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GENERALIZED MULTIPLE PERRON INTEGRALS AND THE GREEN-GOURSAT THEOREM FOR DIFFERENTIABLE VECTOR FIELDS

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1. INTRODUCTION

The divergence theorem for vectors fields of class C^1 in $\mathbb{R}^n$ is well known and its proof can be found in many textbooks. In 1955, Bochner considered the possibility of reducing the regularity of the vector fields to a mere assumption of differentiability. In [2], he proved such a theorem under the additional condition of continuity for the divergence of the field, and weakened in [3] this assumption into Lebesgue integrability together with an auxiliary condition. This last one was shown to be superfluous by Shapiro [20], using a rather involved argument based on multiple trigonometric series.

The divergence theorem can be viewed as a n-dimensional generalization of the fundamental theorem of calculus telling that

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)\,, \tag{1}$$

for a differentiable function of one variable, at least when the left-hand member has a meaning. As it is well known neither the Riemann nor even the Lebesgue integral is powerful enough to be able to integrate an arbitrary derivative. One needs the more general theory of Denjoy [4] or of Perron [18] to obtain a type of integral for which (1) holds for every differentiable f. Kurzweil [12], in 1957, and independently Henstock [6], in 1961, have shown that those type of integrals could be recovered by a technically slight but conceptually basic modification of Riemann's original definition. This modification can be easily adapted to the n-dimensional case (see eg. [7, 13, 15, 17]) and raises the question of knowing if the obtained integral allows the proof of a divergence theorem under a mere differentiability assumption on the vector field. This does not seem to be the case and is the reason why we introduce here a new concept of integral (on an interval) for functions from $\mathbb{R}^n$ into a Banach space. We modify the Kurzweil-Henstock definition by taking the limit of the

Riemann sums non-uniform with respect to the so called irregularity of the partition, which measures the way in which the stretching of the intervals of the partition differs from that of the original interval (see Section 2). Such an approach was initiated in our paper [16], with the so-called RP-integral, but this integral is in fact too general to have the nice properties usually wanted. The one introduced here, and called the *GP-integral* (see Section 3) has the linearity, positivity, restriction and additivity properties of the integral, except that it is unknown if the GP-integrability over members of a finite partition in intervals of an interval implies the GP-integrability over the interval (see Section 4). We also show that a Levi's type monotone convergence theorem holds for the GP-integral but it is still unknown if a dominated convergence theorem is valid (see Section 6). Section 5 is devoted to the proof of the divergence theorem and of a corresponding Stokes theorem for differential forms with differentiable coefficients. Some applications of those theorems given in [16] can be immediately adapted to the present setting and other will be given in another paper.

Finally let us notice that more general abstract versions of the Kurzweil-Henstock approach have been given by Henstock [9, 10, 11], McShane [17] and others, but they do not seem to contain the present integral when specialized to the considered situation.

2. RIGHT-CLOSED INTERVALS, L- AND P-PARTITIONS AND RIEMANN SUMS

As usual, a *right-closed interval* $I =]a, b]$ in $\mathbb{R}^n$ will be defined as the cartesian product of n right-closed intervals $]a_i, b_i]$ of $\mathbb{R}$ where

$$a_i < b_i \ (1 \leqq i \leqq n)$$

and the (n-dimensional Lebesgue) *measure* $m(I)$ of $I =]a, b]$ will be given by

$$m(I) = \prod_{i=1}^{n} (b_i - a_i) \, .$$

If

$$b_1 - a_1 = b_2 - a_2 = \ldots = b_n - a_n \, ,$$

$]a, b]$ will be called a *right-closed cube* in $\mathbb{R}^n$ we shall denote by $|x|$ the norm $|x| = \max |x_i|$ of x in $\mathbb{R}^n$.

Definition 1. The *rate of stretching* $\sigma(I)$ of the right-closed interval $I =]a, b]$ is defined by

$$\sigma(I) = [\max_{1 \leqq i \leqq n} (b_i - a_i)]/[\min_{1 \leqq i \leqq n} (b_i - a_i)]$$

Clearly, $\sigma(I) \geqq 1$, $\sigma(I) = 1$ when $n = 1$ and, for $n \geqq 2$, $\sigma(I) = 1$ if and only if I is a right-closed cube.

The following concept is due to McShane [17]. We denote by int A, cl A, bdry A the interior, closure and boundary of a set A.

Definition 2. A *L-partition* of a right-closed interval $I =]a, b]$ is a finite family

$$\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$$

where the I^j are right-closed intervals such that $\{I^1, \ldots, I^m\}$ is a partition of I and where

$$x^j \in \operatorname{cl} I \quad (1 \leqq j \leqq m) .$$

A special case of L-partitions, is introduced in Riemann integration.

Definition 3. A *P-partition* of a right-closed interval $I =]a, b]$ is a L-partition

$$\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$$

such that

$$x^j \in \operatorname{cl} I^j \quad (1 \leqq j \leqq m) .$$

It is important to measure how the I^j of a L-partition differ in shape from I.

Definition 4. Let

$$\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$$

be a L-partition of the right-closed interval I. The *irregularity* $\Sigma(\Pi)$ of Π is the positive number defined by

$$\Sigma(\Pi) = [\max_{1 \leqq j \leqq m} \sigma(I^j)]/\sigma(I) .$$

We see that $\Sigma(\Pi)$ is independent from the choice of the x^j $(1 \leqq j \leqq m)$ and so is really a characteristic of the set partition $\{I^1, \ldots, I^m\}$ of I. The following special type of L-partition was introduced in [16].

Definition 5. A L-partition $\Pi = \{(x^1, I^1), \ldots, x^m, I^m)\}$ of I is called *regular* (shortly a *RL-partition* or a *RP-partition* in the special case of a P-partition) if I^j is similar to I for each $1 \leqq j \leqq m$.

Obviously, for a RL-partition Π, $\sigma(I^j) = \sigma(I)$ $(1 \leqq j \leqq m)$, so that, $\Sigma(\Pi) = 1$.

