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COMPLETION OF A CYCLICALLY ORDERED GROUP

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A cyclic order on a set P is defined to be a ternary relation on P fulfilling certain conditions (E. Čech [1]; for definitions, cf. Section 1 below (the ternary relation under consideration will be denoted by $[x, y, z]$)).

V. Novák [6], [7] studied completions of cyclically ordered sets by means of regular cuts. The method is analogous to that applied for ordered sets ("Dedekind cuts").

The notion of cyclically ordered group is due to L. Rieger [12]. (Cf. also L.Fuchs [3], Chap. IV, § 6.) A representation theorem for cyclically ordered groups was proved by S. Swierczkowski [13]. Further results in this field were established by A. I. Zabarina [14], A. I. Zabarina and G. G. Pestov [15] and B. C. Olticar [10]. G. Pringerová [11] studied radical classes of cyclically ordered groups.

Each linearly ordered group can be considered as being cyclically ordered.

In the present paper the completion of a cyclically ordered group will be dealt with. This completion is constructed by means of certain subsets of the set of all regular cuts. A cyclically ordered group is said to be complete if it is equal to its completion.

Each cyclically ordered group G possesses a largest linearly ordered subgroup G_0 (cf. [11]). Let $a(G_0)$ be the archimedean kernel of G_0 (cf., e.g., [4]).

Let us denote by Z and R the additive group of all integers or all reals, respectively (with the natural linear order). Next, let K be the group of all reals a with $0 \leqq a < 1$, the group operation being the addition mod 1. For $a, b, c \in K$ we put $[a, b, c]$ if

$$(1) \qquad a < b < c \quad \text{or} \quad b < c < a \quad \text{or} \quad c < a < b$$

is valid. Then K is a cyclically ordered group ([12], [13]).

For a linearly ordered group H we denote by $m(H)$ the maximal (Dedekind) completion of H (cf. Černák [2]).

It turns out that if $G_0 \neq \{0\}$, then the completion of the cyclically ordered group G is an amalgam of G and $m(G_0)$ with the common subgroup G_0.

It will be shown that a cyclically ordered group G is complete if and only if some of the following conditions (i), (ii), (iii) is fulfilled:

(i) G is finite.

(ii) G is isomorphic to K.

(iii) $G_0 \neq \{0\}$ and $m(G_0) = G_0$.

If $a(G_0) \neq \{0\}$, then G is complete if and only if

(iv) $a(G_0)$ is isomorphic either to Z or to R.

1. PRELIMINARIES

Let A be a nonempty set. Let $[x, y, z]$ be a ternary relation defined on A such that the following conditions are fulfilled:

I. If $[x, y, z]$ holds, then x, y and z are distinct; if x, y and z are distinct, then either $[x, y, z]$ or $[z, y, x]$.

II. $[x, y, z]$ implies $[y, z, x]$.

III. $[x, y, z]$ and $[y, u, z]$ imply $[x, u, z]$.

Then the relation under consideration (we shall often denote it by $[\ ]$) is a *cyclic order* on A (cf. Čech [1]). The set A equipped with this relation is called a *cyclically ordered set.* Each nonempty subset of A is cyclically ordered by the inherited cylic order.

A generalization of this notion was investigated in a series of papers by V. Novák and M. Novotný (cf., e.g., [8], [9]; cf. also [6] and the papers quoted there). In their terminology, the cyclic order (in the sense defined above) is called "*linear cyclic order*"; in a "*cyclic order*" (in the sense of [6]) there can exist distinct elements x, y, z such that neither $[x, y, z]$ nor $[z, y, x]$ is valid. This generalized notion could by called a *partial cyclic order.* For groups with such a partial cyclic order cf. S. D. Želeva [16], [17], [18].

Let L be a linearly ordered set. Then a cyclic order $[\ ]$ is defined on L by

$$(2) \qquad [x, y, z] \equiv x < y < z \quad \text{or} \quad y < z < x \quad \text{or} \quad z < x < y\,.$$

We shall say that this cyclic order is *generated* by the linear order on L.

Let G be a cyclically ordered set. Suppose that a binary operation $+$ is defined on G such that $(G; +)$ is a group (G need not be abelian). Further, assume that for any $x, y, z, a, b \in G$,

$$[x, y, z] \quad \text{implies} \quad [a + x + b,\ a + y + b,\ a + z + b]\,.$$

Then G is said to be a *cyclically ordered group.* In particular, in view of the above remark on linearly ordered sets, each linearly ordered group G is, at the same time, a cyclically ordered group (with respect to the cyclic order generated by the linear order on G).

Let us consider the following examples of cyclically ordered groups.

Example 1. Let K be as in the introduction. (For the application of the cyclically ordered group K cf. Theorem 1.1 below.)

Example 2. (Cf. [13].) Let L be a linearly ordered group; hence we can consider L as cyclically ordered. We define a cyclic order on the direct product $L \times K$ as follows.

