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# A REMARK ON LINEAR CODIMENSIONS OF FUNCTION ALGEBRAS

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## I. ANTISYMMETRIC FUNCTION SPACES

A *function space* $A$ on a compact Hausdorff space $X$ is a closed subspace of $C(X)$ (i.e., the Banach space of all complex-valued continuous functions on $X$) which separates the points of $X$ and contains the constant functions.

A function space $A$ is said to be *antisymmetric* provided it satisfies the following condition:

Whenever a function in $A$ is real, then it is constant.

Denote by $M(X)$ the set of all complex Borel regular measures on $X$, i.e. by the Riesz Representation Theorem, the dual space of $C(X)$.

Whenever $A$ is a function space on $X$, let $A^\perp$ be the *annihilator* of $A$, or the set of all measures $m$ in $M(X)$ such that $\int f \, \mathrm{d}m = 0$ for any $f$ in $A$.

Finally, denote by $M_0$ the set of all $m$ in $M(X)$ such that $\int \mathrm{d}m = 0$; $M_0$ is then the annihilator of constant functions on $X$, and $\operatorname{Re} M_0 = \{\operatorname{Re} m: m \text{ in } M_0\}$ is the real annihilator of real constant functions on $X$.

Remark 1. The real space $\operatorname{Re} M(X)$ is the dual space of the real Banach space $\operatorname{Re} C(X)$. Endow $\operatorname{Re} M(X)$ with the weak-star topology: it is well known that $\operatorname{Re} M(X)$ becomes a locally convex topological linear space with the dual space $\operatorname{Re} C(X)$.

**Theorem 1.** *Let $A$ be a function space on $X$. Then $A$ is antisymmetric if and only if the set* $\operatorname{Re} A^\perp = \{\operatorname{Re} m: m \text{ in } A^\perp\}$ *is dense in* $\operatorname{Re} M_0$ *with respect to the weak-star topology.*

Proof. Let $m$ be in $A^\perp$. Then

$$0 = \int 1 \, \mathrm{d}m = \int \mathrm{d} \operatorname{Re} m + \mathrm{i} \int \mathrm{d} \operatorname{Im} m$$

and both $\operatorname{Re} m$, $\operatorname{Im} m$ are in $\operatorname{Re} M_0$; so $\operatorname{Re} A^\perp \subset \operatorname{Re} M_0$.

Suppose $\operatorname{Re} A^\perp$ is not dense in $\operatorname{Re} M_0$ and fix an $m$ in $\operatorname{Re} M_0 - \overline{\operatorname{Re} A^\perp}$ where the bar denotes the weak-star closure. Take, by the Hahn-Banach Separating Theorem for locally convex topological linear spaces (cf. Remark 1) and $f$ in Re $C(X)$ such that

$$\int f\,dm \neq 0 \quad \text{and} \quad \int f\,d\,\mathrm{Re}\,n = 0 \quad \text{for any } n \text{ in } A^{\perp}\,.$$

If $n$ is in $A^{\perp}$, then both Re $n$, Im $n$ are in Re $A^{\perp}$, so that

$$\int f\,dn = \int f d\,\mathrm{Re}\,n + i \int f d\,\mathrm{Im}\,n = 0$$

and $f$ annihilates $A^{\perp}$; by the Hahn-Banach Theorem $f$ lies in $A$. But $m$ annihilates constants and does not annihilate $f$; this implies that $f$ is not constant, so $A$ is not antisymmetric.

Conversely, let $f$ be a non-constant real function in $A$. There is, by the Hahn-Banach Theorem, a measure $m$ in Re $M(X)$ such that

$$\int f\,\mathrm{d}m \neq 0 \quad \text{and} \quad \int dm = 0\,.$$

Then $m$ is in $M_0$. For an arbitrary $n$ in $A^{\perp}$,

$$\int f d\,\mathrm{Re}\,n = \mathrm{Re} \int f\,dn = 0\,;$$

so $f$ annihilates Re $A^{\perp}$ and then, by continuity, it annihilates the whole $\overline{\mathrm{Re}\,A}^{\perp}$ as well. But $f$ does not annihilate $m$, thus $m$ is not in $\overline{\mathrm{Re}\,A}^{\perp}$; consequently, Re $A^{\perp}$ is not dense in Re $M_0$, and Theorem 1 is proved.