With McShane [17], let us call *gauge on* cl I any positive mapping δ defined on cl I, and introduce the following

Definition 6. If δ is a gauge on cl I, a *L-partition* $\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$ is called *δ-fine* if

$$I^j \subset B[x^j; \delta(x^j)] , \quad 1 \leqq j \leqq m ,$$

where $B[a; r] = [a_1 - r, a_1 + r] \times \ldots \times [a_n - r, a_n + r]$ is the closed ball of center $a \in \mathbb{R}^n$ and radius $r > 0$ relative to our norm.

The following simple result has been implicitely used or explicitely rediscovered many times under various forms (see eg. [15] for its elementary proof based on the theorem of nested closed intervals, its historical development and its use in unifying the treatment of several theorems in basic analysis).

Cousin's lemma. *If I is a right-closed interval of $\mathbb{R}^n$ and δ is a gauge on* $\operatorname{cl} I$, *then there exists at least one δ-fine RP-partition of I.*

Consequently, there will always exist a δ-fine P-partition Π of I such that $\Sigma(\Pi) = 1$.

Let now V be a vector space over $\mathbb{R}$ or $\mathbb{C}$, and f be a function of $\mathbb{R}^n$ into V defined on $\operatorname{cl} I$.

Definition 7. If $\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$ is a L-partition of I, the *Riemann sum* $S(I, f, \Pi)$ *associated to f and Π* is the element of V defined by

$$S(I, f, \Pi) = \sum_{j=1}^{m} f(x^j)\, m(I^j)\,.$$

We can now prove the following.

Lemma 1. *Let δ be a gauge on* $\operatorname{cl} I$. *Then, for every δ-fine L-partition Π of I there exists a δ-fine L-partition $\tilde{\Pi}$ of I such that*

$$\Sigma(\tilde{\Pi}) \leqq 2/\sigma(I)$$

and

$$S(I, f, \Pi) = S(I, f, \tilde{\Pi})\,.$$

Proof. Let $\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$ with $I^j =]a^j, b^j]$ $(1 \leqq j \leqq m)$. Then there exists integers $r_{j,i} \geqq 1$ such that,

$$r_{j,i} \min_{1 \leqq k \leqq n} (b^j_k - a^j_k) \leqq b^j_i - a^j_i < (r_{j,i} + 1) \min_{1 \leqq k \leqq n} (b^j_k - a^j_k)\,,$$

$$1 \leqq j \leqq m\,, \quad 1 \leqq i \leqq n\,.$$

Hence if we partition I^j into $r_j = r_{j,1} \times \ldots \times r_{j,n}$ right-closed intervals $I^{j,k}$, $1 \leqq k \leqq r_j$ by dividing the i^{th} side in $r_{j,i}$ equal parts, we obtain the L-partition

$$\tilde{\Pi} = \{(x^j, I^{j,k}) : 1 \leqq k \leqq r_j,\ 1 \leqq j \leqq m\}$$

which is still δ-fine and moreover is such that

$$\sigma(I^{j,k}) < \max_{1 \leqq i \leqq n} \frac{r_{j,i} + 1}{r_{j,i}} \leqq 2\,, \quad 1 \leqq k \leqq r_j\,, \quad 1 \leqq j \leqq m\,,$$

and

$$S(I, f, \tilde{\Pi}) = \sum_{j=1}^{m} \sum_{k=1}^{r_j} f(x^j)\, m(I^{j,k}) = \sum_{j=1}^{m} f(x^j) \Big(\sum_{k=1}^{r_j} m(I^{j,k})\Big) =$$

$$= \sum_{j=1}^{m} f(x^j)\, m(I^j) = S(I, f, \Pi)\,.$$

Consequently

$$\Sigma(\tilde{\Pi}) \leqq 2/\sigma(I)$$

and the proof is complete.

Remark 1. One shall check easily that a result like Lemma 1 does not hold for P-partitions.

3. GENERALIZED RIEMANN INTEGRALS AND THEIR RELATIONS WITH CLASSICAL INTEGRALS

The following definition was introduced independently by Kurzweil [12] (who assumed $n = 1$ and showed its equivalence with the Perron integral [18]) and Henstock [6, 7], and, for the L-integral by McShane [17].

See [8, 9, 10, 11, 13, 17] for more details and much more general settings and see [15] for an elementary and systematic treatment of integration in $\mathbb{R}^n$ using this approach. Let $I \subset \mathbb{R}^n$ be a right-closed interval, X be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with norm $\|\cdot\|$, and let f be a function of $\mathbb{R}^n$ into X defined on $\operatorname{cl} I$.

Definition 8. We say that f is *P-integrable* (resp. *L-integrable*) over I if there exists $J \in X$ such that, for each $\varepsilon > 0$ there exists a gauge δ on $\operatorname{cl} I$ with the property that for every δ-fine P-partition (resp. L-partition) Π of I, one has

$$\|S(I, f, \Pi) - J\| \leqq \varepsilon .$$

Clearly, Riemann integrability of f over I corresponds to restricting δ to be a *constant* gauge in the definition of P-integrability. Moreover, every L-integrable function over I is P-integrable over I and the two following basic results were proved by McShane [17]:

a) the above definition of L-integrability restricted to constant gauges is equivalent to the Riemann integrability over I;

b) f is L-integrable over I if and only if f is Lebesgue integrable over I.

Denoting respectively by $R(I, X)$, $L(I, X)$ and $P(I, X)$ the class of Riemann-Graves [5], Lebesgue-Bochner [1] and of P-integrable functions, we have the following inclusions

$$R(I, X) \subset L(I, X) \subset P(I, X),$$

and classical examples [15] show that the inclusions are strict ones.

We now introduce a more general concept of integral, the GP-integral (for Gauss-Green-Goursat-Perron because of its links with definition 8 and of its importance in the divergence theorem an in the Goursat version of the Cauchy theorem for holomorphic functions) and its corresponding GL-integral.

Definition 9. We say that f is *GP-integrable* (resp. *GL-integrable*) *over* I if there exists $J \in X$ such that, for each $\varepsilon > 0$ and for each $\eta \geqq 1$, there exists a gauge δ on $\mathrm{cl}\, I$ with the property that for every δ-fine P-partition (resp. L-partition) Π of I with

$$\Sigma(\Pi) \leqq \eta$$

one has

$$\|S(I, f, \Pi) - J\| \leqq \varepsilon .$$

Clearly every P-integrable function over I is GP-integrable over I and the two concepts coincide with the Perron integral if $n = 1$. In fact the P-integrability is a GP-integrability which is uniform with respect to the irregularity of the P-partitions.

