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Ján Jakubík

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RADICAL SUBGROUPS OF LATTICE ORDERED GROUPS

JÁN JAKUBÍK, Košice

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The lattice $c(G)$ of all convex l-subgroups of a lattice ordered group G was studied in [3]. A lattice ordered group $H \in c(G)$ will be said to be a *radical sugroup of* G (shortly: r-subgroup of G) if, whenever $G_1 \in c(G)$ and $H_1 \in c(H)$ such that G_1 is isomorphic to H_1, then $G_1 \subseteq H$. The system $R(G)$ of all r-subgroups of G is partially ordered by inclusion.

Radical classes of lattice ordered groups were investigated in [2], [6], [7], [8] and [9]. The collection of all radical classes of lattice ordered groups will be denoted by $\mathscr{R}$; this collection is partially ordered by inclusion. Let $\mathscr{G}$ be the class of all lattice ordered groups. For $G \in \mathscr{G}$ and $A \in \mathscr{R}$ we denote by $A(G)$ the largest convex l-subgroup of G belonging to A.

It turns out that the partially ordered set $R(G)$ is a closed sublattice of the lattice $c(G)$ and that for each $H \in c(G)$ the following conditions are equivalent: (i) H is an r-subgroup of G; (ii) there exists $A \in \mathscr{R}$ such that $H = A(G)$.

If G has no nontrivial r-subgroup (i.e., if card $R(G) \leqq 2$), then G is said to be *r-homogeneous*. G will be said to be *totally r-inhomogeneous* if, whenever $\{0\} \neq H \in R(G)$, then there exists $H_1 \in R(G)$ such that $\{0\} \subset H_1 \subset H$ (i.e., the lattice $R(G)$ has no atom).

The main results of this paper concern the lattice $R(G)$ for the case when G is a complete lattice ordered group. Let us mention the following existence results:

For each cardinal $\alpha > 0$ there is a proper class A_α of mutually nonisomorphic complete lattice ordered groups such that for each $G \in A_\alpha$, $R(G)$ is isomorphic to the Boolean algebra 2^α. (Hence, in particular, there exists a proper class of mutually nonisomorphic r-homogeneous complete lattice ordered groups.) For each ordinal δ there is a complete lattice ordered group G such that $R(G)$ is a chain isomorphic to δ. For each complete lattice ordered group G there exists a complete lattice ordered group G_1 such that $G \in R(G_1)$ and G is covered by G_1 in the lattice $R(G_1)$. There exists a proper class of mutually nonisomorphic totally r-inhomogeneous lattice ordered groups. The question whether there exists a complete totally r-inhomogeneous lattice ordered group $G \neq \{0\}$ remains open. Some results on the lattice $\mathscr{R}_c$ of all radical classes of complete lattice ordered groups will be also established; e.g., it will be shown that $\mathscr{R}_c$ is a Stone lattice.

1. PRELIMINARIES

The standard notations for lattice ordered groups will be applied (cf. [1] and [4]). The group operation will be written additively.

When considering a subclass Y of $\mathcal{G}$ we always assume that Y is closed with respect to isomorphisms and that the zero group $\{0\}$ belongs to Y.

Let $G \in \mathcal{G}$. The system $c(G)$ is a complete lattice (the operation $\wedge$ in $c(G)$ coincides with the set theoretical intersection; the join in $c(G)$ will be denoted by $\vee^c$).

A subclass X of $\mathcal{G}$ is said to be a *radical class* if it is closed with respect to

a) convex l-subgroups, and

b) joins of convex l-subgroups.

Hence $\mathcal{G}$ is the largest element in $\mathcal{R}$. The class containing one-element lattice ordered groups only is the least in $\mathcal{R}$; this class will be denoted by 0^-.

For $X \subseteq \mathcal{G}$ we denote by

Sub X – the class of all convex l-subgroups of lattice ordered groups belonging to X;

$\mathrm{Join}_c\, X$ – the class of all lattice ordered groups G having a system $\{G_i\}_{i \in I} \subseteq c(G)$ with $G_i \in X$ for each $i \in I$ such that $\bigvee_{i \in I}^c G_i = G$.

The following three propositions were proved in [6].

1.1. Proposition. *$\mathcal{R}$ is a complete lattice in which the meet coincides with the intersection of classes. Let I be a nonempty class and for each $i \in I$ let $X_i \in \mathcal{R}$. Then $\bigvee_{i \in I} X_i = \mathrm{Join}_c (\bigcup_{i \in I} X_i)$.*

For $X \subseteq \mathcal{G}$ we denote by $T(X)$ the intersection of all $Y \in \mathcal{R}$ with $X \subseteq Y$. In view of 1.1, $T(X)$ belongs to $\mathcal{R}$; it is said to be the *radical class generated by X*.

1.2. Proposition. *Let $X \subseteq \mathcal{G}$. Then $T(X) = \mathrm{Join}_c\, \mathrm{Sub}\, X$.*

1.3. Proposition. *The lattice $\mathcal{R}$ satisfies the infinite distributive law*

$$X \wedge (\bigvee_{i \in I} Y_i) = \bigvee_{i \in I} (X \wedge Y_i)\,. \tag{1}$$

If $X_1, X_2 \in \mathcal{R}$ and $X_1 \leqq X_2$, then $[X_1, X_2]$ denotes the collection of all $Y \in \mathcal{R}$ with $X_1 \leqq Y \leqq Y_2$.