## II. A CODIMENSION OF A FUNCTION ALGEBRA

By a *function algebra* we mean a closed subalgebra of the commutative Banach algebra $C(X)$ on a compact Hausdorff space $X$, which separates points of $X$ and contains constant functions.

Function algebras are then function spaces of a special type.

$A$ being a function algebra on $X$, a nonvoid subset $F$ of $X$ is called an *antisymmetric set* for $A$, whenever $A/F$, the algebra of all restrictions of functions in $A$ to the set $F$, is antisymmetric, i.e., contains no real functions but constants.

It is obvious that the union of two antisymmetric sets which meet is an antisymmetric set, likewise the closure of an antisymmetric set. Any singleton in $X$ is an antisymmetric set. Then, via the Zorn Lemma, the system of all maximal antisymmetric sets for a function algebra $A$ on $X$ is a closed disjoint covering of $X$.

Remark 2. Any function algebra $A$ contains the constants and separates the points of $X$. It follows that for any natural $n$ and for an arbitrary $n$-tuple of distinct points $x_1, \ldots, x_n$ in $X$ the system of equations

$$f(x_i) = y_i\,, \quad i = 1, \ldots, n$$

has a solution $f$ in $A$ for any $n$-tuple of complex numbers $y_1, \ldots, y_n$. Consequently, any maximal antisymmetric set is either a singleton or infinite.

The following theorem can be held as well-known (see [1] or [2]):

**Bishop Decomposition Theorem.** *Let $A$ be a function algebra on $X$, and let $\mathscr{K}$ be the system of all its antisymmetric sets. Then, for an arbitrary $K$ in $\mathscr{K}$, the restriction algebra $A|K$ is a function algebra, or a closed subalgebra of $C(K)$. Whenever $f$ is a continuous complex-valued function on $X$, and $f|K$ is in $A|K$ for all $K$ in $\mathscr{K}$, then $f$ is in $A$.*

Remark 3. Whenever all maximal antisymmetric sets of a function algebra $A$ on $X$ are singletons, then, by the Bishop Theorem, $A$ is equal to the whole $C(X)$. So, the Bishop Theorem is a generalization of the classical Weierstrass-Stone Theorem. In compliance with Remark 2, any function algebra on $X$ which is not equal to all the $C(X)$ has an infinite maximal antisymmetric set.

**Theorem 2.** *Let $A$ be a function algebra on $X$ which is not equal to $C(X)$. Then $A$ has an infinite linear codimension in $C(X)$.*

Proof. According to the Bishop Theorem and to Remarks 2 and 3, it suffices to prove Theorem 2 in case of $A$ being antisymmetric and $X$ infinite.

Suppose $A$ has a finite linear codimension in $C(X)$. Take an $n$-tuple $f_1, \ldots, f_n$ of functions in $C(X)$ with the following two properties:
(i) $f_1, \ldots, f_n$ and $A$ generate, in the linear sense, the whole $C(X)$;
(ii) the only linear combination of $f_1, \ldots, f_n$ which belongs to $A$ is the trivial one.
There is, by the Riesz and Hahn-Banach Theorems, an $n$-tuple $m_1, \ldots, m_n$ of measures in $A^\perp$ such that

$$\int f_i \, \mathrm{d}m_j = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

Now, it is easy to see that $m_1, \ldots, m_n$ generate all the $A^\perp$: for an arbitrary $m$ in $A^\perp$ put

$$\boldsymbol{m} = m - \sum \left( \int f_i \, \mathrm{d}m \right) m_i .$$

Then $\boldsymbol{m}$ is in $A^\perp$ and, moreover, $\boldsymbol{m}$ annihilates any function $f_i$, $i = 1, \ldots, n$. The condition (i) implies that $\boldsymbol{m}$ has to be the zero measure.

