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ON STRICTLY POSITIVE LATTICE ORDERED SEMIGROUPS

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In this note a question proposed by M. Anderson [1] concerning subdirect product decompositions of lattice ordered semigroups will be dealt with.

Let us recall the basic notions. By an l-semigroup we will mean a semigroup equipped with a lattice order such that the multiplication distributes over each of the lattice operations, from both the left and the right (this definition is stronger than that applied in [2]). Let S be an l-semigroup.

S is said to be *strictly positive* if $ab \wedge ba \geqq b$ for all elements a and b in S. All l-semigroups considered in this note are assumed to be strictly positive.

S is called *a-simple* if for any elements a and b of S there exist positive integers m and n for which

$$a \leqq b^m \quad \text{and} \quad b \leqq a^n .$$

S is said to be a *nil-l-semigroup* if it has a zero element 0, and some finite power of every other element equals 0.

We denote by $\mathscr{A}$ the class of all a-simple nil-l-semigroups. If $S \in \mathscr{A}$ and if the element 0 is finitely join irreducible (that is, if $a \vee b = 0$ implies that a or b is 0) then S is said to be a *step*.

Let R be a congruence relation on S. The corresponding factor l-semigroup is denoted by S/R. The symbols R_0 and R_1 always denote the least and the greatest congruence relation, respectively. The congruence relation R is nontrivial if $R_0 \neq R \neq R_1$. For $x \in S$, $x(R)$ is the set of all $y \in S$ with $x\, R\, y$.

If the semigroup operation of S is not taken into account, then the corresponding lattice will be denoted by $(S; \wedge, \vee)$. The l-semigroup S is said to be *distributive* if the lattice $(S; \wedge, \vee)$ is distributive.

If x and y are elements of a lattice such that x is less than y, then the relation $x \leqq y$ is called a *nontrivial comparability relation.* A lattice is said to be *discrete* if each of its bounded chains is finite.

The following theorem was proved in [1]:

(A) *Let $S \in \mathscr{A}$ and let S be distributive. Then S is a subdirect product of steps.*

Also, in [1] the following question is proposed:

Question. *Is Theorem* (A) *true for non-distributive l-semigroups?*

Let us denote by $\mathscr{C}$ the class of all l-semigroups $S \in \mathscr{A}$ such that S cannot be represented as a subdirect product of steps. The above question consists in asking whether the class $\mathscr{C}$ is empty.

The following existence results (B) and (C) show that the class $\mathscr{C}$ is rather large.

(B) *For each cardinal $\alpha \geqq 5$ there exists an l-semigroup S such that the lattice $(S, \wedge, \vee)$ is modular, $S \in \mathscr{C}$ and* card $S = \alpha$.

Proof. Let β be a cardinal such that $\alpha = \beta + 2$. Let $S = \{u, v, a_i\}_{i \in I}$, card $I = \beta$. The partial order $\leqq$ on S is defined as follows: $u < a_i < v$ for each $i \in I$, and a_i

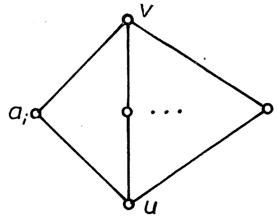

Fig. 1

is incomparable with a_j whenever i and j are distinct elements of S. (Cf. Fig. 1.) Hence $(S, \wedge, \vee)$ is a modular lattice. For any $x, y \in S$ put $xy = v$. Then S is an l-semigroup belonging to the class $\mathscr{C}$ and card $S = \alpha$.

S fails to be a step, since v is finitely join-reducible. By way of contradiction, suppose that S does not belong to $\mathscr{C}$. Hence S can be represented as a subdirect product of steps S_k $(k \in K)$. For each $k \in K$ there exists a congruence relation R_k on S such that S_k is isomorphic to S/R_k. Since S is not a step, we must have $R_k \neq R_0$ for each $k \in K$; without loss of generality we may assume that $R_k \neq R_1$ for each $k \in K$. Each R_k is also a congruence relation on the lattice $(S, \wedge, \vee)$. Because this lattice is discrete and any two of its prime intervals are projective, we have Con $((S; \wedge, \vee)) =$ $= \{R_0, R_1\}$; thus $R_k \in \{R_0, R_1\}$ for each $k \in K$, which is a contradiction.

