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PERMUTABLE GROUPOIDS

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1. INTRODUCTION

A groupoid satisfying the identity $x \,.\, yz = y \,.\, xz$ is said to be *left permutable*; it is said to be *right permutable* if it satisfies the dual identity $xy \,.\, z = xz \,.\, y$; it is said to be *bi-permutable* if it is both left and right permutable. Equivalently, a groupoid is left permutable iff its left translations commute. The aim of the present paper is to investigate the variety of left permutable groupoids and the variety of bi-permutable groupoids.

Both these varieties are in a close connection with the variety of commutative semigroups. Every commutative semigroup is clearly bi-permutable. On the other hand, every left permutable groupoid can be embedded into a groupoid obtained in a natural way from a commutative semigroup and its fixed transformation (see Theorem 3.1). Notice that every commutative permutable groupoid is a commutative semigroup; also, every left permutable groupoid containing a right unit is a commutative semigroup.

1.1. Example. Let $S(+)$ be a commutative semigroup and let f be a transformation of S. Define a new binary operation on S by $ab = f(a) + b$. We obtain a left permutable groupoid.

1.2. Example. The set of non-negative integers together with the binary operation $(a, b) \mapsto a^b$ is a right permutable groupoid. As proved in [4] (cf. [1], p. 384), this groupoid generates the variety of right permutable groupoids.

1.3. Example. The set of all subsets of a given set, together with the binary operation $(a, b) \mapsto a - b$, is a right permutable groupoid satisfying the following identities:

$$x \,.\, xy = y \,.\, yx\,, \quad xx = yy\,, \quad x \,.\, xx = x\,.$$

Right permutable groupoids satisfying these identities were studied under the name commutative BCK-algebras by several authors (see e.g. [5], [6], [7]).

The variety of left permutable quasigroups was studied in [2] and [3]. It turned out that this variety is equivalent to the variety of algebras $A(+, -, 0, p, p^{-1})$ such that $A(+, -, 0)$ is an abelian group and p is a permutation of A preserving the zero element. We proceed by a summary of results on left permutable quasigroups obtained in [2]: Every countable left permutable quasigroup Q can be embedded into a cyclic left permutable quasigroup P such that P is finite if Q is so; every left permutable quasigroup Q can be embedded into a simple left permutable quasigroup P such that P is finite if Q is so; if Q is a left permutable quasigroup then a congruence of the groupoid Q need not be a congruence of the quasigroup Q but any two groupoid congruences of Q commute; the variety of left permutable quasigroups has uncountably many minimal subvarieties, it has the strong amalgamation property, the finite embeddability property and the Schreier property; a quasigroup is bi-permutable iff it is an abelian group.

Some of these properties and some others are considered in the sequel for the variety of left permutable and the variety of bi-permutable groupoids. Nevertheless, the following questions remain open:

(1) Has the variety of left permutable groupoids uncountably many minimal subvarieties?

(2) Has the variety of bi-permutable groupoids only countably many subvarieties?

2. FREE LEFT PERMUTABLE GROUPOIDS

2.1. Lemma. *Let n be a positive integer and p a permutation of $\{1, \ldots, n\}$. Then every left permutable groupoid satisfies the identity $x_1(x_2(\ldots(x_n y))) = x_{p(1)}(x_{p(2)}(\ldots(x_{p(n)}y)))$.*

Proof. Obvious.

Denote by CS1T the variety of algebras with one binary operation $+$ and one unary operation f satisfying the identities $x + (y + z) = (x + y) + z$ and $x + y = y + x$. Thus the algebras from CS1T are just commutative semigroups with a fixed transformation.

2.2. Proposition. *Let $A(+, f)$ be an algebra from* CS1T; *put $ab = f(a) + b$ for all $a, b \in A$. Then $A(.)$ is a left permutable groupoid.*

Proof. Obvious.

Let X be a non-empty set. Our aim is to construct the free left permutable groupoid over X. For this purpose, it turns out to be useful first to construct the free CS1T-algebra over X.

Define a chain $A_0(+) \subseteq A_1(+) \subseteq A_2(+) \subseteq \ldots$ of commutative semigroups and a chain $f_1 \subseteq f_2 \subseteq \ldots$ of mappings $f_i : A_{i-1} \to A_i$ as follows: $A_0(+)$ is the free commutative semigroup over X; if $i \geqq 1$ then fix a bijection g_i of $A_{i-1} \setminus A_{i-2}$ onto a set disjoint with A_{i-1}, put $f_i = f_{i-1} \cup g_i$ (here $A_{-1} = f_0 = \emptyset$) and let $A_i(+)$ be the free commutative semigroup over $X \cup f_i(A_{i-1})$. Denote by $P_X(+)$ the union of the chain $A_i(+)$ $(i = 0, 1, \ldots)$ and by f the union of the chain f_i $(i = 1, 2, \ldots)$.

2.3. Proposition. *Let X be a non-empty set. Then:*
(1) *$P_X(+, f)$ is a free* CS1T-*algebra over X.*
(2) *f is an injective transformation of P_X and $X \cap f(P_X) = \emptyset$.*
(3) *$P_X(+)$ is a free commutative semigroup over $X \cup f(P_X)$.*

Proof. Easy.

Evidently, there exists a unique mapping λ of P_X into the set of positive integers such that $\lambda(x) = 1$ for all $x \in X$, $\lambda(f(a)) = 1 + \lambda(a)$ and $\lambda(a + b) = \lambda(a) + \lambda(b)$ for all $a, b \in P_X$. The number $\lambda(a)$ will be called the *length of an element $a \in P_X$*.

By a lifting sequence we shall mean a finite (possibly empty) sequence of elements of $P_X \cup \{0\}$ (where $0 \notin P_X$). Given an element $a \in P_X$ and a lifting sequence $s = (u_1, \ldots, u_n)$, we define an element $a * s$ of P_X by $a * s = a$ if $n = 0$, $a * s = f(a * (u_1, \ldots, u_{n-1}))$ if $n \geqq 1$ and $u_n = 0$ and $a * s = (a * (u_1, \ldots, u_{n-1})) + u_n$ if $n \geqq 1$ and $u_n \in P_X$.

Let $a, b \in P_X$. We shall say that a is a part of b if $b = a * s$ for a lifting sequence s. The following lemma shows that the notion of a part of an element $u \in P_X$ can be equivalently defined by induction on the length of u.