Also GL-integrable functions over I are GP-integrable over I and L-integrable functions over I are GL-integrable over I. We now show that the converse of the last assertion holds, so that the classes $GL(I, X)$ and $L(I, X)$ coincide.

Proposition 1. *If f is GL-integrable over I, then f is L-integrable over I, with the same integral.*

Proof. Let $\varepsilon > 0$,

$$\eta = \max\left(1, 2/\sigma(I)\right),$$

and let δ be the corresponding gauge on $\mathrm{cl}\, I$ such that for each δ-fine L-partition $\tilde{\Pi}$ of I with $\Sigma(\tilde{\Pi}) \leqq \eta$, one has

$$\|S(I, f, \tilde{\Pi}) - J\| \leqq \varepsilon .$$

Let now Π be a δ-fine L-partition of I; by Lemma 1, there exists a δ-fine L-partition $\tilde{\Pi}$ of I such that

$$\Sigma(\tilde{\Pi}) \leqq 2/\sigma(I) \quad \text{and} \quad S(I, f, \tilde{\Pi}) = S(I, f, \Pi) .$$

Consequently, $\Sigma(\tilde{\Pi}) \leqq \eta$ and hence

$$\|S(I, f, \Pi) - J\| = \|S(I, f, \tilde{\Pi}) = J\| \leqq \varepsilon$$

which completes the proof.

Finally the following concept had been introduced in [16].

Definition 10. We say that f is *RP-integrable over* I if there exists $J \in X$ such that, for each $\varepsilon > 0$, there exists a gauge δ on $\mathrm{cl}\, I$ with the property that for every δ-fine RP-partition Π of I, one has

$$\|S(I, f, \Pi) - J\| \leqq \varepsilon .$$

As, for a RP-partition Π, $\Sigma(\Pi) = 1$, we see that every GP-integrable over I is RP-integrable, so that we have the following chain of inclusions

$$R(I, X) \subset L(I, X) = GL(I, X) \subset P(I, X) \subset GP(I, X) \subset RP(I, X) .$$

We shall show in the next sections that, in contrast with the RP-integral, the GP-integral conserves most of basic properties of the P-integral but is still general enough to allow a divergence theorem for arbitrary differentiable vector fields.

4. BASIC PROPERTIES OF THE GP-INTEGRAL

We conserve the notations of Section 3. By using the fact that if δ_1 and δ_2 are gauges on $\mathrm{cl}\, I$, then δ defined by

$$\delta(x) = \min(\delta_1(x), \delta_2(x)), \quad x \in \mathrm{cl}\, I$$

is a gauge on $\mathrm{cl}\, I$ such that every δ-fine L-partition of I is both δ_1-fine and δ_2-fine, it is easy to prove the following *uniqueness result.*

Proposition 2. *There exists at most one $J \in X$ which satisfies the conditions of Definition* 9 *for the GP-integrability.*

Consequently, this J will be called the *GP-integral* (shortly the integral) *of f over I and denoted by*

$$\int_I f \quad \text{or} \quad \int_I f(x)\, \mathrm{d}x .$$

Similar elementary arguments immediately lead to the *linearity properties* of the GP-integral.

Proposition 3. *If f and g are functions from $\mathbb{R}^n$ into X defined on* $\mathrm{cl}\, I$ *and GP-integrable on I, then $f + g$ is GP-integrable on I as well as cf for every $c \in \mathbb{K}$, and, moreover*

$$\int_I (f + g) = \int_I f + \int_I g, \quad \int_I (cf) = c\left(\int_I f\right).$$

The following *inequalities* for the GP-integral are immediate consequences of the corresponding properties of the Riemann sums and of Definition 9.

Proposition 4. *If f and $\|f\|$ are GP-integrable over I then one has*

$$\left\|\int_I f\right\| \leqq \int_I \|f\| .$$

Proposition 5. *If f and g are real functions defined on* $\mathrm{cl}\, I$, *GP-integrable over I and such that*

$$f(x) \leqq g(x), \quad x \in \mathrm{cl}\, I ,$$

then one has

$$\int_I f \leqq \int_I g .$$

As 0 is clearly GP-integrable over I with an integral equal to zero, it follows from Proposition 5 that the GP-integral of non-negative function will be non-negative. Notice also that in Proposition 4, the GP-integrability of f and $\|f\|$ has to be assumed, the GP-integral being in general not absolutely integrable, like Perron-type integrals.

Of importance is the following *Cauchy criterion of GP-integrability.*

Proposition 6. *f is GP-integrable over I if and only if for each $\varepsilon > 0$ and for each $\eta \geqq 1$, there exists a gauge δ on $\mathrm{cl}\, I$ with the property that for all δ-fine P-partitions Π and Π' of I with*

$$\Sigma(\Pi) \leqq \eta\,, \quad \Sigma(\Pi') \leqq \eta\,,$$

one has

$$\|S(I, f, \Pi) - S(I, f, \Pi')\| \leqq \varepsilon\,. \tag{2}$$

Proof. Necessity is an easy consequenee of Definition 9. For sufficiency, let $\eta \geqq 1$ be fixed. Then, by taking successively $\varepsilon = 1/k$ $(k = 1, 2, \ldots)$ in (2), we obtain a sequence $(\delta_k)_{k \in N^*}$ of gauges on $\mathrm{cl}\, I$ such that

$$\delta_{k+1}(x) \leqq \delta_k(x)\,, \quad x \in \mathrm{cl}\, I\,, \quad k \in \mathbb{N}^*\,,$$

and such that for all δ_k-fine P-partitions Π_k and Π'_k of I with

$$\Sigma(\Pi_k) \leqq \eta\,, \quad \Sigma(\Pi'_k) \leqq \eta\,,$$

one has

$$\|S(I, f, \Pi_k) - S(I, f, \Pi'_k)\| \leqq \frac{1}{k}\,. \tag{3}$$

For each $k \in \mathbb{N}^*$, select such a P-partition Π_k; then if $q \geqq k$, Π_q is δ_q-fine and hence δ_k-fine so that

$$\|S(I, f, \Pi_k) - S(I, f, \Pi_q)\| \leqq \frac{1}{k}\,, \tag{4}$$

which shows that $(S(I, f, \Pi_k))_{k \in N^*}$ is a Cauchy-sequence in X. Set

$$J = \lim_{k \to \infty} S(I, f, \Pi_k)\,,$$

so that by (4) one has

$$\|S(I, f, \Pi_k) - J\| \leqq \frac{1}{k}\,, \quad k \in \mathbb{N}^*\,. \tag{5}$$