The real space Re $A^\perp$ is then finitely generated by Re $m_1, \ldots,$ Re $m_n$, so it is weak-star closed. Hence, by Theorem 1, $A$ being antisymmetric, Re $A^\perp$ is equal to Re $M_0$. To complete the proof, it is enough to construct an infinite sequence of linearly independent measures in Re $M_0$. Let $\{x_i\}$ be an infinite sequence of distinct points in $X$. Denote by $d_i$ the one-point mass 1 in $x_i$ and put $m_i = d_{2i-1} - d_{2i}$.

Theorem 2 is proved.

## III. PERVASIVE FUNCTION SPACES

A function space $A$ on $X$ is called *pervasive* (for algebras see [3]) provided it satisfies the following condition:

Whenever $F$ is a nonvoid proper closed subset of $X$, then $A/F$, the linear space of all restrictions of the functions in $A$ to the set $F$, is dense in $C(X)$.

The classical disc algebra $D$, the uniform closure of all complex polynomials on the unit circle in the $z$-plane, is pervasive. There is a closed connection between any two nonzero measures annihilating $D$: the classical $F$. and $M$. Riesz Theorem affirms they are mutually absolutely continuous.

We have dealt with pervasive algebras in [4, 5]. The problem settled in [5] may be restated, in a rather bolder manner, as follows:

**Problem 1.** Is it true that any two nonzero measures annihilating a pervasive algebra $A$, are mutually absolutely continuous?

Is it true at least under some additional conditions on $A$?

We cannot answer this question (which may be called a "general F. and M. Riesz conjecture"). Nevertheless, the following theorem gives some information:

**Theorem 3.** *A function space $A$ on $X$ is pervasive if and only if the closed support of any nonzero measure in $A^\perp$ is the whole $X$.*

Proof. Let $A$ be pervasive. Fix an arbitrary $m$ in $M(X)$ such that $\emptyset \neq \operatorname{spt} m = F \neq X$ and prove that $m$ does not annihilate $A$. Without loss of generality we may suppose that $|m|$, the total mass of $m$ on $X$, is equal to one. Take an $f$ in $C(F)$ such that $\int_F f \, dm = 1$ and choose $g$ in $A$ such that $|f - g|_F < 1$. Then

$$\left|1 - \int g \, dm\right| = \left|\int (f - g) \, dm\right| = |f - g|_F \, |m| < 1 .$$

It follows that $\int g \, dm \neq 0$.

Let, on the contrary, $A$ be not pervasive. Then there is a closed nonvoid proper subset $F$ of $X$ and a function $f$ in $C(F)$ which is not in the closure of $A/F$ in the sup-norm of $F$. By the Hahn-Banach Theorem there is an $m$ in $M(F)$ such that $\int_F f \, dm \neq 0$ while $\int_F g \, dm = 0$ for any $g$ in $A$. If $\boldsymbol{m}$ is a trivial extension of $m$ from $F$ to $X$, then $\boldsymbol{m}$ annihilates $A$. But $\operatorname{spt} \boldsymbol{m} = \operatorname{spt} m \subset F \neq X$ and, as $m$ is not zero, $\boldsymbol{m}$ is nonzero as well.

Remark 4. Now, the way of constructing pervasive spaces on $X$, which (by Theorem 2) are not algebras, is obvious. It suffices to choose an $n$-tuple of measures $m_1, \ldots, m_n$ in $M(X)$ which are "locally linearly independent", i.e., which have the property that any their linear combination is either the trivial measure or it has the whole $X$ as its closed support, and take as $A$ the set of all those functions in $C(X)$ which are annihilated by all of $m_j$'s.

Also it is simple to construct pervasive function spaces without the "general F. and M. Riesz property".

In connection with the relations between function algebras and function spaces the solution of the following problem may be of interest:

**Problem 2.** Given $A^{\perp}$, how to distinguish, by inner properties of this space only, whether $A$ is an algebra or merely a space?

A partial answer is, of course, Theorem 2.

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