Let $u = (x, a)$, $v = (y, b)$, $w = (z, c)$ be distinct elements of $L \times K$. We put $[u, v, w]$ if some of the following conditions is fulfilled:

(i) $[a, b, c]$;

(ii) $a = b \neq c$ and $x < y$;

(iii) $b = c \neq a$ and $y < z$;

(iv) $c = a \neq b$ and $z < x$;

(v) $a = b = c$ and $[x, y, z]$.

Then the group $L \times K$ equipped with the relation $[u, v, w]$ is a cyclically ordered group; this cyclically ordered group will be denoted by $L \otimes K$. (Cf. [13].)

An isomorphism of cyclically ordered groups is defined in the natural way. Each subgroup of a cyclically ordered group is again a cyclically ordered group.

The following theorem is the main result of [13].

1.1. Theorem. (Swierczkowski) *Let G be a cyclically ordered group. Then there exists a linearly ordered group L such that G is isomorphic to a cyclically ordered subgroup of $L \otimes K$.*

A subgroup H of a cyclically ordered group G is said to be *linearly ordered* if there exists a linear order $\leqq$ on H such that

(i) $(H; \leqq)$ is a linearly ordered group;

(ii) the cyclic order on H generated by the linear order $\leqq$ coincides with the original cyclic order defined on H.

1.2. Lemma. (Cf. [11], Chap. III, Lemma 2.2.) *Let G be a cyclically ordered group. Then the following conditions are equivalent:*

(i) *G is a linearly ordered group.*

(ii) *Each nonzero subgroup of G is infinite, and for each $g \in G$ and each positive integer n, the relation $[-g, 0, g]$ implies $[-g, 0, ng]$.*

Let G and L be as in 1.1. Let f be an isomorphism of G into $L \otimes K$. Let us denote by G_0 the set of all elements $g \in G$ having the property that there exists $x \in L$ with $f(g) = (x, 0)$. Then G_0 is, evidently, a subgroup of G.

1.3. Lemma. (Cf. [11], Chap. III, 2.9.) *Let H be a subgroup of a cyclically ordered group G. Then the following conditions are equivalent:*

(i) *H is a linearly ordered group.*

(ii) *$H \subseteq G_0$.*

From 1.3 it follows that G_0 is the largest linearly ordered subgroup of G. Moreover, G_0 is clearly a normal subgroup of G.

Let L_1 be the projection of G into L with respect to f (i.e., L_1 is the set of all elements $x \in L$ having the property that there exist $g \in G$ and $a \in K$ with $f(g) = (x, a)$). Then L_1 is a subgroup of L; let us remark that G_0 is always isomorphic to a subgroup of L_1, but G_0 need not be isomorphic to L_1. (Cf. [11], Chap. III, Section 2.)

Let H be a convex subgroup of G_0. Suppose that H is a normal subgroup of G.

Consider the factor group $G/H = G'$. For $x', y', z' \in G'$ we put $[x', y', z']$ if the following conditions are fulfilled:

(i) x', y', z' are distinct;

(ii) there exist $x_1 \in x'$, $y_1 \in y'$, $z_1 \in z'$ such that $[x_1, y_1, z_1]$.

1.4. Lemma. (Cf. [11], Lemma 4.3, Lemma 4.4.) *The group G' equipped with the ternary relation $[x', y', z']$ is a cyclically ordered group. If $[x', y', z']$ is valid for some $x', y', z' \in G'$, then for all $x_2 \in x'$, $y_2 \in y'$ and $z_2 \in z'$ the relation $[x_2, y_2, z_2]$ is valid in G.*

Let L and f be as above. Let K_1 be the set of all $a \in K$ having the property that there exist $x \in L$ and $g \in G$ with $f(g) = (x, a)$. Clearly K_1 is a subgroup of K. The following lemma is easy to verify.

1.5. Lemma. *Under the above notation, let $f(g) = (x, a)$. Then the mapping $f_1 \colon g + G_0 \to a$ is an isomorphism of the cyclically ordered group G/G_0 onto the cyclically ordered group K_1.*

As a corollary we obtain that if G is given, then K_1 is defined uniquely up to isomorphism.

2. COMPLETIONS

In this section the definition of the completion of a cyclically ordered group will be introduced, some auxiliary results will be proved and some examples will be presented.

We start by recalling the basic definitions on completions of cyclically ordered sets (cf. [6], [7]).

Let G be a cyclically ordered set. For each $g \in G$ there exists a uniquely determined linear order $<_g$ on G such that (i) the cyclic order on G determined by $<_g$ coincides with the original cyclic order as defined on G, and (ii) g is the least element of G with respect to $<_g$. (Cf. [6].)

A regular cut h in G is defined to be a *linear order* (we will denote it also by $<_{(h)}$) on G such that the cyclic order on G generated by the linear order $<_{(h)}$ coincides with the original cyclic order defined on G, and some of the following conditions is fulfilled:

(i) $(G; <_{(h)})$ has neither the least nor the greatest element;

(ii) there exists $g \in G$ such that $<_{(h)} = <_g$. (Cf. [6], [7].)