2.4. Lemma. (1) *If $x \in X$ and $a \in P_X$ then a is a part of x iff $a = x$.*
(2) *If $a, b \in P_X$ then a is a part of $f(b)$ iff either $a = f(b)$ or a is a part of b.*
(3) *If $n \geqq 2$, $a_1, \ldots, a_n \in X \cup f(P_X)$ and $a \in P_X$ then a is a part of $a_1 + \ldots + a_n$ iff either a is a part of at least one of the elements $a_1, \ldots, a_n$ or $a = a_{i_1} + \ldots + a_{i_k}$ for some $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$.*

Proof. Easy.

Now, define a multiplication on P_X by $ab = f(a) + b$ for all $a, b \in P_X$. We obtain a groupoid $P_X(.)$; by 2.2, this groupoid is left permutable. We denote by F_X the subgroupoid of $P_X(.)$ generated by X.

2.5. Theorem. *Let X be a non-empty set. Then:*
(1) *F_X is a free left permutable groupoid over X.*
(2) *An element $a \in P_X$ belongs to F_X iff the following condition is satisfied: If $b \in P_X$ is such that either $b = a$ or $f(b)$ is a part of a then $b = x + f(c_1) + \ldots + f(c_n)$ for some $n \geqq 0$, $x \in X$ and $c_1, \ldots, c_n \in P_X$.*

Proof. (1) Denote by W the absolutely free groupoid over X and by h the homomorphism of W onto F_X such that $h(x) = x$ for every $x \in X$. It suffices to show that $h(a) = h(b)$ for $a, b \in W$ iff the identity $a = b$ is satisfied in all left permutable groupoids. The converse implication is trivial, since F_X is left permutable. Now, let $h(a) = h(b)$. We shall proceed by induction on the length of the term ab. If one of the terms a, b belongs to X then clearly $a = b$. In the opposite case, $a =$

$= a_1(\dots(a_n x))$ and $b = b_1(\dots(b_m y))$ for some $n, m \geqq 1$, $x, y \in X$ and $a_1, \dots, a_n$, $b_1, \dots, b_m \in W$. Since $h(a) = h(b)$, we have $f\,h(a_1) + \dots + f\,h(a_n) + x = f\,h(b_1) + \dots + f\,h(b_m) + y$. Consequently, $n = m$, $x = y$ and there is a permutation p of $\{1, \dots, n\}$ with $h(a_i) = h(b_{p(i)})$ for all $1 \leqq i \leqq n$. The rest follows from the induction hypothesis and 2.1.

(2) Denote by E the set of all $a \in P_X$ satisfying the condition. Then E is a subgroupoid of F_X, $X \subseteq E$ and so $F_X \subseteq E$. Conversely, proceeding by induction on the length of a, we can show that $a \in F_X$ for every $a \in E$.

2.6. Lemma. *F_X is a block of a congruence of the algebra $P_X(+, f)$.*

Proof. Define two binary relations R and S on P_X as follows: $(a, b) \in R$ iff there exist elements $u, v \in F_X$ and a lifting sequence s such that $a = u * s$ and $b = v * s$; $(a, b) \in S$ iff there exists a finite sequence $a_1, \dots, a_k$ of elements of P_X such that $a = a_1$, $b = a_k$ and $(a_i, a_{i+1}) \in R$ for all $1 \leqq i \leqq k - 1$. Clearly, S is a congruence of $P_X(+, f)$ and F_X is contained in a block of S. Now, let $a \in F_X$, $b \in P_X$ and $(a, b) \in R$; we are going to show that $b \in F_X$. There are $u, v \in F_X$ and a lifting sequence $s = (u_1, \dots, u_n)$ with $a = u * s \in F_X$ and $b = v * s$. We shall proceed by induction on n. Everything is clear for $n = 0$. Let $n \geqq 1$. By 2.5(2), $u_n \in P_X$. If $n \geqq 2$ and $u_{n-1} \in P_X$ then we can use the induction hypothesis for $a = u * r$ and $b = v * r$, where $r = (u_1, \dots, u_{n-2}, u_{n-1} + u_n)$. If $n \geqq 2$ and $u_{n-1} = 0$ then (by 2.5(2)) $u * r \in F_X$ where $r = (u_1, \dots, u_{n-2})$; by the induction hypothesis, $v * r \in F_X$ and we have $b = f(v * r) + u_n = (v * r) \,.\, u_n \in F_X$. If $n = 1$ then $a = u + u_1$, $b = v + u_1$ and $b \in F_X$ is an easy consequence of 2.5(2).

2.7. Proposition. *Let Q be a free left permutable quasigroup over X and let G be the subgroupoid of Q generated by X. Then G is a free left permutable groupoid over X.*

Proof. Since the commutative semigroup $P_X(+)$ is cancellative and f is an injective transformation, there exist an abelian group $A(+)$ and its permutation g such that $P_X(+, f)$ is a subalgebra of $A(+, g)$. Setting $ab = g(a) + b$ for all $a, b \in A$, we obtain a left permutable quasigroup A and there is a homomorphism h of Q into A such that $h(x) = x$ for each $x \in X$. Hence the restriction of h to G is a homomorphism of G onto F_X, and it is clearly an isomorphism.

2.8. Corollary. *Every free left permutable groupoid is cancellative and can be embedded into a left permutable quasigroup.*

2.9. Proposition. *Let X be a non-empty subset of a left permutable groupoid G. Then G is a free left permutable groupoid over X iff the following two conditions are satisfied:*

(1) *G is generated by X;*

(2) *If $n, m \geqq 0$, $x, y \in X$, $a_1, \dots, a_n, b_1, \dots, b_m \in G$ and $a_1(\dots(a_n x)) = b_1(\dots(b_m y))$ then $n = m$, $x = y$ and there is a permutation p of $\{1, \dots, n\}$ with $a_i = b_{p(i)}$ for all $1 \leqq i \leqq n$.*

Proof. The direct implication follows from 2.5. Now, suppose that the conditions (1) and (2) are satisfied, denote by W the absolutely free groupoid over X and consider the homomorphism h of W onto G such that $h(x) = x$ for each $x \in X$. It is enough to show by induction on the length of ab that if a, b are two terms (elements of W) such that $h(a) = h(b)$ then the equation $a = b$ is satisfied in all left permutable groupoids. If $a, b \in X$, then this is clear. Otherwise we can write $a = a_1(\dots(a_n x))$ and $b = b_1(\dots(b_m y))$ for some $a_1, \dots, a_n, b_1, \dots, b_m \in W$ and $x, y \in X$. Then $h(a_1)(h(a_2)(\dots(h(a_n)x))) = h(b_1)(\dots(h(b_m)y))$; we can use (2), the induction hypothesis and 2.1.