2. BASIC PROPERTIES OF THE LATTICE $R(G)$

Let $G \in \mathcal{G}$ and $A \in \mathcal{R}$. Let $\{H_i\}_{i \in I}$ be the set of all elements of $c(G)$ which belong to A. According to the definition of the notion of a radical class (cf. the condition b) in Section 1) the lattice ordered group $A(G) = \bigvee_{i \in I}^c H_i$ belongs to A. We obviously have $A(G) \in R(G)$.

If $G_1 \in \mathcal{G}$ and if X is the class of all lattice ordered groups G_2 such that either G_2 is a zero group or G_2 is isomorphic to G_1, then we denote $T(X) = T(G_1)$. The radical class $T(G_1)$ is said to be *principal* (and *generated by G_1*).

Now let $H \in R(G)$. Put $A = T(H)$. Clearly $H \in A$, hence $H \subseteq A(G)$. Because

$A(G) \in A$, in view of 1.2 there are elements H_i $(i \in I)$ of $c(H)$ such that $A(G) = \bigvee^c_{i\in I} H_i$. Thus $A(G) \subseteq H$ and therefore $A(G) = H$. We obtain:

2.1. Proposition. *Let $G \in \mathcal{G}$ and $H \in c(G)$. Then the following conditions are equivalent:*

(i) *H belongs to $R(G)$.*

(ii) *There exists $A \in \mathcal{R}$ such that $H = A(G)$.*

2.2. Proposition. *Let $G \in \mathcal{G}$. Then $R(G)$ is a closed sublattice of $c(G)$.*

Proof. Let $I \neq \emptyset$ be a set and for each $i \in I$ let $H_i \in R(G)$. In view of the definition of $R(G)$ we have $\bigcap_{i\in I} H_i \in R(G)$.

Put $\bigvee^c_{i\in I} H_i = H$. We have to verify that H belongs to $R(G)$. Let $H_1 \in c(H)$, $G_1 \in c(G)$ and suppose that φ is an isomorphism of H_1 onto G_1. It is well-known (cf., e.g., [3]) that for any $G_0 \in c(G)$ and $\{G_j\}_{j\in J} \subseteq c(G)$ the following infinite distributive law is valid:

(1a) $$G_0 \wedge (\bigvee^c_{j\in J} G_j) = \bigvee^c_{j\in J} (G_0 \wedge G_j) .$$

Hence

$$H_1 = H_1 \wedge H = H_1 \wedge (\bigvee^c_{i\in I} H_i) = \bigvee^c_{i\in I} (H_1 \wedge H_i) .$$

Put $G_i = \varphi(H_1 \wedge H_i)$. From $H_i \in R(G)$ we infer that $G_i \subseteq H_i$; moreover, $G_1 = \bigvee^c_{i\in I} G_i$. Thus $G_1 \subseteq H$ and therefore $H \in R(G)$.

From 2.2 and 1.3 we obtain:

2.2.1. Corollary. *Let $G \in \mathcal{G}$. The lattice $R(G)$ satisfies the infinite distributive law* (1).

From 2.1 and [6], Corollary 2 of Proposition 4.2 we infer:

2.3. Proposition. *Let $G \in \mathcal{G}$. Then $R(G)$ is isomorphic to the interval $[0^-, T(G)]$ of the lattice $\mathcal{R}$.*

Let us remark that Corollary 2.2.1 can be obtained also as a consequence of 2.3 and 1.3.

The following example shows that the lattice $R(G)$ need not satisfy the infinite distributive law dual to (1a).

2.4. Example. Let R_0 be the additive group of all reals with the natural linear order. Let P be the set of all positive primes and for each $p \in P$ let G_p be the l-subgroup of R consisting of all elements of R_0 which can be written in the form mp^{-n}, where m and n are integers, $n > 0$. Let G be the (complete) direct product

$$G = \prod_{p\in P} G_p .$$

We denote by H the discrete direct product (= direct sum) of the system $\{G_p\}_{p\in P}$. For each $p \in P$ let $I(p) = \{q \in P: q > p\}$ and

$$H_p = \prod_{i\in I(p)} G_i .$$

Then $H \in R(G)$ and $H_p \in R(G)$. We have

$$\bigwedge_{p\in P} H_p = \{0\}$$

and

$$H \vee^c H_p = G \quad \text{for each} \quad p \in P .$$

Therefore

$$H \vee^c (\bigwedge_{p\in P} H_p) = H ,$$

$$\bigwedge_{p\in P} (H \vee^c H_p) = G .$$

Since $G \neq H$, the infinite distributive law dual to (1a) does not hold in the lattice $R(G)$.

Let $G \in \mathscr{G}$ and $M \subseteq G$. The set

$$M^{\perp} = \{g \in G: |g| \wedge |m| = 0 \text{ for each } m \in M\}$$

is a *polar* of G; $M^{\perp}$ and $M^{\perp\perp}$ are *complementary polars* of G.

Let $X \subseteq \mathscr{G}$. We denote by X^{δ} the class of all lattice ordered groups G such that, whenever $H \in c(G) \cap X$, then $H = \{0\}$.

From 1.2 we infer that for each $Y \in \mathscr{R}$ the relation

$$T(X) \wedge Y = 0^- \Leftrightarrow Y \leqq X^{\delta}$$

is valid. Hence $X^{\delta\delta\delta} = X^{\delta}$ for each $X \subseteq \mathscr{G}$.