Let $C(G)$ be the set of all regular cuts in G. Let us have distinct cuts $h_1, h_2, h_3 \in C(G)$. We denote $<_{(h_i)} = <_i$ $(i = 1, 2, 3)$. We put $[h_1, h_2, h_3]$ if there exist $x, y, z \in G$ such that

$$x <_1 y <_1 z\,, \quad y <_2 z <_2 x \quad \text{and} \quad z <_3 x <_3 y\,.$$

For each $g \in G$ let $\varphi(g) = <_g$.

2.1. Theorem. (Cf. [7].) *The set $C(G)$ equipped with the ternary relation $[h_1, h_2, h_3]$ is a cyclically ordered set. The mapping φ is an isomorphism of the cyclically ordered set G into $C(G)$.*

We shall often identify the elements g and $\varphi(g)$; hence we consider G as a subset of $C(G)$. The cyclically ordered set $C(G)$ is said to be the *completion of the cyclically ordered set G.*

A cut $h \in C(G)$ will be called *proper* if h does not belong to G.

In what follows we assume that G is a cyclically ordered group. The notations introduced in the introduction and in Section 1 will be applied.

Let $\emptyset \neq G_1$ be a subset of $C(G)$ with $G \subseteq G_1$. Suppose that a binary operation $+_1$ is defined on G_1 such that the following conditions are fulfilled:

(i) $(G_1, +_1)$ is a cyclically ordered group (under the cyclic order inherited from $C(G)$).

(ii) $(G; +)$ is a subgroup of $(G_1, +_1)$.

Then $(G_1; +_1)$ is said to be an *extension* of G in $C(G)$. We shall often write G_1 instead of $(G_1; +_1)$. Let $\mathscr{C}(G)$ be the set of all extensions of G in $C(G)$. For $G_1, G_2 \in \mathscr{C}(G)$ we put $G_1 \leqq G_2$ if G_1 is a subgroup of G_2. Then $\mathscr{C}(G)$ is a partially ordered set. If $\mathscr{C}(G)$ possesses a greatest element $d_1(G)$, then $d_1(G)$ is said to be a *completion of the cyclically ordered group G.*

Let $a(G_0)$ be the archimedean kernel of G_0. Because $a(G_0)$ is an archimedean linearly ordered group, it is isomorphic to a subgroup R_1 of R (with the inherited linear order). We shall often identify $a(G_0)$ and R_1.

From 1.3 it follows that G_0 is a characteristic subgroup of G (in the sense that $\chi(G_0) = G_0$ whenever χ is an automorphism of the cyclically ordered group G). Moreover, $a(G_0)$ is the largest archimedean convex subgroup of G_0. Thus $a(G_0)$ is a characteristic subgroup of G as well. In particular, $a(G_0)$ is a normal subgroup of G.

A cut $h \in C(G)$ will be said to be of *type $a(G_0)$* if there are $g_1, g_2 \in G$ such that (i) $0 < g_2 - g_1 \in a(G_0)$, and (ii) $[g_1, h, g_2]$ in $C(G)$. Otherwise h will be said to be of *type $a'(G_0)$*.

When investigating the cyclically ordered set $C(G)$ we distinguish two cases.

First suppose that $G_0 = \{0\}$. Then G/G_0 is isomorphic to G, whence in view of 1.5, G is isomorphic to K_1; thus G is isomorphic to a subgroup of K. If G is finite, then clearly $C(G) = G$; if G is infinite then it is easy to verify that the cyclically ordered set $C(G)$ is isomorphic to the cyclically ordered set K. Conversely, if the cyclically ordered set G is isomorphic to K, then there are no proper cuts in G. We arrive at the following result:

2.2. Lemma. *Assume that $G_0 = \{0\}$.* (i) *If G is finite, then $C(G) = G$.* (ii) *If G is infinite, then $C(G)$ is isomorphic to K.* (iii) *If the cyclically ordered set G is isomorphic to K, then $C(G) = G$.*

Now suppose that $G_0 \neq \{0\}$. The following three examples illustrate some typical situations which may occur.

2.3. Example. Let L be the additive group of all rational numbers with the natural linear order. Let K_1 be the subgroup of K consisting of the elements $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Put $G_1 = L \otimes K_1$. Further, let $G_1' = R \otimes K_1$. Let y_1 be an irrational number. Put $v_0 = (y_1, 0)$. For $v_1, v_2 \in G_1$ we put $v_1 < v_2$ if the relation $[v_0, v_1, v_2]$ is valid in G_1'. Then $<$ is a linear order on G_1. The linearly ordered set $(G_1; <)$ has no least and no greatest element. The cyclic order on G_1 generated by the linear order $<$ coincides with the original cyclic order defined on G_1. Hence $<$ is a proper regular cut in G_1. This cut is of type $a(G_0)$.