Let now $\varepsilon > 0$ be given, $m \in \mathbb{N}^*$ be such that

$$2/m \leqq \varepsilon\,,$$

and let $\delta(x) = \delta_m(x)$. Then if Π is a δ-fine P-partition of I with $\Sigma(\Pi) \leqq \eta$, we have by (3) and (5)

$$\|S(I,f,\Pi) - J\| \leqq \|S(I,f,\Pi) - S(I,f,\Pi_m)\| + \|S(I,f,\Pi_m) - J\| \leqq \frac{2}{m} \leqq \varepsilon\,,$$

and the proof is complete.

The Cauchy criterion is useful in establishing the *restriction property* of the GP-integral.

Proposition 7. *If f is GP-integrable on I then f is GP-integrable on every right-closed interval $K \subset I$.*

Proof. We can write

$$I \setminus K = \bigcup_{i=1}^{q} L^i$$

where the L^i are right-closed intervals. Let $\varepsilon > 0$ and $\eta \geqq 1$ and let

$$\eta' = [\sigma(I)]^{-1} \max\left(\eta\, \sigma(K), \sigma(L^1), \ldots, \sigma(L^q), \sigma(I)\right). \tag{6}$$

By Cauchy criterion there exists a gauge δ on cl I such that for all δ-fine P-partitions Π and $\tilde{\Pi}$ of I with

$$\Sigma(\Pi) \leqq \eta'\,, \quad \Sigma(\tilde{\Pi}) \leqq \eta'\,, \tag{7}$$

one has

$$\|S(I,f,\Pi) - S(I,f,\tilde{\Pi})\| \leqq \varepsilon\,. \tag{8}$$

The restriction of δ to cl K and to cl L^i ($1 \leqq i \leqq q$) being a gauge on those sets, let Π_{L^i} be a δ-fine RP-partition of L^i ($1 \leqq i \leqq q$) and let Π_K and $\tilde{\Pi}_K$ be two δ-fine P-partitions of K with

$$\Sigma(\Pi_K) \leqq \eta\,, \quad \Sigma(\tilde{\Pi}_K) \leqq \eta\,. \tag{9}$$

If, explicitely,

$$\Pi_K = \{(x^1, K^1), \ldots, (x^m, K^m)\}\,,$$

$$\tilde{\Pi}_K = \{(\tilde{x}^1, \tilde{K}^1), \ldots, (\tilde{x}^m, \tilde{K}^m)\}\,,$$

$$\Pi_{L^i} = \{(x^{i,1}, L^{i,1}), \ldots, (x^{i,m_i}, L^{i,m_i})\}\,, \quad 1 \leqq i \leqq q\,,$$

then

$$\Pi = \{(x^j, K^j),\ 1 \leqq j \leqq m;\ (x^{i,j_i}, L^{i,j_i}),\ 1 \leqq j_i \leqq m_i,\ 1 \leqq i \leqq q\}$$

and

$$\tilde{\Pi} = \{(\tilde{x}^j, \tilde{K}^j),\ 1 \leqq j \leqq m;\ (x^{i,j_i}, L^{i,j_i}),\ 1 \leqq j_i \leqq m_i,\ 1 \leqq i \leqq q\}$$

are δ-fine P-partitions of I such that

$$S(I,f,\Pi) - S(I,f,\tilde{\Pi}) = S(K,f,\Pi_K) - S(K,f,\tilde{\Pi}_K)\,. \tag{10}$$

Moreover, the Π_{L^i} being regular, one has by (6) and (9),

$$\Sigma(\Pi) = [\sigma(I)]^{-1} \max_{\substack{1 \leqq j \leqq m \\ 1 \leqq j_i \leqq m_i \\ 1 \leqq i \leqq q}} [\sigma(K^j), \sigma(L^{i,j_i})] =$$

$$= [\sigma(I)]^{-1} \max_{\substack{1 \leqq j \leqq m \\ 1 \leqq i \leqq q}} [\sigma(K^j), \sigma(L^i)] \leqq [\sigma(I)^{-1} \max [\sigma(K)\, \Sigma(\Pi_K), \max_{1 \leqq i \leqq q} \sigma(L^i)] \leqq \eta'$$

and similarly

$$\Sigma(\tilde{\Pi}) \leqq \eta' .$$

Using (8) and (10), we get

$$\|S(K, f, \Pi_K) - S(K, f, \tilde{\Pi}_K)\| \leqq \varepsilon$$

and the proof is complete.

We can now prove the (finite) *additivity property* of the GP-integral.

Proposition 8. *Let $\{K^1, \ldots, K^r\}$ be a partition of I into right-closed intervals K^l $(1 \leqq l \leqq r)$. Then, if f is GP-integrable on I, one has*

$$\int_I f = \sum_{l=1}^{r} \int_{K^l} f .$$

Proof. The existence of the $\int_{K^l} f$ follows from Proposition 7. Let $\varepsilon > 0$ be given and let

$$\eta = \max [1, [\sigma(I)]^{-1} \max_{1 \leqq l \leqq r} \sigma(K^l)] .$$

Then there exists a gauge δ_l on adh I such that for each δ_l-fine P-partition Π_0 of I with $\Sigma(\Pi_0) \leqq \eta$ one has

$$\left\| S(I, f, \Pi_0) - \int_I f \right\| \leqq \frac{\varepsilon}{r + 1}$$

and there exists a gauge δ_l on cl K^l such that for each δ_l-fine P-partition Π_{K^l} of K^l with $\Sigma(\Pi_{K^l}) \leqq 1$, one has

$$\left\| S(K^l, f, \Pi_{K^l}) - \int_{K^l} f \right\| \leqq \frac{\varepsilon}{r + 1}, \quad (1 \leqq l \leqq q) .$$