2.5. Lemma. (Cf. [6], Lemma 2.1.) *Let $X \subseteq \mathscr{G}$. Then $X^{\delta} \in \mathscr{R}$.*

For each $g \in G$ we denote by $[g]$ the convex l-subgroup of G generated by g. If $g > 0$, then g is a strong unit in $[g]$; in particular, for each $0 < g_1 \in [g]$ we have $0 < g_1 \wedge g$.

2.6. Lemma. *Let $X \subseteq \mathscr{G}$ and $G \in \mathscr{G}$. Then $X^{\delta}(G)$ and $X^{\delta\delta}(G)$ are complementary polars of G.*

Proof. We obviously have $X^{\delta} \wedge X^{\delta\delta} = 0^-$, whence

$$X^{\delta}(G) \wedge X^{\delta\delta}(G) = (X^{\delta} \wedge X^{\delta\delta})(G) = 0^-(G) = 0^- .$$

Thus $X^{\delta\delta}(G) \subseteq (X^{\delta}(G))^{\perp}$ and $X^{\delta}(G) \subseteq (X^{\delta\delta}(G))^{\perp}$.

We shall show that

$$(2) \qquad (X^{\delta}(G))^{\perp} \subseteq X^{\delta\delta}(G)$$

is valid. Let $0 < y \in (X^{\delta}(G))^{\perp}$. For proving that y belongs to $X^{\delta\delta}(G)$ it suffices to verify that $[y]$ belongs to the class $X^{\delta\delta}$. By way of contradiction, assume that $[y]$ does not belong to $X^{\delta\delta}$. Hence there exists $H \in c([y]) \cap X^{\delta}$ such that $H \neq \{0\}$. Choose $0 < y_1 \in H$. Then $y_1 \in [y]$, hence $y_1 \wedge y > 0$. On the other hand, we have $H \in X^{\delta}$, whence $H \subseteq X^{\delta}(G)$, thus $y_1 \in X^{\delta}(G)$ and therefore $y_1 \wedge y = 0$, which is a contradiction. Thus (2) is valid and hence

$$(3) \qquad (X^{\delta}(G))^{\perp} = X^{\delta\delta}(G)$$

holds. By putting X^{δ} instead of X in (3) we obtain

$$(X^{\delta\delta}(G))^{\perp} = X^{\delta\delta\delta}(G) = X^{\delta}(G) ,$$

completing the proof.

3.1. Lemma. (Cf. [6], Corollary 2 to Proposition 4.1.) *Let $G \in \mathcal{G}$ and $Y \in \mathcal{R}$. Assume that $Y \leqq T(G)$. Then $Y = T(G_1)$, where $G_1 = Y(G)$.*

3.2. Lemma. *Let $G \in \mathcal{G}$. For each $G_1 \in R(G)$ we put $\varphi(G_1) = T(G_1)$. Then φ is an isomorphism of the lattice $R(G)$ onto the interval $[0^-, T(G)]$ of $\mathcal{R}$.*

Proof. In view of 3.1 and 2.1, the mapping φ is an epimorphism. If $G_1, G_1' \in R(G)$ such that $\varphi(G_1) = \varphi(G_1')$, then $G_1' \in T(G_1)$, whence (in view of 2.1) $G_1' \subseteq G_1$; similarly we have $G_1 \subseteq G_1'$. Thus φ is a monomorphism.

Let $G_1, G_2 \in R(G)$ be such that $G_1 \subseteq G_2$. According to 1.2 we have $\varphi(G_1) \leqq \varphi(G_2)$. Now let $Y_1, Y_2 \in [0^-, T(G)]$ be such that $Y_1 \leqq Y_2$. Put $G_1 = Y_1(G)$, $G_2 = Y_2(G)$. Hence $G_1 = \varphi^{-1}(Y_1)$ and $G_2 = \varphi^{-1}(Y_2)$. Because of $G_1 \in Y_2$ we infer that $G_1 \subseteq G_2$ (by applying 1.2 again). Thus φ is an isomorphism.

3.3. Corollary. *A lattice ordered group $G \neq \{0\}$ is r-homogeneous if and only if $T(G)$ is an atom of the lattice $\mathcal{R}$.*

If $M \subseteq \mathcal{R}$ ($\mathcal{G}_1 \subseteq \mathcal{G}$) and if there exists an injective mapping of the class of all cardinals into M (or $\mathcal{G}_1$, respectively), then M is said to be a *proper collection of radical classes* (*a proper class of lattice ordered groups*).

3.4. Proposition. *There exists a proper collection $\mathcal{A} \subseteq \mathcal{R}$ such that* (i) *for each $X \in \mathcal{A}$ there is a linearly ordered group G xuch that $X = T(G)$*; (ii) *each $X \in \mathcal{A}$ is an atom in $\mathcal{R}$.*

From 3.3 and 3.4 we infer:

3.5. Theorem. *There exists a proper class $\mathcal{G}_1$ of linearly ordered groups such that*

(i) *if G_1 and G_2 are distinct elements of $\mathcal{G}_1$, then G_1 is not isomorphic to G_2;*

(ii) *if $G \in \mathcal{G}_1$, then G is r-homogeneous.*

The class of all nonisomorphic types of complete linearly ordered groups fails to be a proper class, hence Theorem 3.5 cannot be sharpened by assuming that all linearly ordered groups of the class $\mathcal{G}_1$ are complete. Thus if we search for a large collection of nonisomorphic complete lattice ordered groups, then we must cancel the assumption of linear ordering.