2.4. Example. Let G_1 be as in 2.3 and $G_1' = L \otimes K$. Let $a_1 = \frac{1}{8}$, $v_0 = (0, \frac{1}{8}) \in G_1'$. For $v_1, v_2 \in G_1$ we put $v_1 < v_2$ if $[v_0, v_1, v_2]$ is valid in G_1'. Then $<$ is again a proper regular cut on G_1; this cut is of type $a'(G_0)$.

2.5. Example. Let L be as in 2.3. Let K_1 be the subgroup of K consisting of all elements $a \in K$ such that a is rational. Put $G_1 = L \otimes K_1$ and let G_1' be as in 2.3. Let $a_1 \in K \setminus K_1$ and $v_0 = (0, a_1)$. For $v_1, v_2 \in G_1$ we put $v_1 < v_2$ if $[v_0, v_1, v_2]$ holds in G_1'. Then $<$ is a proper regular cut on G_1 of type $a'(G_0)$.

2.6. Example. Let L be the additive group of all reals with the natural linear order. Put $G = L \otimes K$. Let $a_0 \in K$, $a_0 \neq 0$. A new element h will be added to G, and on the set $G' = G \cup \{h\}$ we define a ternary relation $[v_1, v_2, v_3]$ as follows: for $v_1, v_2, v_3 \in G$ the new relation on G' has its original meaning. If $v_1, v_3 \in G$, $v_1 = (x_1, a_1)$, $v_3 = (x_3, a_3)$, $v_1 \neq v_3$, $v_2 = h$, then we put $[v_1, v_2, v_3]$, $[v_2, v_3, v_1]$ and $[v_3, v_1, v_2]$ if some of the following relations is valid:

(i) $a_1 \leqq a_0 < a_3$;
(ii) $a_0 < a_3 < a_1$.

This ternary relation is a cyclic order on G'. For $g_1, g_2 \in G$ we put $g_1 < g_2$ if $[h, g_1, g_2]$ is valid in G'. Then $<$ is a proper cut in G; this cut is of type $a'(G_0)$.

Let G be a cyclically ordered group and let w be a fixed element of G. Then the mapping $\varphi_w^0\colon G \to G$ defined by $\varphi_w^0(v) = w + v$ for each $v \in G$ is an automorphism of the cyclically ordered set G. Let $h = <$ be a regular cut on G. For $v_1, v_2 \in G$ we put $v_1 <' v_2$ if $(\varphi_w^0)^{-1}(v_1) < (\varphi_w^0)^{-1}(v_2)$ is valid. Then $<' = h'$ is again a regular cut on G. We denote $h' = \varphi_w(h)$. The cut h' is proper (or of type $a(G_0)$, or of type $a'(G_0)$) iff h is proper (or of type $a(G_0)$, or of type $a'(G_0)$, respectively).

Let X, Y, Z be nonempty subsets of G. We put $[X, Y, Z]$ if, whenever $x \in X$, $y \in Y$ and $z \in Z$, then $[x, y, z]$ is valid. If, e.g., $X = \{x\}$ is a one-element set, then we write $[x, Y, Z]$ instead of $[\{x\}, Y, Z]$.

In the following Lemmas 2.7–2.10 we assume that h is a proper cut of type $a'(G_0)$ and that $a(G_0) \neq \{0\}$.

We denote by X the set of all $g \in G$ having the property that there exists $g_0 \in a(G_0)$

such that $[g_0, g, h]$ is valid in $C(G)$. Put $Y = G \setminus X$. From the assumptions just mentioned we immediately obtain the following three lemmas:

2.7. Lemma. $G_0 \subseteq X$. *If* $g_1 \in X$, $g \in G$ *and* $[0, g, g_1]$, *then* $g \in X$.

2.8. Lemma. *Assume that* $Y \neq \emptyset$. *Then* Y *is the set of all* $y \in G$ *such that* $[X, h, y]$ *is valid in* $C(G)$. *Thus* $[X, h, Y]$ *in* $C(G)$.

2.9. Lemma. $X + g_0 = X = g_0 + X$ *for each* $g_0 \in a(G_0)$. *If* $Y \neq \emptyset$, *then* $Y + g_0 = Y = g_0 + Y$ *for each* $g_0 \in a(G_0)$.

2.10. Lemma. *Let* $Y \neq \emptyset$. *Let* h_1 *be a proper regular cut in* G *such that* $[X, h_1, Y]$ *is valid in* $C(G)$. *Then* $h_1 = h$.

Proof. Let $h_1 = <_1$ and let $g_1, g_2 \in G$. Because $[X, h_1, Y]$, the relation $g_1 <_1 g_2$ holds if and only if some of the following conditions is valid:

(i) $g_1 \in Y$ and $g_2 \in X$;

(ii) $g_1, g_2 \in Y$ and $[g_1, g_2, g_3]$ for each $g_3 \in X$;

(iii) $g_1, g_2 \in X$ and $[g_3, g_1, g_2]$ for each $g_3 \in Y$.