2.10. Proposition. *Let G be a free left permutable groupoid over X and let Y be a non-empty subset of G; denote by H the subgroupoid of G generated by Y. Then H is a free left permutable groupoid over Y iff the following condition is satisfied: If $n, m \geqq 0$, $a_1, \dots, a_n, b_1, \dots, b_m \in H$, $a \in G$, $a_1(\dots(a_n a)) \in Y$ and $b_1(\dots(b_m a)) \in Y$ then $n = m$ and there is a permutation p of $\{1, \dots, n\}$ such that $a_i = b_{p(i)}$ for all $1 \leqq i \leqq n$.*

Proof. First, let H be free over Y. Put $b = a_1(\dots(a_n a))$ and $c = b_1(\dots(b_m a))$. By 2.1 we have $b_1(\dots(b_m b)) = a_1(\dots(a_n c))$; by 2.9 we get $b = c$, $n = m$ and $a_i = b_{p(i)}$ for some permutation p. Now, let the condition be satisfied. Let $b, c \in Y$, $n, m \geqq 0$, $a_1, \dots, a_n, b_1, \dots, b_m \in H$ and $a_1(\dots(a_n b)) = b_1(\dots(b_m c))$. By 2.9 it is enough to prove that $n = m$, $b = c$ and $a_i = b_{p(i)}$ for a permutation p of $\{1, \dots, n\}$. This will be done by induction on $n + m$. Everything is clear if $n = m = 0$. If the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ are not disjoint then we can assume $a_1 = b_1$ and we have $a_2(\dots(a_n b)) = b_2(\dots(b_m c))$, so that the induction hypothesis works. Hence we can assume that the two sets are disjoint. The elements b and c can be expressed in the form $b = c_1(\dots(c_k x))$ and $c = d_1(\dots(d_j x))$ for some $k, j \geqq 0$, $x \in X$ and $c_1, \dots, c_k, d_1, \dots, d_j \in G$. The sequences $(a_1, \dots, a_n, c_1, \dots, c_k)$ and $(b_1, \dots, b_m, d_1, \dots, d_j)$ coincide up to the order of their members; since $\{a_1, \dots, a_n\}$ is disjoint with $\{b_1, \dots, b_m\}$, up to the order of members we have $(c_1, \dots, c_k) = (b_1, \dots, b_m, e_1, \dots, e_r)$ and $(d_1, \dots, d_j) = (a_1, \dots, a_n, e_1, \dots, e_r)$ for some $e_1, \dots, e_r$. Hence $b = b_1(\dots(b_m a))$ and $c = a_1(\dots(a_n a))$ for some $a \in G$, and we can use our condition.

2.11. Example. Let G be a free left permutable groupoid over $X = \{x, y, z\}$ and let H be the subgroupoid of G generated by $\{x, y, xz, yz\}$. Then it follows from 2.10 that H is not a free left permutable groupoid.

2.12. Corollary. *The variety of left permutable groupoids does not have the Schreier property.*

2.13. Example. Let G be a free left permutable groupoid over $X = \{x\}$ and let H be the subgroupoid of G generated by the set $Y = \{xx, x \, . \, xx, x(x \, . \, xx), \dots\}$. Then it follows from 2.10 that H is a free left permutable groupoid over the infinite set Y.

2.14. Corollary. *The free left permutable groupoid of countable rank can be embedded into the free left permutable groupoid of rank* 1.

3. SEMIGROUP REPRESENTATIONS OF LEFT PERMUTABLE GROUPOIDS

In this section we shall prove the following basic result.

3.1. Theorem. *Let G be a left permutable groupoid. Then there exist a commutative semigroup $S(+)$ and a permutation p of S such that $G \subseteq S$ and $ab = p(a) + b$ for all $a, b \in G$.*

Proof. Denote by X the underlying set of G, by $\circ$ the operation of G and consider the free CS1T-algebra $P_X(+, f)$ and the free left permutable groupoid F_X from the preceding section. Further, define two binary relations R and T on P_X as follows: $(a, b) \in R$ iff there exist $x, y, z \in X$ and a lifting sequence s such that $z = x \circ y$, $a = z * s$ and $b = (f(x) + y) * s$; $(a, b) \in T$ iff there exists a finite sequence $a_1, \ldots, a_k$ of elements of P_X such that $a = a_1$, $b = a_k$ and $(a_i, a_{i+1}) \in R \cup R^{-1}$ for all $1 \leqq i \leqq k - 1$. It is easy to see that T is a congruence of the algebra $P_X(+, f)$. Now, we shall prove the following five lemmas.

3.2. Lemma. *Let $(a, b) \in R$, $(c, b) \in R$ and let b be such that $x + y$ is not a part of b for any $x, y \in X$. Then either $a = c$ or there exists a $d \in P_X$ with $(d, a) \in R$ and $(d, c) \in R$.*

Proof. We have $a = (x \circ y) * (e_1, \ldots, e_n)$, $b = (f(x) + y) * (e_1, \ldots, e_n) = (f(u) + v) * (f_1, \ldots, f_m)$, $c = (u \circ v) * (f_1, \ldots, f_m)$ for some $x, y, u, v, e_1, \ldots, f_m$. We shall proceed by induction on $n + m$. Distinguish the following cases.

Case 1: Let $n = 0$ (resp. $m = 0$). Then $b = f(x) + y = (f(u) + v) * (f_1, \ldots, f_m)$ implies $m = 0$, $x = u$, $y = v$ and $a = c$.

Case 2: Let $n, m \geqq 1$ and $e_n = 0$ (resp. $f_m = 0$). Then $f_m = 0$ and we can use the induction hypothesis for the triple $(x \circ y) * (e_1, \ldots, e_{n-1})$, $(f(x) + y) * (e_1, \ldots, e_{n-1}) = (f(u) + v) * (f_1, \ldots, f_{m-1})$, $(u \circ v) * (f_1, \ldots, f_{m-1})$.

Case 3: Let $n \geqq 2$, $m \geqq 1$, $e_n \neq 0 \neq f_m$ and $e_{n-1} \neq 0$ (resp. $n \geqq 1$, $m \geqq 2$, $e_n \neq 0 \neq f_m$ and $f_{m-1} \neq 0$). Then we can use the induction hypothesis for the triple $(x \circ y) * (e_1, \ldots, e_{n-2}, e_{n-1} + e_n)$, $(f(x) + y) * (e_1, \ldots, e_{n-2}, e_{n-1} + e_n) = (f(u) + v) * (f_1, \ldots, f_m)$, $(u \circ v) * (f_1, \ldots, f_m)$.