Let B be a Boolean algebra. Let us recall the notion of Carathéodory functions corresponding to B (cf. [5], or [10], p. 97).

Let $E(B)$ be the system consisting of all forms

$$f = a_1 b_1 + \ldots + a_n b_n \tag{4}$$

(where $a_i \neq 0$ are reals and $b_i \in B$, $b_i > 0$, $b_{i_1} \wedge b_{i_2} = 0$ for any $i_1, i_2 \in \{1, 2, \ldots, n\}$, $i_1 \neq i_2$) and of the empty form; if g is another such form,

$$g = a_1' b_1' + \ldots + a_m' b_m' ,$$

then f and g are considered equal if $\bigvee_{i=1}^{n} b_i = \bigvee_{j=1}^{m} b'_j$ and $a_i = a'_j$ whenever $b_i \wedge b'_j \neq 0$. For any $b, b' \in B$ let $b - b'$ be the relative complement of $b \wedge b'$ in the interval $[0, b]$. The operation $+$ in $E(B)$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a'_j)(b_i \wedge b'_j) + \sum_{i=1}^{n} a_i(b_i - \bigvee_{j=1}^{m} b'_j) + \\ + \sum_{j=1}^{m} a'_j(b'_j - \bigvee_{i=1}^{n} b_i),$$

where in the summations only those terms are taken into account in which $a_j + a'_j \neq 0$ and the elements $b_i \wedge b'_j$, $b_i - \bigvee_{j=1}^{m} b'_j$ or $b'_j - \bigvee_{i=1}^{n} b_i$ are non-zero. The multiplication by a real $a \neq 0$ is defined by $af = (aa_1) b_1 + \ldots + (aa_n) b_n$; $0f$ is the empty form. The form (4) is positive if $a_i > 0$ for $i = 1, 2, \ldots, n$. Then $E(B)$ is a vector lattice; in particular, $E(B)$ is a lattice ordered group. Elements of $E(B)$ are said to be the *elementary Carathéodory functions.*

Let us denote by $G_c(B)$ the subset of $E(B)$ consisting of the empty form and of all forms (4) such that a_i are integers ($i = 1, 2, \ldots, n$). Then $G_c(B)$ is an l-subgroup of the l-group $E(B)$. The empty form is the zero element of $G_c(B)$. If $0 \neq b \in B$, then the form $1b$ will be identified with b.

It is easy to verify that if B is a complete Boolean algebra, then $G_c(B)$ is a complete lattice ordered group.

From the definition of $G_c(B)$ we immediately obtain:

3.6. Lemma. *Let $0 < b \in B$. Then $[b] = G_c([0, b])$.*

A Boolean algebra B is said to be *homogeneous* if for each $0 < b \in B$, the Boolean algebra $[0, b]$ is isomorphic to B.

The following proposition is a consequence of [11] (Corollaries 3.12 and 3.14).

3.7. Proposition. *For each cardinal α there exists a homogeneous Boolean algebra B such that* (i) *B is complete, and* (ii) card $B \geqq \alpha$.

3.8. Lemma. *Let B be a homogeneous Boolean algebra. Then the lattice ordered group $G_c(B)$ is r-homogeneous.*

Proof. Let $G_1 \in R(G_c(B))$, $G_1 \neq \{0\}$. Choose $0 < g_1 \in G_1$. There exists $0 < b \in B$ such that $b \leqq g_1$, hence $[b] \subseteq G_1$. Let $0 < g \in G_c(B)$. There are nonzero elements $b_1, b_2, \ldots, b_n$ in B and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $g = \alpha_1 b_1 + \alpha_2 b_2 + \ldots \ldots + \alpha_n b_n$. In view of 3.6, each lattice ordered group $[b_i]$ ($i = 1, 2, \ldots, n$) is isomorphic to $[b]$, hence $[b_i] \subseteq G_1$. Thus $g \in G_1$. We infer that $G_1 = G_c(B)$; hence $G_c(B)$ is r-homogeneous.

From 3.7 and 3.8 we obtain:

3.9. Theorem. *There exists a proper class $\mathscr{G}_2$ of nonzero complete lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathscr{G}_2$, $0 < g_1 \in G_1$, $0 < g_2 \in G_2$, then $[g_1]$ is not isomorphic to $[g_2]$;* (ii) *if $G \in \mathscr{G}_2$, then G is r-homogeneous.*

In this section we will construct complete lattice ordered groups whose lattice of radical subgroups is isomorphic to the Boolean algebra 2^α, where α is a given cardinal. Further it will be shown that the lattice $R(G)$ corresponding to a nonzero lattice ordered group G is an atomic Boolean algebra if and only if G is a direct sum of r-homogeneous lattice ordered groups belonging to $R(G)$.

4.1. Lemma. *Let $K \neq \{0\}$ be an r-homogeneous lattice ordered group, $G \in \mathcal{G}$, $K \in R(G)$. Then K is an atom in $R(G)$.*

This is an immediate consequence of the definition of r-homogeneity.

The direct sum G of lattice ordered groups G_i $(i \in I)$ is denoted by $\sum_{i \in I} G_i$. For $g \in G$ we denote by $g(i)$ the i-th component of g; we put $I(g) = \{i \in I \colon g(i) \neq 0\}$. For $H \subseteq G$ we set $I(H) = \bigcup_{h \in H} I(h)$.