Let δ be the gauge defined on adh I

$$\delta(x) = \min (\delta_0(x), \delta_1(x), \ldots, \delta_q(x))$$

where for each $x \in$ adh I, the minimum is taken only on those $\delta_i(x)$ which are defined at x. Taking then δ-fine RP-partitions Π_{K^l} of K^l $(1 \leqq l \leqq q)$, say

$$\Pi_{K^l} = \{(x^{l,j_l}), \quad K^{l,j_l} : 1 \leqq j_l \leqq m_l\}$$

and letting

$$\Pi = \{(x^{l,j_l}, K^{l,j_l}) : 1 \leqq j_l \leqq m_l,\ 1 \leqq l \leqq q\}\,,$$

we obtain a δ-fine P-partition Π of I with

$$\Sigma(\Pi) = [\sigma(I)]^{-1} \max_{\substack{1 \leqq j \leqq m \\ 1 \leqq l \leqq r}} \sigma(K^{l,j_l}) = [\sigma(I)]^{-1} \max_{1 \leqq l \leqq r} \sigma(K^1) \leqq \eta\,.$$

Consequently,

$$\left\| \int_I f - \sum_{l=1}^{r} \int_{K^l} f \right\| \leqq \left\| \int_I f - S(I, f, \Pi) \right\| + \sum_{l=1}^{r} \left\| S(K^l, f, \Pi_{K^l}) - \int_{K^l} f \right\| \leqq \varepsilon\,,$$

and the result follows, $\varepsilon > 0$ being arbitrary.

Remark 2. The P-integral has, with respect to additivity, the property that if I is partitioned into the right-closed intervals I^1 and I^2 and if f is P-integrable over I^1 and over I^2, then f is P-integrable over I. (see e.g. [7, 8, 10, 11, 14, 15, 17]). The corresponding proof does not seem to extend to the GP-integral. The reason is that if δ_1 and δ_2 are respective gauges on $\mathrm{cl}\, I^1$ and $\mathrm{cl}\, I^2$, and if we define δ on $\mathrm{cl}\, I$ by

$$\delta(x) = \min\left(\delta_1(x),\ \tfrac{1}{2} \text{ distance } (x, \mathrm{cl}\, I^2)\right) \quad \text{if} \quad x \in \mathrm{cl}\, I^1 \setminus \mathrm{cl}\, I^2\,,$$

$$\delta(x) = \min\left(\delta_2(x),\ \tfrac{1}{2} \text{ distance } (x, \mathrm{cl}\, I^1)\right) \quad \text{if} \quad x \in \mathrm{cl}\, I^2 \setminus \mathrm{cl}\, I^1\,,$$

$$\delta(x) = \min\left(\delta_1(x), \delta_2(x)\right) \text{ if } x \in \mathrm{cl}\, I^1 \cap \mathrm{cl}\, I^2\,,$$

this gauge forces every $(\tilde{x}^j, \tilde{I}^j)$ of a δ-fine P-partition $\tilde{\Pi}$ of I to be such that $\tilde{I}^j \subset I^1$ if $\tilde{x}^j \in \mathrm{cl}\, I^1 \setminus \mathrm{cl}\, I^2$ and $\tilde{I}^j \subset I^2$ if $\tilde{x}^j \in \mathrm{cl}\, I^2 \setminus \mathrm{cl}\, I^1$. Then such a δ-fine P-partition nicely restricts into a δ_1-fine P-partition $\tilde{\Pi}_1$ of I^1 and a δ_2-fine P-partition $\tilde{\Pi}_2$ of I^2 giving easily the result. The problem with the GP-integral is that when we start with such a δ-fine P-partition $\tilde{\Pi}$ with $\Sigma(\tilde{\Pi}) \leqq \eta$, we have no control for the rate of stretching of the $\tilde{I}^j \cap I^1$ or $\tilde{I}^j \cap I^2$ associated to an $x^j \in \mathrm{cl}\, I^1 \cap \mathrm{cl}\, I^2$, and hence no control on $\Sigma(\tilde{\Pi}_1)$ and $\Sigma(\tilde{\Pi}_2)$.

5. A DIVERGENCE THEOREM FOR DIFFERENTIABLE FUNCTIONS

Let $I =]a, b]$ be a right-closed interval in $\mathbb{R}^n$, $U = [0, 1]$ and let $\mathscr{I}$ be the n-simplex defined by

$$\mathscr{I} : U^n \to \mathbb{R}^n\,, \quad u = (u_1, \ldots, u_n) \mapsto \tag{11}$$

$$\mapsto (a_1 + u_1(b_1 - a_1), \ldots, a_n + u_n(b_n - a_n))\,,$$

so that

$$\mathscr{I}(U^n) = \mathrm{cl}\, I \quad \text{and} \quad \det \mathscr{I}'_u = m(I)\,,$$

for every $u \in U^n$, where $\mathscr{I}'_u$ denotes the differential of $\mathscr{I}$ at u.

Let f be a function from $\mathbb{R}^n$ into $\mathbb{K}^n$ which is differentiable on an open domain Ω containing $\operatorname{cl} I$. Thus, f is continuous on Ω and the integral of the $(n-1)$-form ω_f defined by

$$\omega_f = \sum_{i=1}^{n} (-1)^{i-1} f_i \, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \tag{12}$$

(where the symbol $\widehat{\ }$ denotes that the corresponding term is missing) over the $(n-1)$-complex $\partial\mathscr{I}$ is defined as usual (see e.g. [15] or [21]) by

$$\int_{\partial\mathscr{I}} \omega_f = \sum_{\alpha=0}^{1} (-1)^{\alpha-1} \int_{U^{n-1}} \sum_{i=1}^{n} f_i(\mathscr{I}^{i,\alpha}(v)) (b_i - a_i)^{-1} m(I) \, dv \, . \tag{13}$$