Case 4: Let $n, m \geqq 2$, $e_n \neq 0 \neq f_m$ and $e_{n-1} = 0 = f_{m-1}$. We can express b in the form $b = b_1 + \ldots + b_k$ where $k \geqq 2$ and $b_1, \ldots, b_k \in X \cup f(P_X)$. Now, $(f(x) + y) * (e_1, \ldots, e_{n-1}) = b_i$ and $(f(u) + v) * (f_1, \ldots, f_{m-1}) = b_j$ for some $1 \leqq i, j \leqq k$. If $i = j$ then $e_n = f_m$ and we can use the induction hypothesis for the triple $(x \circ y) * (e_1, \ldots, e_{n-1})$, $(f(x) + y) * (e_1, \ldots, e_{n-1}) = (f(u) + v) * (f_1, \ldots, f_{m-1})$, $(u \circ v) * (f_1, \ldots, f_{m-1})$. Hence we can assume that $i = 1$ and $j = 2$. Then $a = (f(u) + v) * (f_1, \ldots, f_{m-1}) + b_3 + \ldots + b_k + (x \circ y) * (e_1, \ldots, e_{n-1})$, $c = (f(x) + y) *$

$* (e_1, \ldots, e_{n-1}) + b_3 + \ldots + b_k + (u \circ v) * (f_1, \ldots, f_{m-1})$ and we can put $d = (x \circ y) * (e_1, \ldots, e_{n-1}) + b_3 + \ldots + b_k + (u \circ v) * (f_1, \ldots, f_{m-1})$.

Case 5: Let $n \geqq 2$, $m = 1$, $e_n \neq 0 \neq f_m$ and $e_{n-1} = 0$ (resp. $n = 1$, $m \geqq 2$, $e_n \neq 0 \neq f_m$ and $f_{m-1} = 0$). We have $b = b_1 + \ldots + b_k$ for some $b_1, \ldots, b_k \in X \cup f(P_X)$ with $k \geqq 3$ and we can assume that $(f(x) + y) * (e_1, \ldots, e_{n-1}) = b_1$, $f(u) = b_2$ and $v = b_3$. Then $a = (x \circ y) * (e_1, \ldots, e_{n-1}) + f(u) + v + b_4 + \ldots + b_k$, $c = (u \circ v) + (f(x) + y) * (e_1, \ldots, e_{n-1}) + b_4 + \ldots + b_k$ and we can put $d = (x \circ y) * (e_1, \ldots, e_{n-1}) + (u \circ v) + b_4 + \ldots + b_k$.

Case 6: Let $n = 1 = m$ and $e_1 \neq 0 \neq f_1$. We have $b = b_1 + \ldots + b_k$ for some $b_1, \ldots, b_k \in X \cup f(P_X)$ with $k \geqq 3$ and $f(x) = b_i, f(u) = b_j, y = b_r, v = b_s$ for some $1 \leqq i, j, r, s \leqq k$. Since $y + v$ is not a part of b, we have $r = s$ and $y = v$. If $i = j$ then $a = c$. Hence, assume that $i = 1$, $j = 2$ and $r = 3$. Then $a = (x \circ y) + f(u) + b_4 + \ldots + b_k$, $c = (u \circ y) + f(x) + b_4 + \ldots + b_k$ and it follows from the left permutability of G that we can put $d = (u \circ (x \circ y)) + b_4 + \ldots + b_k = (x \circ (u \circ y)) + b_4 + \ldots + b_k$.

3.3. Lemma. *Let $(a, b) \in T$. Then $a \in F_X$ iff $b \in F_X$.*

Proof. It is easily seen that T is just the congruence of $P_X(+, f)$ generated by all the pairs $(x \circ y, f(x) + y)$ where $x, y \in X$. Hence $T \subseteq V$ where V is a congruence of $P_X(+, f)$ such that F_X is a block of V (see 2.6) and the result is clear.

3.4. Lemma. *Let $x, y \in X$ and $(x, y) \in T$. Then $x = y$.*

Proof. There is a sequence $a_1, \ldots, a_k$ of elements of P_X such that $x = a_1$, $y = a_k$ and $(a_i, a_{i+1}) \in R \cup R^{-1}$ for all $1 \leqq i \leqq k - 1$. We shall proceed by induction on $\lambda(a_1) + \ldots + \lambda(a_k)$. Evidently, we can assume that $k \geqq 3$ and $(a_i, a_{i+1}), (a_{i+2}, a_{i+1}) \in R$ for some $1 \leqq i \leqq k - 2$. By 3.3, $a_{i+1} \in F_X$, and hence $u + v$ is not a part of a_{i+1} for any $u, v \in X$ (use 2.5). According to 3.2, either $a_i = a_{i+2}$ and we can use the indunction hypothesis for the sequence $a_1, \ldots, a_i, a_{i+3}, \ldots, a_k$ or $(d, a_i) \in R$ and $(d, a_{i+2}) \in R$ for some $d \in P_X$ and then we can use the induction hypothesis for the sequence $a_1, \ldots, a_i, d, a_{i+2}, \ldots, a_k$, since $\lambda(d) < \lambda(a_{i+1})$.

3.5. Lemma. *Let $a, b \in P_X$ be such that $(f(a), f(b)) \in T$. Then $(a, b) \in T$.*

Proof. Easy.

Now, we are ready to finish the proof of 3.1. Applying 3.4 and 3.5, we see that there exists a CS1T-algebra $A(+, q)$ isomorphic to $P_X(+, f)/T$ and such that q is an injective transformation of A, $X \subseteq A$ and $a \circ b = q(a) + b$ for all $a, b \in X$. Now, $A(+)$ can be embedded into a commutative semigroup $S(+)$ such that $\mathrm{Card}(S \setminus A) = \mathrm{Card}(A)$, and q can be extended to a permutation p of S.

This completes the proof of 3.1.

3.6. Lemma. *Let G be a left permutable groupoid and let $a, b \in G$ be elements* *such that* $ab = b$ *and the right translation* R_b *(i.e. the mapping* $x \mapsto xb$*) is surjective. Then* a *is a left unit of* G *(i.e.* $ax = x$ *for all* $x \in G$*).*

Proof. We have $a . cb = c . ab = cb$ for every $c \in G$ and so $ax = x$ for all $x \in G$.

3.7. Proposition. *The following three conditions are equivalent for a groupoid* G:
(1) G *is left permutable and there exist elements* $a, b \in G$ *such that* $ab = b$ *and both the right translations* R_a *and* R_b *are surjective.*
(2) G *is left permutable and contains a left unit* e *such that the right translation* R_e *is surjective.*
(3) *There exist a commutative semigroup* $G(+)$ *with a neutral element* 0 *and a surjective transformation* f *of* G *such that* $f(0) = 0$ *and* $xy = f(x) + y$ *for all* $x, y \in G$.