4.2. Lemma. *Let $\{0\} \neq G_i$ $(i \in I \neq \emptyset)$ be r-homogeneous lattice ordered groups and let $G = \sum_{i \in I} G_i$. Assume that $G_i \in R(G)$ for each $i \in I$. Let $H \in R(G)$, $H \neq \{0\}$. Then $H = \sum_{i \in I(H)} G_i$.*

Proof. If $i \in I(H)$, then $H \cap G_i \neq \{0\}$, hence in view of 4.1 we have $H \supseteq G_i$. If $i \in I \setminus I(H)$, then $H \cap G_i = \{0\}$. Therefore $H = \sum_{i \in I(H)} G_i$.

4.3. Lemma. *Let G_i $(i \in I)$ be as in 4.2. Let $\emptyset \neq I_1 \subseteq I$, $H_1 = \sum_{i \in I_1} G_i$. Then $H_1 \in R(G)$.*

Proof. Let $\{0\} \neq K \in c(H_1)$ and $K' \in c(G)$. Assume that φ is an isomorphism of K onto K'. Clearly $G_i \in R(H_1)$ for each $i \in I_1$. Moreover, in view of 4.2 we have

$$K = \sum_{i \in I(K)} G_i\,, \quad K' = \sum_{i \in I(K')} G_i\,.$$

Let $j \in I(K')$. Then $\{0\} \neq \varphi^{-1}(G_j) \in c(K) \subseteq c(H_1)$. Because $G_j \in R(G)$ we have $\varphi^{-1}(G_j) \subseteq G_j$, thus $G_j \cap H_1 \neq \{0\}$; therefore $j \in I_1$. Hence $K' \subseteq H_1$ and so $H_1 \in R(G)$.

From 4.2 and 4.3 we obtain:

4.4. Lemma. *Let G and G_i $(i \in I)$ be as in 4.2. Then the lattice $R(G)$ is isomorphic to the Boolean algebra 2^α, where $\alpha = \operatorname{card} I$.*

4.5. Lemma. *Let G_i $(i \in I)$ be nonzero r-homogeneous lattice ordered groups and let $G = \sum_{i \in I} G_i$. Then the following conditions are equivalent:* (i) *all G_i belong to $R(G)$;* (ii) *if $i_1, i_2 \in I$, $i_1 \neq i_2$, $0 < g_1 \in G_{i_1}$, $0 < g_2 \in G_{i_2}$, then $[g_1]$ is not isomorphic to $[g_2]$.*

Proof. Let (i) be valid. Let $i_1, i_2 \in I$, $i_1 \neq i_2$, $0 < g_1 \in G_{i_1}$, $0 < g_2 \in G_{i_2}$. Assume that $[g_1]$ is isomorphic to $[g_2]$. Because $G_{i_1} \in R(G)$ we infer that $[0, g_2] \subseteq G_{i_1}$, hence $G_{i_1} \cap G_{i_2} \neq \{0\}$, which is a contradiction; thus (ii) holds. Conversely, assume that (ii) is fulfilled. Let $i \in I$. By way of contradiction, assume that G_i does not belong to $R(G)$. Hence there are $H_1 \in c(G_i)$ and $H \in c(G)$ such that H_1 is isomorphic to H but H is not a subset of G_i. Hence there is $0 < h \in H \setminus G_i$. If $h(G_j) = 0$ for each

$j \in I \setminus \{i\}$, then we should have $h \in G_i$; thus there is $j \in I \setminus \{i\}$ such that $h(G_j) > 0$. Because H_1 and H are isomorphic there is $0 < h_i \in G_i$ such that $[h(G_j)]$ is isomorphic to $[h_i]$, which contradicts (ii). Thus (i) must be valid.

From 4.4, 4.5 and 3.9 we infer:

4.6. Theorem. *Let α be a cardinal. There exists a proper class $\mathcal{G}_\alpha$ of complete lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathcal{G}_\alpha$, then G_1 is not isomorphic to G_2;* (ii) *if $G \in \mathcal{G}_\alpha$, then the lattice $R(G)$ is isomorphic to 2^α.*

4.7. Lemma. *Let $G \neq \{0\}$ be a lattice ordered group such that $R(G)$ is an atomic Boolean algebra. Let $\{G_i\}_{i\in I}$ be the set of all atoms of $R(G)$. Then all G_i are r-homogeneous and $G = \sum_{i\in I} G_i$.*

Proof. Since G_i is an atom in $R(G)$, it is r-homogeneous. If i, j are distinct elements in I, then $G_i \cap G_j = \{0\}$; hence whenever $g_i \in G_i$ and $g_j \in G_j$, then $g_i + g_j = g_j + g_i$. Because $R(G)$ is atomic, we have $G = \bigvee_{i\in I} G_i$. Therefore for each nonzero element $g \in G$ there are distinct indices $i_1, i_2, \ldots, i_n \in I$ and elements $g_1 \in G_{i_1}, \ldots, g_n \in G_{i_n}$ such that $g = g_1 + \ldots + g_n$. Hence $G = \sum_{i\in I} G_i$.