In this expression, $\mathscr{I}^{k,\alpha}$ is the $(n-1)$-simplex defined by

$$\mathscr{I}^{k,\alpha} : U^{n-1} \to \mathbb{R}^n \,, \quad v = (v_1, \ldots, v_{n-1}) \mapsto \tag{14}$$

$$\mapsto (a_1 + v_1(b_1 - a_1), \ldots, a_{k-1} + v_{k-1}(b_{k-1} - a_{k-1}) \,, \quad a_k + \alpha(b_k - a_k) \,,$$

$$a_{k+1} + v_k(b_{k+1} - a_{k+1}), \ldots, a_n + v_{n-1}(b_n - a_n)) \,,$$

$$(1 \leqq k \leqq n;\ \alpha = 0, 1) \,,$$

and the integrals are Riemann integrals of continuous functions. We also define as usual the divergence of f by the formula

$$\operatorname{div} f = \sum_{i=1}^{n} (\partial f / \partial x_i) \, .$$

We can now state and prove the following version of the *divergence theorem.*

Theorem 1. *Let f be a function of $\mathbb{R}^n$ into $\mathbb{K}^n$ which is differentiable on an open domain Ω. Then, for every right-closed interval I in $\mathbb{R}^n$ such that $\operatorname{cl} I \subset \Omega$, $\operatorname{div} f$ is GP-integrable over I and*

$$\int_I \operatorname{div} f = \int_{\partial\mathscr{I}} \omega_f \,, \tag{15}$$

where $\mathscr{I}$ is the n-simplex defined by (11) and the right-hand member is defined by (13).

Proof. Let $I =]a, b] =]a_1. b_1] \times \ldots \times]a_n, b_n]$ be a right-closed interval in $\mathbb{R}^n$ with closure in Ω, and let $\varepsilon > 0$ and $\eta \geqq 1$ be given; then, by the differentiability of f, for each $x \in \operatorname{cl} I$, there exists $\delta(x) > 0$ such that, for every $y \in B[x; \delta(x)]$, one has

$$\|f(y) - f(x) - f'_x(y - x)\| \leqq \varepsilon / 2n\eta\, m(I)\, \sigma(I) \,, \tag{16}$$

where f'_x denotes the differential of f at x. We define in this way a gauge $\delta : x \mapsto \delta(x)$ on $\operatorname{cl} I$. Let $\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$ be a δ-fine P-partition of I, with $\Sigma(\Pi) \leqq \eta$ and

$$I^j =]a^j, b^j] =]a_1^j, b_1^j] \times \ldots \times]a_n^j, b_n^j] \, .$$

If $\mathscr{I}^j$ denotes the n-simplex defined by

$$\mathscr{I}^j : U^n \to \mathbb{R}^n\,, \quad (u_1, \ldots, u_n) \mapsto (a_1^j + u_1(b_1^j - a_1^j), \ldots, a_n^j + u_n(b_n^j - a_n^j))\,,$$

so that $\mathscr{I}^j(U^n) = I^j$ $(1 \leqq j \leqq m)$, then, by a well known result about integrals of (continuous) differential forms, one has

$$\int_{\partial \mathscr{I}} \omega_f = \sum_{j=1}^{m} \int_{\partial \mathscr{I}^j} \omega_f\,.$$

Consequently, if we define the functions g^j and h^j from $\mathbb{R}^n$ into $\mathbb{K}^n$ by the relations

$$g^j(y) = f(x^j) + f'_{x^j}(y - x^j)\,, \quad h^j(y) = f(y) - g^j(y) \quad (1 \leqq j \leqq m)\,,$$

and if we define the corresponding differential form ω_{g^j} and ω_{h^j} according to (12), we obtain

$$(17) \qquad \left\| S(I, \operatorname{div} f, \Pi) - \int_{\partial \mathscr{I}} \omega_f \right\| = \left\| \sum_{j=1}^{m} \left[\operatorname{div} f(x^j)\, m(I^j) - \int_{\partial \mathscr{I}^j} \omega_f \right] \right\| =$$

$$= \left\| \sum_{j=1}^{m} \left[\operatorname{div} f(x^j)\, m(I^j) - \int_{\partial \mathscr{I}^j} \omega_{g^j} - \int_{\partial \mathscr{I}^j} \omega_{h^j} \right] \right\|.$$

Now, ω_{g^j} is a $(n - 1)$-differential form of class C^1 and

$$\mathrm{d}\omega_{g^j} = \operatorname{div} f(x^j)\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n\,,$$

so that the classical Stokes theorem (see e.g. [15] or [20]) or a direct computation gives

$$\int_{\partial \mathscr{I}^j} \omega_{g^j} = \int_{\mathscr{I}^j} \mathrm{d}\omega_{g^j} = \operatorname{div} f(x^j)\, m(I^j)\,, \quad (1 \leqq j \leqq m)\,.$$

Therefore, the relation (7) becomes

$$(18) \qquad \left\| S(I, \operatorname{div} f, \Pi) - \int_{\partial \mathscr{I}} \omega_f \right\| = \left\| \sum_{j=1}^{m} \int_{\partial \mathscr{I}^j} \omega_{h^j} \right\| \leqq$$

$$\leqq \sum_{j=1}^{m} \sum_{\alpha=0}^{1} \sum_{i=1}^{n} \int_{U^{n-1}} \left\| h_i^j(\mathscr{I}^{j,i}(v)) \right\| (b_i^j - a_i^j)^{-1}\, m(I^j)\, \mathrm{d}v\,,$$

where the $\mathscr{I}^{j,i,\alpha}$ are defined from the $\mathscr{I}^j$ according to formula (14). But, by (16) and the fact that Π is δ-fine, we have, for all $v \in U^{n-1}$,

$$\left\| h_i^j(\mathscr{I}^{j,i,\alpha}(v)) \right\| \leqq [\varepsilon/2n\eta\, m(I)\, \sigma(I)] \left\| \mathscr{I}^{j,i,\alpha}(v) - x^j \right\| \leqq$$

$$\leqq [\varepsilon/2n\eta\, m(I)\, \sigma(I)] \left(\max_{1 \leqq i \leqq n} (b_i^j - a_i^j) \right), \quad (1 \leqq j \leqq m;\ 1 \leqq i \leqq n;\ \alpha = 0, 1)\,.$$

Introducing these inequalities in (18) and using the fact that $\Sigma(\Pi) \leqq \eta$, we obtain finally,

$$\left\| S(I, \operatorname{div} f, \Pi) - \int_{\partial \mathscr{I}} \omega f \right\| \leqq [\varepsilon/\eta \; m(I)] \; \Sigma(\Pi) \left[\sum_{j=1}^{m} m(I^j) \right] \leqq \varepsilon ,$$

and the proof is complete.