Thus $h_1 = h$.

3. CUTS OF TYPES $a'(G_0)$ AND G_0' IN G

Let G and $C(G)$ be as above.

We are interested in the following question: for which $h \in C(G)$ does there exist $G_1 \in \mathscr{C}(G)$ such that $h \in G_1$?

If $h \in G$, then we can take $G_1 = G$. Let $h \notin G$. We begin with the case when h is of type $a'(G_0)$.

3.1. Lemma. *Let* $a(G_0) \neq \{0\}$, $a(G_0) \neq G$. *Let* h *be a proper regular cut of type* $a'(G_0)$ *in* G. *Let* G_1 *be an extension of* G *in* $C(G)$. *Then* h *does not belong to* G_1.

Proof. By way of contradiction, assume that h belongs to G_1. Let X and Y be as in Section 2. Suppose that $Y \neq \emptyset$. In view of 2.8 we have

$$[X, h, Y] \quad \text{in} \quad G_1 .$$

Let $0 \neq g_0 \in a(G_0)$. Then

$$[X + g_0, \ h +_1 g_0, \ Y + g_0] \quad \text{in} \quad G_1 .$$

Thus according to 2.9,

$$[X, h +_1 g_0, Y] \quad \text{in} \quad G_1 .$$

The element $h +_1 g_0$ determines a proper cut of type $a'(G_0)$ in G. Therefore according to 2.10 we infer that $h +_1 g_0 = h$, which is a contradiction.

Now suppose that $Y = \emptyset$. Choose $0 < g_0 \in a(G_0)$. Put $Y_0 = G \setminus a(G_0)$. Then $Y_0 \neq \emptyset$ and we have

$$[a(G_0), Y_0, h] \quad \text{in} \quad G_1 ,$$

thus

$$[a(G_0) + g_0,\ Y_0 + g_0,\ h +_1 g_0] \quad \text{in} \quad G_1 .$$

Because $a(G_0) + g_0 = a(G_0)$ and $Y_0 + g_0 = Y_0$, we obtain

$$[a(G_0),\ Y_0,\ h +_1 g_0] \quad \text{in} \quad G_1 .$$

Hence we infer that $h = h +_1 g_0$, which is a contradiction.

3.2. Lemma. *Let $\{0\} \neq a(G_0) = G$. Then there is one (and only one) proper cut of type $a'(G_0)$ in G; this cut coincides with the linear order given on G.*

This is an immediate consequence of the definition of a cut of type $a'(G_0)$.

3.3. Lemma. *Let $\{0\} \neq a(G_0) = G$. Let $G_1 \in \mathscr{C}(G)$. Let h be a proper cut in G of type $a'(G_0)$. Then h does not belong to G_1.*

Proof. By way of contradiction, assume that h belongs to G_1. Let $g_0 \in a(G_0)$, $g_0 \neq 0$. Then $h +_1 g_0 \in G_1$ and $h +_1 g_0$ determines a proper cut of type $a'(G_0)$ in G. Hence in view of 3.2, $h +_1 g_0 = h$, which is a contradiction.

Now 3.1 and 3.3 imply:

3.4. Proposition. *Let G be a cyclically ordered group and let $G_1 \in \mathscr{C}(G)$. Let $a(G_0) \neq \{0\}$. Let h be a proper cut of type $a'(G_0)$ in G. Then h does not belong to G_1.*

Now suppose that $G_0 \neq \{0\}$. Let $h \in C(G)$. If there exist $g_1, g_2 \in G$ such that

(i) $0 < g_2 - g_1 \in G_0$, and

(ii) $[g_1, h, g_2]$ is valid in $C(G)$,

then h is said to be a *cut of type* G_0. Otherwise h is said to be *of type* G_0'.

By the same method as above (with $a(G_0)$ replaced by G_0) we obtain the following result:

3.5. Proposition. *Let G be a cyclically ordered group and let $G_1 \in \mathscr{C}(G)$. Let $G_0 \neq \{0\}$. Let h be a cut of type G_0' in G. Then h does not belong to G_1.*

4. PARTIAL ORDER ON $\mathscr{C}(G)$

In this section it will be shown that the partial order on the set $\mathscr{C}(G)$ introduced in Section 2 coincides with the set-theoretical inclusion.

Let L be a linearly ordered set. For a subset A of L we denote by A^u and A^l the set of all upper bounds or lower bounds of A, respectively. Let $D(L)$ be the system of all sets $(A^u)^l$, where A runs over the family of all nonempty upper bounded subsets of L. The elements of $D(L)$ will be called *Dedekind cuts* of L. The system $D(L)$ is partially ordered by inclusion; in fact, $D(L)$ is a linearly ordered set. For $x \in L$, the element x will be identified with $(\{x\}^u)^l$. In this way, the linearly ordered set L is considered to be embedded into $D(L)$.