Proof. Putting $a = b = 0$, we see that (3) implies (1). By 3.6, (1) implies (2). It remains to prove that (2) implies (3). There is a transformation g of G with $g(e) = e$ and $g(x) e = x$ for every $x \in G$. Put $x + y = g(x) . y$ for all $x, y \in G$. Further, put $0 = e$. Then 0 is a neutral element of $G(+)$ and $x + (y + z) = g(x) . g(y) z = g(y) . g(x) z = y + (x + z)$ for all $x, y, z \in G$. Consequently, $G(+)$ is a commutative semigroup. Moreover, $g(xe) . ye = y(g(xe) e) = y . xe = x . ye$ for all $x, y \in G$. Hence we see that $R_e(x) + y = g(xe) . y = xy$ and we can put $f = R_e$.

A groupoid is said to be *right divisible* (*right cancellative*) if all its right translations are surjective (resp. injective).

3.8. Corollary. *The following conditions are equivalent for a groupoid* G:
(1) G *is a left permutable and right divisible groupoid.*
(2) G *is a left permutable, divisible and left cancellative groupoid.*
(3) *There exist an abelian group* $G(+)$ *and a surjective transformation* f *of* G *such that* $f(0) = 0$ *and* $xy = f(x) + y$ *for all* $x, y \in G$.

3.9. Corollary. *The following conditions are equivalent for a groupoid* G:
(1) G *is a left permutable right quasigroup.*
(2) G *is a left permutable quasigroup.*
(3) *There exist an abelian group* $G(+)$ *and a permutation* f *of* G *such that* $f(0) = 0$ *and* $xy = f(x) + y$ *for all* $x, y \in G$.

4. SEVERAL PROPERTIES OF THE VARIETY OF LEFT PERMUTABLE GROUPOIDS

4.1. Proposition. *Every countable left permutable groupoid can be embedded into a cyclic (i.e. one-generated) left permutable groupoid.*

Proof. Let G be a countable left permutable groupoid. By 3.1, there exist a countable commutative semigroup $S(+)$ and a permutation p of S such that $G \subseteq S$ and $ab = p(a) + b$ for all $a, b \in G$. Put $T(+) = S_0(+) \times N(+)$ where $S_0(+)$ is the commutative semigroup obtained from $S(+)$ by adding a neutral element 0 and $N(+)$

is the additive semigroup of non-negative integers. Clearly, there is a transformation f of T with the following three properties:

(1) $f(a, 0) = (p(a), 0)$ for all $a \in S$;

(2) $f(0, 0) = (0, 1)$;

(3) every element of T is equal to $f(0, n)$ for some $n \in N$.

Now, define a multiplication on T by $xy = f(x) + y$ for all $x, y \in T$. We obtain a left permutable groupoid, the map $a \mapsto (a, 0)$ is an embedding of G into T and it suffices to show that the groupoid T is generated by the element $(0, 0)$. However, $(0, 0) . (0, n) = (0, n + 1)$ and $f(a, n) = (a, n) . (0, 0)$ for all $a \in S_0$ and $n \in N$. The rest is clear.

4.2. Proposition. *Every left permutable groupoid can be embedded into a simple left permutable groupoid.*

Proof. Let G be a left permutable groupoid. It suffices to show that for any three different elements a, b, c of G there exists a left permutable groupoid H such that G is a subgroupoid of H and (a, c) belongs to the congruence of H generated by (a, b). Let $S(+, p)$ be as in 3.1. Without loss of generality, we can assume that $S(+)$ contains a neutral element 0 such that $p(0) = 0$ and $0 \notin \{a, b, c\}$. Denote by $D(+)$ the two-element group $\{0, 1\}$, put $T(+) = S(+) \times D(+)$ and define a transformation g of T by $g(x, 0) = (p(x), 0)$, $g(a, 1) = (a, 0)$, $g(b, 1) = (c, 0)$ and $g(y, 1) = (y, 1)$ for all $x, y \in S$, $a \neq y \neq b$. Further, define a multiplication on T by $xy = g(x) + y$ and let r be a congruence of T with $((a, 0), (b, 0)) \in r$. Then $((a, 0), (c, 0)) \in r$, since $(a, 0) = ((0, 1) . (a, 0)) (0, 0)$ and $(c, 0) = ((0, 1) . (b, 0)) (0, 0)$. The mapping $x \mapsto (x, 0)$ is an embedding of G into T and T is a left permutable groupoid.

4.3. Proposition. *Every left permutable divisible groupoid is a homomorphic image a of a left permutable quasigroup.*

Proof. Let G be a left permutable divisible groupoid. By 3.8, there are an abelian group $G(+)$ and a surjective transformation f of G such that $ab = f(a) + b$ for all $a, b \in G$. Consider an infinite cardinal number k such that $\mathrm{Card}(A) \leqq k$ whenever A is a block of $\ker(f)$. There exist an abelian group $H(+)$ and a surjective homomorphism h of $H(+)$ onto $G(+)$ such that $\mathrm{Card}(B) = k$ for every block B of $\ker(h)$. Now, it is easy to see that there is a permutation p of H with $h\, p(a) = f\, h(a)$ for every $a \in H$. Define a multiplication on H by $ab = p(a) + b$ for all $a, b \in H$. Then H becomes a left permutable quasigroup and h is a homomorphism of H onto G.

In contrast to 4.3, it is not true that every left permutable cancellative groupoid can be embedded into a left permutable quasigroup. A counterexample will be constructed in the next section.

A variety V of universal algebras is said to have the *amalgamation property* if for any triple A, B, C of algebras from V and any pair $f : A \to B$, $g : A \to C$ of injective homomorphisms there exist an algebra $D \in V$ and two injective homomorphisms $h : B \to D$, $k : C \to D$ such that $hf = kg$.

4.4. Proposition. *Let V be any variety contained in the variety of left permutable groupoids and containing the variety of commutative semigroups satisfying $xyz = uu$. Then V does not have the amalgamation property.*

Proof. Following the wellknown Kimura's proof of the fact that the variety of semigroups does not have the amalgamation property, define three groupoids A, B, C as follows: $A = \{0, a, b, c\}$, $B = \{0, a, b, c, d\}$, $C = \{0, a, b, c, e\}$; $xy = 0$ in all cases except for $bd = db = c$ in B and $ae = ea = b$ in C. Then evidently $A, B, C \in V$ and A is a subgroupoid of both B and C. Suppose that there is a left permutable groupoid D and two injective homomorphisms $h : B \to D$, $k : C \to D$ coinciding on A. Then $h(c) = h(d)\, h(b) = h(d)\, k(b) = h(d)\,(k(e)\, k(a)) = k(e)\,.$ $.\,(h(d)\, k(a)) = k(e)\,(h(d)\, h(a)) = k(e)\, h(0) = k(e)\, k(0) = k(0) = h(0)$, a contradiction.