This theorem improves earlier results of Bochner [2, 3] and Shapiro [20]. By using the concept of GP-integral instead of the Lebesgue integral, we can avoid any unnatural integrability assumption in the divergence theorem in the same way as Perron integral allows integrating all derivatives. In fact, when $n = 1$ and with the usuals conventions for 0-differential forms and boundaries of 1-simplexes, Theorem 1 above just reduces to the Perron's form of the fundamental theorem of calculus [18].

We can deduce from Theorem 1 a more general version of the Skotes theorem. Let $\varphi : \Delta \to \mathbb{R}^m$ be a twice differentiable mapping where $\Delta \subset \mathbb{R}^n$ is open and contains U^n, and let ω be a $(n - 1)$-form with coefficients differentiable on an open set Ω containing $\varphi(\Delta)$. Denoting as usual [15, 21] by $\varphi^* \lambda$ the pullback by φ of the k-form λ on Ω, it is known [19] that under the above assumptions one will have

$$\varphi^*(\mathrm{d}\omega) = \mathrm{d}(\varphi^* \omega) \tag{19}$$

Now $\varphi^* \omega$ is a $(n - 1)$-form which is differentiable on Δ and then it can be written

$$\varphi^* \omega = \sum_{i=1}^{n} (-1)^{i-1} g_i \, \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \wedge \mathrm{d}x_n$$

where the real functions g_i are differentiable on Δ. By Theorem 1, $\operatorname{div} g$ is GP-integrable on $]0, 1]^n$ and one has

$$\int_{]0,1]^n} \operatorname{div} g = \int_{\partial \mathscr{U}^n} \varphi^* \omega \tag{20}$$

where $\mathscr{U}^n$ is the standard n-cube defined by

$$\mathscr{U}^n : U^n \to \mathbb{R}^n , \quad u \to u .$$

Now, as

$$\mathrm{d}(\varphi^* \omega) = \operatorname{div} g \, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n ,$$

$\int_{\mathscr{U}^n} \mathrm{d}(\varphi^* \omega)$ is well defined by the usual formula

$$\int_{\mathscr{U}^n} \mathrm{d}(\varphi^* \omega) = \int_{\mathscr{U}^n} \operatorname{div} g ,$$

so that (20) can be written, using moreover (19), as

$$\int_{\mathscr{U}^n} \varphi^*(\mathrm{d}\omega) = \int_{\partial \mathscr{U}^n} \varphi^* \omega ,$$

i.e. by definition (see e.g. [21]) of the integral of a k-form over a k-simplex,

$$\int_{\varphi} \mathrm{d}\omega = \int_{\partial\varphi} \omega .$$

We have thus proved the following version of the *Stokes theorem*.

Corollary 1. *Let $\Delta \subset \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^m$ be open sets such that $U^n \subset \Delta$ and $\varphi(\Delta) \subset \Omega$. Then for every twice differentiable $(n - 1)$-simplex $\varphi : \Delta \to \mathbb{R}^m$ and every $(n - 1)$-form ω which is differentiable on Ω one has*

$$\int_{\varphi} \mathrm{d}\omega \int_{\partial\varphi} \omega .$$

One shall notice that in formula (21), the left hand member is the GP-integral defined in (20) and the right-hand member is a usual sum of Riemann integrals of continuous functions.

6. A MONOTONE CONVERGENCE THEOREM FOR THE GP-INTEGRAL

We shall show in this section that a monotone convergence theorem of Levi's type holds for the GP-integral. Its proof, modelled on that given by Henstock [8] for the corresponding result for the P-integral, depends on the following proposition, extending the *Saks-Henstock lemma* [8] to the GP-integral. The notations are those of Section 3.

Proposition 9. *Let f be a function of $\mathbb{R}^n$ into X defined on $\operatorname{cl} I$ and GP-integrable on I. Let $\varepsilon > 0$, $\eta \geqq 1$ be given and let δ be a corresponding gauge according to Definition 9.*

Let

$$\{(x^1, K^1), \ldots, (x^r, K^r)\}, \quad \{L^1, \ldots, L^s\}$$

be such that

$$x^l \in \operatorname{cl} K^l ,$$

$$K^l \subset B_\infty[x^l, \delta(x^l)] , \quad (1 \leqq l \leqq r) ,$$

the K^l $(1 \leqq l \leqq r)$ and L^j $(1 \leqq j \leqq r)$ being right-closed intervals which partition I and satisfy the condition

$$[\sigma(I)]^{-1} \max_{\substack{1 \leqq l \leqq r \\ 1 \leqq j \leqq s}} [\sigma(K^l), \sigma(L^j)] \leqq \eta .$$

Then

$$\left\| \sum_{l=1}^{r} \left[f(x^l)\, m(K^l) - \int_{K^l} f \right] \right\| \leqq \varepsilon .$$

Proof. By Proposition 7 and 8, f is GP-integrable on K^l $(1 \leqq l \leqq r)$ and on L^j $(1 \leqq j \leqq s)$ and

$$\int_I f = \sum_{l=1}^{r} \int_{K^l} f + \sum_{j=1}^{s} \int_{L^j} f.$$

Let $\varepsilon > 0$, $\eta \geqq 1, \delta$ and $\{(x^1, K^1), \ldots, (x^r, K^r), \{L^1, \ldots, L^s\}$ be like in the assertion and let $\xi > 0$ be given.