Let L_1 be a nonempty subset of L. For $A \subseteq L_1$ let $A^{r(u)}$ and $A^{r(l)}$ be the set of all upper bounds or all lower bounds, respectively, of A in L_1. We construct $D(L_1)$

analogously as $D(L)$ (by means of the sets $(A^{r(u)})^{r(l)}$, where A is a nonempty upper bounded subset of L). The mapping

$$i\colon (A^{r(u)})^{r(l)} \to (A^u)^l$$

is an injection of $D(L_1)$ into $D(L)$; by using this injection we can consider $D(L_1)$ to be a subset of $D(L)$. Let us remark that the injection i preserves the linear order, but it need not preserve, in general, suprema and infima.

A subset M of L_1 will be said to be *dense* in L_1 if $L_1 \subseteq D(M)$. If M is dense in L_1 and M_1 is dense in M, then M_1 is dense in L_1.

Let G be a nonzero cyclically ordered group. Let $h_0 = <_0$ be the regular cut on G generated by the element 0. Let $h \in D((G; <_0))$. Put

$$A = \{g \in G\colon 0 \leqq g < h\}\,, \quad B = \{g \in G\colon h \leqq g\}$$

and let $K = B \oplus A$ be the ordinal sum of the linearly ordered sets A and B. Then the linear order k on K is a regular cut in G. Put $k = f(h)$. For each regular cut k of G there exists $h \in D((G; <_0))$ such that $k = f(h)$.

Let k_1, k_2, k_3 be distinct elements of $C(G)$ and let $k_i = f(h_i)$ $(i = 1, 2, 3)$. Then the relation $[k_1, k_2, k_3]$ is valid if and only if some of the following conditions holds in $D((G; <_0))$:

$$h_1 < h_2 < h_3\,; \quad h_2 < h_3 < h_1\,; \quad h_3 < h_1 < h_2\,.$$

Therefore $C(G)$ is uniquely determined by $D((G; <_0))$; we shall often identify the elements h and $f(h)$.

4.1. Lemma. *Let A and B be cyclically ordered sets. Let φ be an isomorphism of A onto B.* (i) *There exists an isomorphism φ' of $C(A)$ onto $C(B)$ such that $\varphi'(a) = \varphi(a)$ for each $a \in A$.* (ii) *Let $A \subseteq A_1 \subseteq C(A)$ and let ψ be an isomorphism of A_1 onto $\varphi'(A_1)$ such that $\varphi(a) = \psi(a)$ for each $a \in A$. Then $\varphi'(a_1) = \psi(a_1)$ for each $a_1 \in A_1$.*

Proof. This is an immediate consequence of the definitions of $C(A)$ and $C(B)$.

Let g be a fixed element of G. For each $t \in G$ we put

$$\varphi_g(t) = g + t\,, \quad \varphi^g(t) = t + g\,.$$

Then φ_g and φ^g are automorphisms of the cyclically ordered set G. In view of 4.1 (i) we can construct automorphisms $(\varphi_g)'$ and $(\varphi^g)'$ of the cyclically ordered set $C(G)$; in view of 4.1 (ii), $(\varphi_g)'$ and $(\varphi^g)'$ are uniquely determined. For $h \in C(G)$ we denote

$$(\varphi_g)'(h) = g +_0 h\,, \quad (\varphi^g)'(h) = h +_0 g\,.$$

Let $G_1 \in \mathscr{C}(G)$. Thus $G \subseteq G_1 \subseteq C(G)$. The mapping

$$\psi\colon t \to g +_1 t \quad (t \in G_1)$$

is an automorphism of the cyclically ordered set G_1. From 4.1 (ii) we obtain:

4.2. Lemma. *Let $G_1 \in \mathscr{C}(G)$ and $g \in G$. Then $g +_1 t = g +_0 t$ for each $t \in G_1$. Analogously, $t +_1 g = t +_0 g$ for each $t \in G_1$.*

4.3. Corollary. *Let* $G_1, G_2 \in \mathscr{C}(G)$, $g \in G$, $t \in G_1 \cap G_2$. *Then* $g +_1 t = g +_2 t$ *and* $t +_1 g = t +_2 g$.

4.4. Lemma. *Let* $G_1 \in \mathscr{C}(G)$, $g_1 \in G_1$. *Then* $C(g_1 +_1 G) = C(G)$.

Proof. Since $G \subseteq G_1 \subseteq C(G)$, we infer that G is dense in G_1. Because the mapping $t \to g_1 +_1 t$ (where t runs over G_1) is an automorphism of the cyclically ordered set G_1, the set $g_1 +_1 G$ is dense in G_1. Moreover, G_1 is dense in $C(G)$; therefore $g_1 +_1 G$ is dense in $C(G)$ as well. Thus $C(g_1 +_1 G) = C(G)$.