4.5. Corollary. *The variety of left permutable groupoids does not have the amalgamation property.*

Proof. This follows immediately from 4.4. However, we shall give yet another proof, showing that D does not exist even in the case when A, B, C are all free. Fix pairwise different elements $a, b, c, d, e, f, g, x, y, z, u, v$ and denote by A, B, C the free left permutable groupoid over $\{x, y, z, u, v\}$, $\{a, b, e, f\}$ and $\{a, c, d, g\}$, respectively. It is an easy consequence of 2.10 that the subgroupoid of B generated by $\{a, e, f, ba, bf\}$ is free over the set, and hence there is an injective homomorphism $\bar{f}$ of A into B with $\bar{f}(x) = a$, $\bar{f}(y) = e$, $\bar{f}(z) = f$, $\bar{f}(u) = ba$, $\bar{f}(v) = bf$. Similarly, the subgroupoid of C generated by $\{a, cd, ca, d, g\}$ is free and there is an injective homomorphism $\bar{g}$ of A into C with $\bar{g}(x) = a$, $\bar{g}(y) = cd$, $\bar{g}(z) = ca$, $\bar{g}(u) = d$, $\bar{g}(v) = g$. Now, suppose that there exist a left permutable groupoid D and injective homormorphisms $h : B \to D$, $k : C \to D$ with $h\bar{f} = k\bar{g}$. We have $h\,\bar{f}(y) = k\,\bar{g}(y) =$ $= k(cd) = k(c)\, k(d) = k(c)\, k\,\bar{g}(u) = k(c)\, h\bar{f}(u) = k(c)\, h(ba) = k(c)\,(h(b)\, h(a)) =$ $= h(b)\,(k(c)\, h(a)) = h(b)\,(k(c)\, h\,\bar{f}(x)) = h(b)\,(k(c)\, k\,\bar{g}(x)) = h(b)\,(k(c)\, k(a)) =$ $= h(b)\, k(ca) = h(b)\, k\,\bar{g}(z) = h(b)\, h\,\bar{f}(z) = h(b)\, h(f) = h(bf) = h\,\bar{f}(v)$ and consequently $y = v$, a contradiction.

4.6. Proposition. *Let S be a (multiplicatively written) cancellative commutative semigroup with unit and let G be its group of quotients. Then the embedding of S into G is an epimorphism in the category of left permutable groupoids.*

Proof. Let f, g be two homomorphisms of G into a left permutable groupoid H such that $f(a) = g(a)$ for each $a \in S$. We have $f(a^{-1}) = f(a^{-1}\,1) = f(a^{-1}) f(1) =$ $= f(a^{-1})\, g(1) = f(a^{-1})\, g(a^{-1}a) = f(a^{-1})\,(g(a^{-1})\, g(a)) = g(a^{-1})\,(f(a^{-1})\, g(a)) =$ $= g(a^{-1})(f(a^{-1}) f(a)) = g(a^{-1}) f(a^{-1}a) = g(a^{-1}) f(1) = g(a^{-1})\, g(1) = g(a^{-1}\,1) =$ $= g(a^{-1})$ and $f(a^{-1}b) = f(a^{-1}) f(b) = g(a^{-1})\, g(b) = g(a^{-1}b)$ for all $a, b \in S$.

4.7. Corollary. *The category of left permutable groupoids has non-surjective epimorphisms.*

In this section we construct a cancellative left permutable groupoid which cannot be embedded into a left permutable quasigroup.

Fix four different elements x, x', y, z, put $X = \{x, x', y, z\}$ and consider the CS1T-algebra $P_X(+, f)$ and the free left permutable groupoid F_X (see Section 2). Further, define two binary relations R and S on P_X as follows: $(a, b) \in R$ iff $a = (y . yx) * s$ and $b = (z . zx) * s$ for some lifting sequence s; $(a, b) \in S$ iff there exists a finite sequence $a_1, \ldots, a_k$ such that $a = a_1$, $b = a_k$ and $(a_i, a_{i+1}) \in R \cup R^{-1}$ for all $1 \leqq i \leqq k - 1$. (A sequence $a_1, \ldots, a_k$ with these properties will be called a *derivation from a to b*.) Clearly, S is just the congruence of $P_X(+, f)$ generated by the pair $(y . yx, z . zx)$. The relation S is also a congruence of the left permutable groupoid $P_X(.)$ and we denote by G the corresponding factor-groupoid.

5.1. Lemma. *The groupoid G is left cancellative.*

Proof. Denote by Q the set of quadruples $q = (a, b, c, d)$ of elements of P_X such that $(a, b) \in S$ and $(f(a) + c, f(b) + d) \in S$. We put $J(q) = (\lambda(a) + \lambda(b) + \lambda(c) + \lambda(d), k)$ where k is the least possible length of a derivation from $f(a) + c$ to $f(b) + d$. Proceeding by induction on $J(q)$ (with respect to the lexicographic ordering of ordered pairs) we are going to show that if $q = (a, b, c, d) \in Q$ then $(c, d) \in S$. Let $a_1, \ldots, a_k$ be a derivation from $f(a) + c$ to $f(b) + d$ of minimal length. If $a_1 = a_k$ then either $c = d$ and $(c, d) \in S$ or $d = f(a) + e$, $c = f(b) + e$ for some $e \in P_X$ and again $(c, d) \in S$. Thus we can assume that $a_1 \neq a_k$ and $k \geqq 2$. Furthermore, without loss of generality, we can assume that $(a_1, a_2) \in R$, so that a_2 is obtained from a_1 by replacing one occurrence of a part $p_1 = f(y) + f(y) + x$ of a_1 by $p_2 = f(z) + f(z) + x$. If the replaced part p_1 of a_1 is a part of c, denote by c' the element obtained from c by replacing p_1 by p_2 and put $q' = (a, b, c', d)$; then $q' \in Q$, $J(q') < J(q)$ in the lexicographic ordering, $(c', d) \in S$ by the induction hypothesis and so $(c, d) \in S$ (we have $(c, c') \in S$). If p_1 is a part of a, we similarly get $(c, d) \in S$. Consider the remaining case. Then evidently $a = y$ and $c = f(y) + x + c'$ for some $c' \in P_X$. Analogously, considering the pair (a_{k-1}, a_k), we see that we can assume that either $b = z$ and $d = f(z) + x + d'$ or $b = y$ and $d = f(y) + x + d'$ for some $d' \in P_X$. The first of these two cases is not possible, since otherwise $(y, z) \in S$, a contradiction. Hence $a = y = b$, $c = f(y) + x + c'$, $d = f(y) + x + d'$, $a_2 = f(z) + f(z) + x + c'$, $a_{k-1} = f(z) + f(z) + x + d'$. The quadruple $q'' = (z, z, f(z) + x + c', f(z) + x + d')$ belongs to Q and $J(q'') < J(q)$; we get $(f(z) + x + c', f(z) + x + d') \in S$ by the induction hypothesis. Now, the quadruple $q''' = (z, z, x + c', x + d')$ belongs to Q and $J(q''') < J(q)$; applying the induction hypothesis once more, we get $(x + c', x + d') \in S$. Hence evidently $(c, d) \in S$.