(C) *For each infinite cardinal α there exists a system $\mathscr{S} = \{S_i\}_{i \in I}$ of l-semigroups S_i such that*

(i) card $I = \alpha$;
(ii) *if $i, j \in I$, then $(S_i, \wedge_i, \vee_i) = (S_j, \wedge_j, \vee_j)$ (i.e., the underlying sets S_i and S_j coincide, and the corresponsing partial orders are the same)*;
(iii) *S_i fails to be isomorphic to S_j whenever i and j are distinct elements of I*;
(iv) $\mathscr{S} \subseteq \mathscr{C}$.

Proof. Let $\alpha_i (i \in I)$ be distinct cardinals and for each $i \in I$ let $J(i)$ be a set of indices with card $J(i) = \alpha_i$. Let

$$S = \{u, u_1, v, a_i, b_{ij}\} \; (i \in I, \; (i, j) \in I \times J(i)) \,.$$

We define a partial order $\leqq$ on S by putting

$$u < u_1 < a_i < b_{ij} < v$$

for each $i \in I$ and each $(i, j) \in I \times J(i)$; no other nontrivial comparability relation is assumed to be valid in S. Then $(S; \leqq)$ is a lattice (cf. Fig. 2).

For each $i \in I$ we now define a binary operation $\circ_i$ on S as follows: let $x, y \in S$; we put $x \circ_i y = a_i$ if $x = y = u$, and $x \circ_i y = v$ otherwise. Put $S_i = (S; \wedge, \vee, \circ_i)$. It is easy to verify that $S_i \in \mathscr{A}$. Hence the conditions (i) and (ii) are valid.

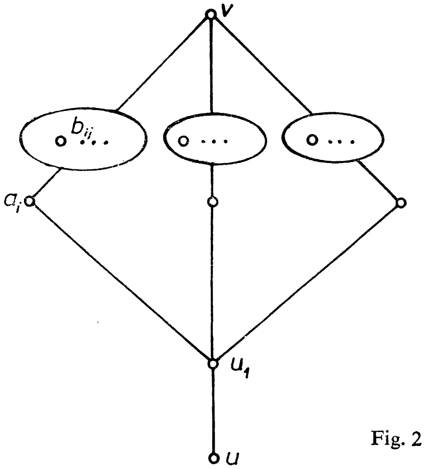

Fig. 2

Suppose that i and j are distinct elements of I and that there exists an isomorphism φ of S_i onto S_j. Then φ is an automorphism of the lattice $(S; \wedge, \vee)$. Thus u, u_1, a_i and a_j are fixed points of φ. Hence we have

$$a_i = \varphi(a_i) = \varphi(u \circ_i u) = \varphi(u) \circ_j \varphi(u) = u \circ_j u = a_j\,,$$

which is a contradiction. Therefore the condition (iii) is fulfilled.

For proving (iv), we proceed by way of contradiction. Suppose that there exists $i \in I$ such that S_i does not belong to $\mathscr{C}$. Hence S_i can be represented as a subdirect product of steps T_m $(m \in M)$. Since S_i fails to be a step, there are non-trivial congruence relations R_m $(m \in M)$ on S_i such that S_i/R_m is isomorphic to T_m for each $m \in M$, and $\bigwedge_{m \in M} R_m = R_0$.

Let $m \in M$ be fixed. Let $j(1), j(2)$ be distinct elements of I.

We have

$$a_{j(1)}(R_m) \vee a_{j(2)}(R_m) = v(R_m)\,.$$

Because S_i/R_m is a step, $v(R_m)$ must be finitely $\vee$-irreducible, hence we have either $a_{j(1)}(R_m) = v(R_m)$ or $a_{j(2)}(R_m) = v(R_m)$. In order to fix the notation, let us suppose that the first of these two possibilities occurs. Let $j(3) \in I$, $j(1) \neq j(3) \neq j(2)$. From

$a_{j(1)} R_m v$ we obtain

$$u_1 = a_{j(1)} \wedge a_{j(2)} R_m v \wedge a_{j(2)} = a_{j(2)},$$

$$u_1 = a_{j(1)} \wedge a_{j(3)} R_m v \wedge a_{j(3)} = a_{j(3)},$$

whence

$$u_1 = u_1 \vee u_1 R_m a_{j(2)} \vee a_{j(3)} = v.$$

Therefore $u_1 R_m v$ for each m. Thus $u_1 R_0 v$, which is a contradiction, because $u_1 \neq v$.

References

[1] *M. Anderson:* Archimedean equivalence for strictly positive lattice ordered semigroups. Czech. Math. J. (submitted).

[2] *L. Fuchs:* Teilweise geordnete algebraische Strukturen, Akadémiai Kiadó, Budapest, 1966.

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