From 4.2 and 4.7 we obtain:

4.8. Proposition. *Let G be a nonzero lattice ordered group. The following conditions are equivalent:* (i) *$R(G)$ is an atomic Boolean algebra.* (ii) *G is a direct sum of r-homogeneous lattice ordered groups belonging to $R(G)$.*

5. AN EXAMPLE

The direct product of lattice ordered groups G_i $(i \in I)$ will be denoted by $\prod_{i\in I} G_i$. Let α be an infinite cardinal. By the α-direct product of the given system $\{G_i\}_{i\in I}$ we shall mean the l-subgroup of $\prod_{i\in I} G_i = G^0$ consisting of all elements $g \in G^0$ such that card $\{i \in I: g(i) \neq 0\} < \alpha$.

By means of α-products we shall construct complete lattice ordered groups whose lattice of radical subgroups is a well-ordered chain having a given cardinality β.

Let $G \in \mathcal{G}$. An element of G will be said to be an *s-element of* G (Sptize in the terminology of [12]) if $g > 0$ and the interval $[0, g]$ is a chain. A system $\{g_j\}_{j\in J}$ of elements of G is said to be *disjoint* if $g_j > 0$ for each $j \in J$ and $g_{j_1} \wedge g_{j_2} = 0$ whenever j_1 and j_2 are distinct elements of J.

Let G_0 be the additive group of all integers with the natural linear order. Let I be an infinite set of indices, card $I = \gamma$, and for each $i \in I$ let G_i be a lattice ordered group isomorphic to G_0. Put $G^0 = \prod_{i\in I} G_i$. Let G be the l-subgroup of G^0 consisting of all bounded elements of G^0 (i.e., an element g of G^0 belongs to G iff there is a positive integer n such that $g(i) \leqq n$ for each $i \in I$). For any $g \in G$ let $I(g)$ be as in Section 4.

Let α be an infinite cardinal, $\alpha \leqq \gamma$. We denote by G^α the set of all $g \in G$ such that card $I(g) < \alpha$ (i.e., G^α is the set of all bounded elements of G^0 which belong to the

α-product of the system $\{G_i\}_{i\in I}$). Then $G^\alpha \in c(G)$. The following lemma is obvious (under the notations as above.).

5.1. Lemma. *Let* $0 < g \in G$. *Then the following conditions are equivalent*:

(i) *g belongs to G^α.*

(ii) *If $\{g_j\}_{j\in J}$ is a disjoint system of s-elements of the lattice ordered group $[g]$, then* card $J < \alpha$.

5.2. Lemma. *Let α be an infinite cardinal, $\alpha \leqq \gamma$. Then $G^\alpha \in R(G)$.*

Proof. Let $H_1 \in c(G^\alpha)$, $H \in c(G)$ and let φ be an isomorphism of H_1 onto H. Let $0 < h \in H$, $g = \varphi^{-1}(h)$. In view of 5.1, the condition (ii) from 5.1 is valid; thus the analogous condition holds for the element h. Therefore $h \in G^\alpha$. This implies that $H \subseteq G^\alpha$ and thus $G^\alpha \in R(G)$.

5.3. Lemma. *Let $G' \in R(G)$, $\{0\} \neq G' \neq G$. Then there is an infinite cardinal α with $\alpha \leqq \gamma$ such that $G' = G^\alpha$.*

Proof. There exists $0 < g \in G'$. Let H be the set of all $h \in G$ such that $I(h) \subseteq I(g)$. There is a positive integer n with $|h| \leqq ng$; hence $H \subseteq G'$. Let $I_2 \subseteq I$, card $I_2 =$ $=$ card $I(g)$. Next, let H' be the l-subgroup of G consisting of all $h' \in G$ with $I(h') \subseteq$ $\subseteq I_2$. Then $H' \in c(G)$ and H' is isomorphic to $H \in c(G')$. Thus $H' \subseteq G$. Hence $G^\beta \subseteq G'$, where $\beta =$ card $I(g)$.

If for each β with $\beta \leqq \gamma$ there exists $0 < g \in G'$ with card $I(g) = \beta$, then we should have $G' = G$, which is a contradiction. Hence there exists a least cardinal $\alpha \leqq \gamma$ with $g_1 \notin G'$ for some g_1 such that card $I(g_1) = \alpha$. Then $G' = G^\alpha$. It is easy to verify that the cardinal α must be infinite.

Let us denote by C_γ the set of all infinite cardinals $\alpha \leqq \gamma$ (with the natural linear order). From 5.2 and 5.3 we obtain:

5.4. Lemma. *Let S be the set of all radical subgroups of G which are distinct from $\{0\}$ and G; S is partially ordered by inclusion. Then S is isomorphic to C_γ.*

Since the infinite cardinal γ considered above was chosen arbitrarily, from 5.4 we infer:

5.5. Theorem. *Let δ be an ordinal. There exists a complete lattice ordered group G such that the lattice $R(G)$ is a chain isomorphic to δ.*

Also, if we consider γ as running over the class of all infinite cardinals, then we obtain:

5.6. Theorem. *There exists a proper class $\mathcal{G}_4$ of complete lattice ordered groups such that the following conditions are valid*: (i) *If G_1 and G_2 are distinct elements of $\mathcal{G}_4$, then G_1 is not isomorphic to G_2; moreover, either G_1 is isomorphic to some radical subgroup of G_2, or G_2 is isomorphic to some radical subgroup of G_1.* (ii) *For each $G \in \mathcal{G}_4$, $R(G)$ is a well-ordered chain.*

The following question remains open: to what extent do the results of this section remain valid if G is an arbitrary nonzero r-homogeneous complete lattice ordered group?