5.2. Lemma. *The groupoid G is right cancellative.*

Proof. Denote by Q the set of quadruples $q = (a, b, c, d)$ such that $(c, d) \in S$ and $(f(a) + c, f(b) + d) \in S$. Define $J(q)$ similarly as in 5.1. Again, by induction on $J(q)$, we are going to show that $(a, b) \in S$. Let $a_1, \ldots, a_k$ be a derivation from $f(a) + c$ to $f(b) + d$ of minimal length. If $a_1 = a_k$ then either $a = b$ and $(a, b) \in S$ or $c = f(b) + e$ and $d = f(a) + e$ for some e; in the latter case we can apply the induction hypothesis for $q' = (a, b, e, e) \in Q$, again receiving $(a, b) \in S$. Hence we can ssume that $a_1 \neq a_k$, $k \geqq 2$ and, proceeding similarly as in 5.1, we can restrict ourselves to the case $a = y$, $c = f(y) + x + c'$, $a_2 = f(z) + f(z) + x + c'$, $b = z$, $d = f(z) + x + d'$, $a_{k-1} = f(y) + f(y) + x + d'$. However, $(a_1, a_{k-1}) \in S$ and hence $(x + c', x + d') \in S$ by 5.1. The quadruple $q'' = (y, z, x + c', x + d')$ belongs to Q and $J(q'') < J(q)$; we get $(y, z) \in S$ by the induction hypothesis and so $(a, b) \in S$.

5.3. Lemma. $(y \,.\, yx', z \,.\, zx') \notin S$.

Proof. It is evident that if $u \in P_X$ and $(y \,.\, yx', u) \in R \cup R^{-1}$ then $u = y \,.\, yx'$.

5.4. Lemma. *If $H(+)$ is a commutative semigroup and h is a transformation of H such that $G \subseteq H$ and $ab = h(a) + b$ for all $a, b \in G$ then $H(+)$ is not cancellative.*

Proof. Denote by g the natural homomorphism of P_X onto G and suppose that $H(+)$ is cancellative. Then $h\, g(y) + h\, g(y) + g(x) = g(y)\,(g(y)\, g(x)) = g(y \,.\, yx) = g(z \,.\, zx) = g(z)\,(g(z)\, g(x)) = h\, g(z) + h\, g(z) + g(x)$ implies $h\, g(y) + h\, g(y) = h\, g(z) + h\, g(z)$ and so $g(y \,.\, ya) = g(z \,.\, za)$ for every $a \in P_X$. In particular, $(y \,.\, yx', z \,.\, zx') \in S$, a contradiction with 5.3.

5.5. Proposition. *G is a cancellative left permutable groupoid and it cannot be embedded into a left permutable quasigroup.*

Proof follows from 5.1, 5.2, 5.4 and 3.9.

6. BI-PERMUTABLE GROUPOIDS

For any groupoid terms $t, s_1, \ldots, s_n$ $(n \geqq 0)$ define two terms as follows:

$$t[s_1, \ldots, s_n] = s_n(s_{n-1}(\ldots (s_2(s_1 t)))),$$
$$t\langle s_1, \ldots, s_n\rangle = ((((t s_1)\, s_2) \ldots)\, s_{n-1})\, s_n .$$

6.1. Lemma. *The following identities are satisfied in all bi-permutable groupoids:*

(1) $xy \,.\, uv = uv \,.\, xy$,

(2) $x(y_1 y_2 \,.\, z) = (x \,.\, y_1 y_2)\, z$,

(3) $y_1[x_1, \ldots, x_n]\, \langle y_2, \ldots, y_m\rangle = y_{q(1)}[x_{p(1)}, \ldots, x_{p(n)}]\, \langle y_{q(2)}, \ldots, y_{q(m)}\rangle$ whenever $n, m \geqq 1$, *p is a permutation of* $\{1, \ldots, n\}$ *and q is a permutation of* $\{1, \ldots, m\}$,

(4) $(x \,.\, yz)\, \langle u_1, \ldots, u_n\rangle = x((yz)\, \langle u_1, \ldots, u_n\rangle)$ *for all* $n \geqq 0$,

(5) $x\langle y_1, \ldots, y_n\rangle \,.\, u\langle v_1, \ldots, v_m\rangle = (x \,.\, uv_1)\, \langle y_1, \ldots, y_n, v_2, \ldots, v_m\rangle$ *for all* $n \geqq 0$ *and* $m \geqq 1$,

(6) $x[y_1, \ldots, y_n] \,.\, u[v_1, \ldots, v_m] = (x[y_1, \ldots, y_n, \; v_1, \ldots, v_m])\, u$ *for all* $n \geqq 1$ *and* $m \geqq 0$,

(7) $y_1[x_1, \ldots, x_n] \langle y_2, \ldots, y_m\rangle \,.\, u_1[v_1, \ldots, v_i] \langle u_2, \ldots, u_j\rangle =$
$= y_1[x_1, \ldots, x_n, v_1, \ldots, v_i] \langle y_2, \ldots, y_m, u_1, \ldots, u_j\rangle$ *for all* $n, m, i, j \geqq 1$.

Proof. (1) $xy \,.\, uv = u(xy \,.\, v) = u(xv \,.\, y) = xv \,.\, uy = (x \,.\, uy)\, v = (u \,.\, xy)\, v = = uv \,.\, xy$.

(2) $x(y_1y_2 \,.\, z) = y_1y_2 \,.\, xz = xz \,.\, y_1y_2 = (x \,.\, y_1y_2)\, z$.

(3) With respect to 2.1 and the dual of 2.1, we can assume that p is the identical permutation of $\{1, \ldots, n\}$, $m = 2$ and $q(1) = 2$. Now, we shall proceed by induction on n. The case $n = 1$ is just the right permutability. If $n \geqq 2$ then

$$y_1[x_1, \ldots, x_n] \,.\, y_2 = x_n(y_1[x_1, \ldots, x_{n-1}] \,.\, y_2) = x_n(y_2[x_1, \ldots, x_{n-1}] \,.\, y_1) = = y_2[x_1, \ldots, x_n] \,.\, y_1,$$

as follows from (2) and the induction hypothesis.