Under the same assumptions as in 4.4, consider the mapping φ of the set G onto $g_1 +_1 G$ defined by $\varphi(t) = g_1 +_1 t$ for each $t \in G$. Then φ is an isomorphism of the cyclically ordered set G onto the cyclically ordered set $g_1 +_1 G$. In view of 4.1 we have the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & g_1 +_1 G \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ C(G) & \xrightarrow{\varphi'} & C(g_1 +_1 G) \end{array},$$

where i_1 and i_2 are embeddings. Moreover, according to 4.1 (ii), φ' is uniquely determined and 4.4 implies that $C(g_1 +_1 G) = C(G)$. Therefore φ' is an automorphism of the cyclically ordered set $C(G)$.

For each $g_2 \in C(G)$ we denote $\varphi'(g_2) = g_1 +_{01} g_2$. Hence we have

4.5. Lemma. *Let* $G_1 \in \mathscr{C}(G)$, $g_1 \in G_1$. *The mapping defined by*

$$\varphi'(g_2) = g_1 +_{01} g_2 \quad (g_2 \in C(G))$$

is an automorphism of the cyclically ordered set $C(G)$. *If* $g_2 \in G_1$, *then* $g_1 +_1 g_2 = g_1 +_{01} g_2$.

Now suppose that the assumptions of 4.5 are fulfilled and that $G_2 \in \mathscr{C}(G)$, $G_1 \subseteq G_2$. We apply the previous construction with the distinction that instead of $g_1 +_1 G$ we now have $g_1 +_2 G$; the corresponding mappings will now be denoted by χ and χ' (instead of φ and φ'). Hence in view of 4.5, χ' is an automorphism of the cyclically ordered set $C(G)$. According to the constructions of φ' and χ' we have $\varphi'(g) = \chi'(g)$ for each $g \in G$. Thus in view of 4.1 (ii) we obtain

$$\varphi'(h) = \chi'(h) \tag{1}$$

for each $h \in C(G)$.

Analogously as in 4.5 we put $\chi'(g_2) = g_1 +_{02} g_2$ for each $g_2 \in C(G)$. According to 4.5 we have

$$g_1 +_2 g_2 = g_1 +_{02} g_2$$

for each $g_2 \in G_1$. Hence in view of (1) we infer that $g_1 +_1 g_2 = g_1 +_2 g_2$ for each $g_2 \in G_1$. Summarizing, we get

4.6. Proposition. *Let* $G_1, G_2 \in \mathscr{C}(G)$, $G_1 \subseteq G_2$. *Then* G_1 *is a subgroup of* G_2.

4.7. Corollary. *Let* $G_1, G_2 \in \mathscr{C}(G)$. *Then* $G_1 \subseteq G_2 \Leftrightarrow G_1 \leqq G_2$.

4.8. Corollary. *If there exists $G_c \in \mathscr{C}(G)$ such that $G_1 \subseteq G_c$ for each $G_1 \in \mathscr{C}(G)$, then G_c is the completion of the cyclically ordered group G.*

Let x be an element of $D(G_0)$. Let X be the set of all elements of G_0 such that $x_i \leqq x$ is valid for each $x_i \in X$. There exists a uniquely determined regular cut $y = <$ in $C(G)$ such that $g < x_i$ whenever $x_i \in X$ and $g \in G \setminus X$. The mapping $x \to y$ is an injection of $D(G_0)$ into $C(G)$; we shall identify the elements x and y. In this sense we consider $D(G_0)$ as a subset of $C(G)$. The following lemma is easy to verify.

4.9. Lemma. *Let $a, b \in D(G_0)$, $c \in C(G)$, such that $a < b$ holds in $D(G_0)$ and $[a, c, b]$ is valid in $C(G)$. Then $c \in D(G_0)$ and $a < c < b$ holds in $D(G_0)$.*

5. THE LINEARLY ORDERED GROUP $m(G_0)$

We continue assuming that $G_0 \neq \{0\}$.

Let g_1 and g_2 be elements of $D(G_0)$. Hence there are subsets X and Y of G_0 such that $X = (X^u)^l$, $Y = (Y^u)^l$, $g_1 = X$, $g_2 = Y$. We define $g_3 = g_1 + g_2$ by putting

$$g = ((X + Y)^u)^l .$$

In particular, if g_1 and g_2 belong to G_0, then the operation $g_1 + g_2$ in $D(G_0)$ coincides with the original operation $g_1 + g_2$ in G_0 (under the natural embedding $G_0 \to D(G_0)$ mentioned in Section 4).

5.1. Lemma. *With respect to the operation $+$, $D(G_0)$ is a linearly ordered semigroup. The set $m(G_0)$ consisting of all elements of $D(G_0)$ which have inverses in $D(G_0)$ is a linearly ordered group.*

Proof. The fact that $D(G_0)$ is linearly ordered was already observed in Section 4. For the remaining assertions of the lemma cf., e.g., [3], Chap. V, Section 10.

