \tag{59}$$

$$\begin{aligned} \mathrm{d}\Gamma_{bi}^a - \Gamma_{bj}^a \omega_i^j - \Gamma_{ci}^a \Omega_b^c + \Gamma_{bi}^c \Omega_c^a - 2\Gamma_i^c \delta_{(b}^a \Omega_{c)} + {} \\ + \Omega_{bi}^a = -2\delta_{(b}^a \Gamma_{c)i} \Omega^c + \Gamma_{bij}^a \omega^j , \end{aligned} \tag{60}$$

$$\mathrm{d}\Gamma_{bi} - \Gamma_{cj} \omega_i^j - \Gamma_{ci} \Omega_b^c + \Gamma_{bi}^c \Omega_c + \Omega_{bi} = \Gamma_{bij} \omega^j . \tag{61}$$

From (48)–(50) it follows that a connection in a projective fibre bundle has the torsion 2-forms Ψ^a, the first order curvature 2-forms Ψ_b^a and the second order curvature forms Ψ_b. The corresponding torsion object R^a_{ij} is given by (48), the curvature objects are

(62) $$R^a_{bij} = \Gamma^a_{b[ij]} + \Gamma^a_{[ci}\Gamma^c_{|b|j]} + 2\delta^a_{(b}\Gamma_{c)[i}\Gamma^c_{j]}\,,$$

(63) $$R_{bij} = \Gamma_{b[ij]} + \Gamma_{c[i}\Gamma^c_{|b|j]}\,.$$

Here a special case is a connection in an affine fibre bundle. We obtain such a connection if there exists a reduction of the structure group $GP(m)$ to the affine subgroup $GA(m)$. In this case $\omega^0_a = A_{ai}\omega^i$ and consequently, the only independent forms are ω^a_0 and $\hat{\omega}^a_b$. In (54)–(57) the π^0_{bi} vanish and therefore the horizontality conditions reduce to $\Gamma^a_{bci} = 0$. Now, in (59)–(61) we have $\Omega_c \equiv 0$, $\Gamma_{ci} \equiv 0$.

The classical formulas for an affine connection on a C^∞-manifold M_n are obtained if $m = n$ and a soldering (in sense of the Cartan connection [1]) is given. Then the indices $a, b, \ldots$ and $i, j, \ldots$ have the same meaning and, for example, (62) gives the classical expression for the curvature tensor:

$$R^l_{kij} = \Gamma^l_{k[ij]} + \Gamma^l_{q[i}\Gamma^q_{|e|j]}\,.$$

The horizontality conditions (58) for a projective connection have been derived in a different way in [9]. In [10] we give their application to a situation, where the base manifold is a manifold of m-planes of a projective space P_N and E is its canonical fibre bundle. As a result we can obtain a purely geometrical description of the parallel displacement for every possible projective connection in canonical fibre bundle in many essential cases.

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Author's address: U.S.S.R., 202400 Tartu, Tartu State University, Faculty of Mathematics.