(4) By induction on n. If $n \geqq 1$ then

$$(x \,.\, yz) \langle u_1, \ldots, u_n\rangle = ((x \,.\, yz) \langle u_1, \ldots, u_{n-1}\rangle)\, u_n = (x((yz) \langle u_1, \ldots, u_{n-1}\rangle))\, u_n = = x((yz) \langle u_1, \ldots, u_{n-1}\rangle \,.\, u_n) = x((yz) \langle u_1, \ldots, u_n\rangle)$$

by (2) and the induction hypothesis.

(5) By induction on n. For $n = 0$, use (4). For $n \geqq 1$,

$$x\langle y_1, \ldots, y_n\rangle \,.\, u\langle v_1, \ldots, v_m\rangle = ((x\langle y_1, \ldots, y_{n-1}\rangle)\, y_n \,.\, u\langle v_1, \ldots, v_m\rangle = = (x\langle y_1, \ldots, y_{n-1}\rangle \,.\, u\langle v_1, \ldots, v_m\rangle)\, y_n = ((x \,.\, uv_1) \langle y_1, \ldots, y_{n-1}, v_2, \ldots, v_m\rangle)\, y_n = = (x \,.\, uv_1) \langle y_1, \ldots, y_n, v_2, \ldots, v_m\rangle$$

by the right permutability, (3) and the induction hypothesis.

(6) By induction on m. For $m = 0$ there is nothing to prove. If $m \geqq 1$ then

$$x[y_1, \ldots, y_n] \,.\, u[v_1, \ldots, v_m] = x[y_1, \ldots, y_n]\, (v_m \,.\, u[v_1, \ldots, v_{m-1}]) = = v_m(x[y_1, \ldots, y_n] \,.\, u[v_1, \ldots, v_{m-1}]) = v_m(x[y_1, \ldots, y_n, v_1, \ldots, v_{m-1}]\, u) = = (v_m \,.\, x[y_1, \ldots, y_n, v_1, \ldots, v_{m-1}])\, u = (x[y_1, \ldots, y_n, v_1, \ldots, v_m])\, u$$

by the left permutability, induction hypothesis and (2).

(7) First, let $j \geqq 2$. Then, by (5), (2) and (3), (6) and (3), the left side equals

$$(y_1[x_1, \ldots, x_n] \,.\, u_1[v_1, \ldots, v_i]\, u_2) \langle y_2, \ldots, y_m, u_3, \ldots, u_j\rangle =$$
$$= (y_1[x_1, \ldots, x_n] \,.\, u_1[v_1, \ldots, v_i]) \langle y_2, \ldots, y_m, u_2, \ldots, u_j\rangle =$$
$$= y_1[x_1, \ldots, x_n, v_1, \ldots, v_i] \langle y_2, \ldots, y_m, u_1, \ldots, u_j\rangle.$$

Now, consider the case $j = 1$. Using the right permutability several times and then (6), we find that the left side equals

$$(y_1[x_1, \ldots, x_n] \,.\, u_1[v_1, \ldots, v_i]) \langle y_2, \ldots, y_m\rangle =$$
$$= y_1[x_1, \ldots, x_n, v_1, \ldots, v_i] \langle u_1, y_2, \ldots, y_m\rangle =$$
$$= y_1[x_1, \ldots, x_n, v_1, \ldots, v_i] \langle y_2, \ldots, y_m, u_1\rangle.$$

6.2. Theorem. *Let X be a non-empty set. Denote by $S(+)$ the free commutative semigroup over X. Put $R_X = X \cup (S \times S)$ and define a multiplication on R_X as follows:*

$$(a, b) \,.\, (c, d) = (a + c,\ b + d),$$
$$(a, b) \,.\, x = (a, b + x),$$
$$x \,.\, (a, b) = (x + a, b),$$
$$x \,.\, y = (x, y)$$

for all $a, b, c, d \in S$ and $x, y \in X$. Then R_X is a free bi-permutable groupoid over X. Moreover, we have

$$(x_1 + \ldots + x_n,\ y_1 + \ldots + y_m) = (((x_n(\ldots(x_2(x_1y_1)))) \, y_2 \ldots) \, y_m$$

for all $x_1, \ldots, x_n,\ y_1, \ldots, y_m \in X$ $(n, m \geqq 1)$.

Proof. The last equality is obvious; it follows that R_X is generated by X. Denote by $F(\circ)$ the free bi-permutable groupoid over X and by f the mapping of R_X into F defined by $f(x) = x$ and $f(x_1 + \ldots + x_n,\ y_1 + \ldots + y_m) = (((x_n \circ (\ldots \circ (x_2 \circ \circ (x_1 \circ y_1)))) \circ y_2) \circ \ldots) \circ y_m$ for all $x, x_1, \ldots, y_m \in X$; this definition is correct by 6.1(3). It suffices to show that f is a homomorphism. Let $x, y \in X$ and $a, b, c, d \in S$. Then $f(xy) = f(x, y) = x \circ y = f(x) \circ f(y)$, $f(x \,.\, (a, b)) = f(x + a, b) = f(x) \circ \circ f(a, b)$ by 6.1(4), also evidently $f((a, b) \,.\, x) = f(a, b) \circ f(x)$ and $f((a, b) \,.\, (c, d)) = = f(a, b) \circ f(c, d)$ by 6.1(7).

6.3. Corollary. *Every free bi-permutable groupoid is cancellative.*

6.4. Corollary. *No free bi-permutable groupoid can be embedded into a bi-permutable quasigroup.*

6.5. Corollary. *The variety of bi-permutable groupoids does not have the Schreier property.*

6.6. Proposition. *Every simple bi-permutable groupoid is a commutative semigroup.*

Proof. Let G be a simple bi-permutable groupoid and $I = GG$. Then I is an ideal of G, $(I \times I) \cup \mathrm{id}_G$ is a congruence of G and so either $I = G$ or I is a one-element set. In the first case G is a commutative semigroup by 6.1(1) and 6.1(2); in the second, G evidently is a commutative semigroup.

6.7. Corollary. *Every minimal variety of bi-permutable groupoids is contained in the variety of commutative semigroups.*

6.8. Proposition. *The variety of bi-permutable groupoids does not have the amalgamation property. The category of bi-permutable groupoids has non-surjective epimorphisms.*

Proof follows from 4.4 and 4.6.

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