There exists a gauge δ_j on cl L^j such that $\delta_j(x) \leqq \delta(x)$, $x \in$ cl L^j and such that

$$\left\| S(L^j, f, \Pi_{L^j}) - \int_{L^j} f \right\| \leqq \frac{\xi}{s}$$

for every δ_j-fine RP-partition Π_{L^j} of L^j $(1 \leqq j \leqq s)$. If, say,

$$\Pi_{L^j} = \{(x^{j,1}, L^{j,1}), \ldots, (x^{j,m_j}, L^{j,m_j})\} \quad (1 \leqq j \leqq s)$$

are such RP-partitions, then

$$\Pi = \{(x^l, K^l),\ 1 \leqq l \leqq r;\ (x^{j,k_j}, L^{j,k_j}) : 1 \leqq k_j \leqq m_j,\ 1 \leqq j \leqq s\}$$

will be a δ-fine P-partition of I. Moreover,

$$\Sigma(\Pi) = [\sigma(I)]^{-1} \max\{\sigma(K^l), \sigma(L^{j,k_j}) : 1 \leqq l \leqq r,\ 1 \leqq k_j \leqq m_j,\ 1 \leqq j \leqq s\} =$$
$$= [\sigma(I)]^{-1} \max\{\sigma(K^l), \sigma(L^j) : 1 \leqq l \leqq r,\ 1 \leqq j \leqq s\} \leqq \eta\,.$$

Consequently,

$$\left\| \sum_{l=1}^{r} \left[f(x^l)\, m(K^l) - \int_{K^l} f \right] \right\| \leqq \left\| S(I, f, \Pi) - \int_I f \right\| +$$
$$+ \sum_{s=1}^{s} \left\| S(L^j, f, \Pi_{L^j}) - \int_{L^j} f \right\| \leqq \varepsilon + \xi$$

and, $\xi > 0$ being arbitrary, the result follows.

We now have the following *Levi's type monotone convergence theorem* for the GP-integral.

Theorem 2. *Let $(f_k)_{k \in N^*}$ be a sequence of real functions defined on* cl I *and such that the following conditions hold*:

1. *For each $k \in \mathbb{N}^*$, f_k is GP-integrable on I.*
2. *For each $k \in \mathbb{N}^*$ and each $x \in$* cl I, *$f_{k+1}(x) \geqq f_k(x)$.*
3. *The sequence $(f_k)_{k \in N^*}$ converges point wise on* cl I *to f.*
4. *The sequence $(\int_I f_k)_{k \in N^*}$ converges to J.*

Then f is GP-integrable on I and

$$\int_I f = J\,.$$

Proof. Let $\varepsilon > 0$ and $\eta \geqq 1$ be given; then, for each $k \in \mathbb{N}^*$ we can find a gauge δ_k on $\mathrm{cl}\, I$ with the property that

$$(22) \qquad \left| S(I, f_k, \Pi_k) - \int_I f_k \right| \leqq \varepsilon/2^{k+1}$$

for every δ_k-fine P-partition Π_k of I with

$$\Sigma(\Pi_k) \leqq \eta \,.$$

Moreover there exists $q \in \mathbb{N}^*$ such that

$$\left| \int_I f_k - J \right| \leqq \varepsilon/4$$

when $k \geqq q$ and for each $x \in \mathrm{cl}\, I$, there exists an integer $p(x) \geqq q$ such that

$$(23) \qquad |f_k(x) - f(x)| \leqq \varepsilon/4\, m(I)$$

when $k \geqq p(x)$. Define the gauge δ on $\mathrm{cl}\, I$ by

$$\delta(x) = \delta_{p(x)}(x)\,, \quad x \in \mathrm{cl}\, I\,,$$

and let $\Pi = \{(x^1, I^1), \ldots, (x^m, I^m)\}$ be a δ-fine P-partition of I with $\Sigma(\Pi) \leqq \eta$. Then,

$$(24) \qquad |S(I, f, \Pi) - J| \leqq \sum_{j=1}^{m} |f(x^j) - f_{p(x^j)}(x^j)|\, m(I^j) +$$

$$+ \left| \sum_{j=1}^{m} \left[f_{p(x^j)}(x^j)\, m(I^j) - \int_{I^j} f_{p(x^j)} \right] \right| + \left| \sum_{j=1}^{m} \int_{I^j} f_{p(x^j)} - J \right| .$$

By (23), the first term in the right-hand member of (24) is clearly smaller than $\varepsilon/4$. For the second we have, if

$$s = \max_{1 \leqq j \leqq m} p(x^j)\,,$$

then, we have

$$\left| \sum_{j=1}^{m} \left[f_{p(x^j)}(x^j)\, m(I^j) - \int_{I^j} f_{p(x^j)} \right] \right| \leqq$$

$$\leqq \sum_{k=1}^{s} \left| \sum_{\substack{j=1 \\ p(x^j)=k}}^{m} \left[f_k(x^j)\, m(I^j) - \int_{I^j} f_k \right] \right| \leqq \sum_{k=1}^{s} \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}$$

by (22) and the Saks-Henstock lemma which can be used because for each $1 \leqq j \leqq m$ such that $p(x^j) = k$, we have

$$I^j \subset B_\infty[x^j, \delta(x^j)] = B_\infty[x^j, \delta_{p(x^j)}(x^j)] = B_\infty[x^j, \delta_k(x^j)]$$

and moreover the I^j $(1 \leqq j \leqq m)$ are such that

$$[\sigma(I)]^{-1} \max_{1 \leqq j \leqq m} \sigma(I^j) = \Sigma(\Pi) \leqq \eta \,.$$

Finally, for the third term in (24) we have, if we set

$$r = \min_{1 \leqq j \leqq m} p(x^j),$$

so that $s \geqq r \geqq q$, using assumption (2) and Propositions 8 and 5,

$$\int_I f_r = \sum_{j=1}^{m} \int_{I^j} f_r \leqq \sum_{j=1}^{m} \int_{I^j} f_{p(x^j)} \leqq \sum_{j=1}^{m} \int_{I^j} f_s = \int_I f_s \leqq J .$$

Consequently,

$$\left| \sum_{j=1}^{m} \int_{I^j} f_{p(x^j)} - J \right| \leqq J - \int_I f_r \leqq \frac{\varepsilon}{4}$$

and hence the proof is complete.

Remark 3. To deduce from Theorem 2 a Lebesgue's type dominated convergence theorem for real functions requires the obtention of a result telling that if f, g, h are integrable, with $h \geqq 0$ and $|f|$, $|g| \leqq h$, then $\max(f, g)$ and $\min(f, g)$ are integrable. Such a result is true for the P-integral (see e.g. [11] or [14]) but again the corresponding proof uses a device of the type described in Remark 2, and which does not work for the GP-integral. The problem of the obtention of a dominated convergence theorem for the GP-integral is therefore open.

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