6. THE COVERING RELATION

Let $G \in \mathcal{G}$. If H is a dual atom of the lattice $R(G)$, then H will be said to be *covered by* G. If G is a nonzero lattice ordered group, then the following questions can be proposed:

(Q_1) Does there exist a lattice ordered group H_1 such that H_1 is covered by G?

(Q_2) Does there exist a lattice ordered group H_2 such that G is covered by H_2?

Both (Q_1) and (Q_2) can be modified in such a way that G, H_1 and H_2 are assumed to be complete.

From 5.5 we obtain as a corollary:

6.1. Proposition. *There exists a proper class $\mathcal{G}_5$ of complete lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathcal{G}_5$, then G_1 is not isomorphic to G_2*; (ii) *if $G \in \mathcal{G}_5$, then no lattice ordered group is covered by G.*

6.2. Lemma. *Let $G \in \mathcal{G}$. There exists a proper class $\mathcal{G}_6(G)$ of nonzero complete r-homogeneous lattice ordered groups such that* (i) *if G_1 and G_2 are distinct elements of $\mathcal{G}_6(G)$ and $0 < g_1 \in G_1$, $0 < g_2 \in G_2$, then $[g_1]$ is not isomorphic to $[g_2]$, and* (ii) *if $G_1 \in \mathcal{G}_6(G)$ and $0 < g_1 \in G$, then no convex l-subgroup of G is isomorphic to $[g_1]$.*

This is an immediate consequence of 3.9.

6.3. Lemma. *Let G and $\mathcal{G}_6(G)$ be as in* 6.2. *Let $G_1 \in \mathcal{G}_6(G)$. Put $H = G \times G_1$. Then G is covered by H.*

Proof. From 6.2 we infer that both G and G_1 belong to $R(H)$ and that $G \cap G_1 = \{0\}$ is valid. Moreover, $G \vee G_1 = H$ holds. As G_1 is r-homogeneous, $\{0\}$ is covered by G_1. In view of the distributivity of $R(H)$, G is covered by H.

6.4. Lemma. *Let G and $\mathcal{G}_6(G)$ be as in* 6.2. *Let $G_1, G_2 \in \mathcal{G}_6(G)$, $G_1 \neq G_2$. Then $G \times G_1$ is not isomorphic to $G \times G_2$.*

Proof. By way of contradiction, assume that φ is an isomorphism of $G \times G_1$ onto $G \times G_2$. Then there are $P \in c(G)$ and $Q \in c(G_2)$ such that $\varphi(G_1) = P \times Q$. In view of 6.2 (i) we must have $Q = \{0\}$. Similarly, according to 6.2 (ii) the relation $P = \{0\}$ must be valid. Hence $G_1 = \{0\}$, which is a contradiction.

Let us remark that if G, G_1 and H are as in 6.3 and if G is complete, then H is complete as well. Thus 6.2, 6.3 and 6.4 yield:

6.5. Theorem. *Let $G \in \mathcal{G}$. There exists a proper class $\mathcal{G}_7(G)$ of lattice ordered groups such that* (i) *the elements of $\mathcal{G}_7(G)$ are mutually nonisomorphic*; (ii) *if*

$H \in \mathscr{G}_7(G)$, *then* G *is covered by* H; (iii) *if* G *is complete, then all elements of* $\mathscr{G}_7(G)$ *are complete.*

Next, we may ask whether there exists a lattice ordered group $G \neq \{0\}$ is covered by no element of $R(G)$; i.e., $R(G)$ has no atoms. Such a lattice ordered group G will called *totally r-inhomogeneous.*

From 2.3 and from the construction established in [6], Section 5 (cf. Proposition 5.4) we obtain:

6.6. Proposition. *There exists a proper class* $\mathscr{G}_8$ *of linearly ordered groups such that* (i) *the elements of* $\mathscr{G}_8$ *are mutually nonisomorphic*; (ii) *if* $G \in \mathscr{G}_8$, *then* G *is totally r-inhomogeneous.*

The question whether there exists a complete totally r-inhomogeneous lattice ordered group remains open.

7. THE LATTICE $\mathscr{R}_c$

We denote by $\mathscr{R}_c$ the collection of all radical classes $A \in \mathscr{R}$ such that each lattice ordered group belonging to A is complete. Similarly as $\mathscr{R}$, the collection $\mathscr{R}_c$ is partially ordered by inclusion.

Let $\mathscr{G}_c$ be the class of all complete lattice ordered groups; then $\mathscr{G}_c$ is a radical class (cf. [6]). Hence $\mathscr{R}_c$ is the interval $[0^-, \mathscr{G}_c]$ of the lattice $\mathscr{R}$.

(For $\mathscr{R}$ and $\mathscr{R}_c$ we apply the usual lattice theoretic notations, though $\mathscr{R}$ and $\mathscr{R}_c$ fail to be sets.) Hence we have:

7.1. Lemma. $\mathscr{R}_c$ *is a closed sublattice of* $\mathscr{R}$; *thus the infinite distributive law* (1) *is valid in* $\mathscr{R}_c$.

In [6] it was shown that no element of R distinct from 0^- and $\mathscr{G}$ has a complement in the lattice $\mathscr{R}$. Thus $\mathscr{R}$ is pseudocomplemented, but it fails to be a Stone lattice.

7.2. Proposition. $\mathscr{R}_c$ *is a Stone lattice.*

Proof. Let $A \in \mathscr{R}_c$. Put $A^{\delta o} = A^\delta \cap \mathscr{G}_c$. Then obviously, $A^{\delta o}$ is a pseudocomplement of A in the lattice $\mathscr{R}_c$. We have to verify that $A^{\delta o} \vee A^{\delta o \delta o} = \mathscr{G}_c$ is valid for each $A \in \mathscr{R}_c$.

We have $A^{\delta o \delta o} = A^{\delta\delta} \cap \mathscr{G}_c$, hence

$$A^{\delta o} \vee A^{\delta o \delta o} = (A^\delta \wedge \mathscr{G}_c) \vee (A^{\delta\delta} \wedge \mathscr{G}_c) = (A^\delta \vee A^{\delta\delta}) \wedge \mathscr{G}_c .$$

Let $G \in \mathscr{G}_c$. Then

$$(A^{\delta o} \vee A^{\delta o \delta o})(G) = ((A^\delta \vee A^{\delta\delta}) \wedge \mathscr{G}_c)(G) = (A^\delta \vee A^{\delta\delta})(G) \cap \mathscr{G}_c(G) =$$
$$= ((A^\delta \vee A^{\delta\delta})(G)) \cap G = (A^\delta \vee A^{\delta\delta})(G) = A^\delta(G) \vee^c A^{\delta\delta}(G) .$$

In view of 2.6, $A^\delta(G)$ and $A^{\delta\delta}(G)$ are complementary polars of G. Since G is complete, $A^\delta(G)$ and $A^{\delta\delta}(G)$ are complementary direct factors of G. Hence $A^\delta(G) \vee^c A^{\delta\delta}(G) = G$. Therefore G belongs to $A^{\delta o} \vee A^{\delta o \delta o}$ and thus $A^{\delta o} \vee A^{\delta o \delta o} = \mathscr{G}_c$.

Since for each nonzero r-homogeneous complete lattice G the radical class $T(G)$ is an atom of $\mathscr{R}_c$, 3.9 implies:

7.3. Proposition. *There exists a proper collection of atoms in* $\mathscr{R}_c$.

7.4. Lemma. *Let* $G \in \mathscr{G}$, $\{G_i\}_{i \in I} \subseteq \mathscr{G}$, $H = \prod_{i \in I} G_i$, $0 < h \in H$, $\operatorname{card} I(h) > \operatorname{card} G$, $A = T(G)$. *Then* h *does not belong to* $A(H)$.

Proof. By way of contradiction, assume that $h \in A(H)$. Hence in view of 1.2 there exist $\{H_j\}_{j \in J} \subseteq c(H)$ and $\{G'_j\}_{j \in J} \subseteq c(G)$ such that for each $j \in J$, H_j is isomorphic to G'_j and $[h] = \bigvee_{j \in J} H_j$. Thus there exists a finite subset J_1 of J such that for some $0 < h_j \in H_j$ $(j \in J_1)$ we have $h = \sum_{j \in J_1} h_j$. For each element $0 \leqq h' \leqq h$ there are $h'_j \in [0, h_j]$ $(j \in J_1)$ with $h' = \sum_{j \in J_1} h'_j$. We obviously have $\operatorname{card} I(h) \leqq \operatorname{card} [0, h]$, whence $\operatorname{card} I(h)$ is equal or less than the product of the cardinals $\operatorname{card} [0, h_j]$ (where j runs over the set J_1). Because $\operatorname{card} [0, h_j] \leqq \operatorname{card} G$ for each $j \in J_1$, we obtain $\operatorname{card} I(h) \leqq \operatorname{card} G$, which is a contradiction.

Next, $\mathscr{R}$ has no dual atom. (This a consequence of Corollary 1 of Propos. 3.4, [6].) Similarly we have:

7.5. Proposition. *The lattice* $\mathscr{R}_c$ *has no dual atom.*

Proof. By way of contradiction, assume that A is a dual atom of $\mathscr{R}_c$. Hence there exists $G \in \mathscr{G}_c$ such that G does not belong to A. Put $B = T(G)$. Let I be a system of indices, $\operatorname{card} I > G$. Denote $H = \prod_{i \in I} G_i$, where each G_i is equal to G. Then H belongs neither to A nor to B. (In fact, the relation $H \in A$ would imply $G \in A$, which is a contradiction; in view of 7.4, H does not belong to $T(G)$.) We have $A \vee B = \mathscr{G}_c$, hence

$$H = \mathscr{G}_c(H) = (A \vee B)(H) = A(H) \vee^c B(H).$$

If $0 < h_1 \in H$ is such that $h_1(i) > 0$ for each $i \in I$, then h does not belong to B (cf. 7.4). There exists $0 < g_0 \in G$ with $g_0 \notin A(G)$. Let $h \in H$ be such that $h(i) = g_0$ for each $i \in I$. We have $h \in H = A(H) \vee^c B(H) = A(H) + B(H)$, hence there are $u \in A(H)$ and $v \in B(H)$ with $h = u + v$. There exists $i \in I$ such that $v(i) = 0$. Hence $h(i) = u(i)$. Because $0 < u(i) \leqq u \in A(H)$, we obtain $g_0 \in A(H)$. Next, from

$$A(G_i) = G_i \cap A(H)$$

we infer that $g_0 \in A(G_i)$, which is a contradiction.

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Author's address: 040 01 Košice, Karpatská 5, Czechoslovakia (Matematický ústav SAV).