Also, from the definition of $D(G_0)$ we immediately obtain the following two lemmas:

5.2. Lemma. *Let $A \subset D(G_0)$, $A \neq \emptyset$. If A is upper bounded (lower bounded) in $D(G_0)$, then* $\sup A$ ($\inf A$) *exists in $D(G_0)$.*

5.3. Lemma. *Let $h_1, h_2 \in D(G_0)$ and let $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ be subsets of G_0 such that the relations $h_1 = \bigvee_{i\in I} x_i$ and $h_2 = \bigvee_{j\in J} y_j$ hold in $D(G_0)$. Then $h_1 + h_2 = \bigvee_{i,j} (x_i + y_j)$.*

(The assertion dual to 5.3 concerning infima is also valid.)

5.4. Lemma. *Let $\{g_i\}_{i\in I}$ be an upper bounded subset of $D(G_0)$ and $g \in D(G_0)$. Then $g + \bigvee_{i\in I} g_i = \bigvee_{i\in I} (g + g_i)$.*

Proof. There exist subsets $\{x_{ij}\}_{j\in J(i)}$ $(i \in I)$ and $\{y_k\}_{k\in K}$ of G_0 such that in $D(G_0)$ we have $\bigvee_{k\in K} y_k = g$ and for each $i \in I$, $\bigvee_{j\in J(i)} x_{ij} = g_i$. Thus in view of Lemma 5.3,

$$g + g_i = \bigvee_{k\in K} y_k + \bigvee_{j\in J(i)} x_{ij} = \bigvee_{k\in K, j\in J(i)} (y_k + x_{ij}) ,$$

hence

$$\bigvee_{i\in I} (g + g_i) = \bigvee_{i\in I, k\in K, j\in J(i)} (y_k + x_{ij}) .$$

Next, we have

$$g + \bigvee_{i\in I} g_i = \bigvee_{k\in K} y_k + \bigvee_{i\in I, j\in J(i)} x_{ij} = \bigvee_{k\in K, i\in I, j\in J(i)} (y_k + x_{ij}) ,$$

completing the proof.

Analogously we have:

5.4′. Lemma. *Let* $\{g_i\}_{i\in I}$ *and* g *be as in* 5.4. *Then*

$$(\bigvee_{i\in I} g_i) + g = \bigvee_{i\in I} (g_i + g) .$$

The assertions dual to 5.4 and 5.4′ are also valid.

5.5. Lemma. *Let* $g \in G_0$, $G_1 \in \mathscr{C}(G)$, $g_1 \in G_1 \cap D(G_0)$. *Then* $g +_1 g_1 = g + g_1$.

Proof. For $g_1 \in G_0$ the assertion is trivial. Let $g_1 \notin G_0$. Let X (and Y) be the set of $x_i \in G_0$ ($y_j \in G_0$) such that $x_i < g_1 < y_j$. Then $X \neq \emptyset \neq Y$ and the relation

$$\bigvee_i x_i = g_1 = \bigwedge_j y_j$$

holds in $D(G_0)$. In view of 5.3 and the assertion dual to 5.3 we have

$$(1) \qquad \bigvee_i (g + x_i) = g + g_1 = \bigwedge_j (g + y_j) .$$

For each $x_i \in X$ and each $y_j \in Y$ the relation $[x_i, g_1, y_j]$ is valid, thus $[g +_1 x_i, g +_1 g_1, g +_1 y_j]$. According to 4.2, $g +_1 x_i = g + x_i$ and $g +_1 y_j = g + y_j$. Hence

$$[g + x_i,\ g +_1 g_1,\ g + y_j]$$

for each $x_i \in X$ and each $y_j \in Y$. Since $g + x_i, g + y_j \in D(G_0)$ and $g + x_i < g + y_j$, in view of 4.9 we infer that $g +_1 g_1 \in D(G_0)$ and

$$g + x_i < g +_1 g_1 < g + y_j .$$

Therefore in view of (1) we have

$$g + g_1 = \bigvee_i (g + x_i) \leqq g +_1 g_1 \leqq \bigwedge_j (g + y_j) = g + g_1 .$$

Hence we have $g +_1 g_1 = g + g_1$.

Similarly we have:

5.5′. Lemma. *Let* g *and* g_1 *be as in* 5.5. *Then* $g_1 +_1 g = g_1 + g$.

5.6. Lemma. *Let* $G_1 \in \mathscr{C}(G)$ *and let* $g_1, g_2, \in G_1 \cap D(G_0)$. *Then* $g_2 +_1 g_1 = g_2 + g_1$.

Proof. For $g_2 \in G_0$ the assertion holds by 5.5. Let $g_2 \notin G_0$. Let Z (and T) be the set of $z_i \in G_0$ ($t_j \in G_0$) such that $z_i < g_2 < t_j$. Then Z and T are nonempty, and

$$\bigvee_i z_i = g_2 = \bigwedge_j t_j$$

is valid in $D(G_0)$. Hence according to 5.4′ and its dual we obtain

$$(1) \qquad \bigvee_i (z_i + g_1) = g_2 + g_1 = \bigwedge_j (t_j + g_1) .$$

For each $z_i \in Z$ and each $t_j \in T$ we have $[z_i, g_2, t_j]$, hence $[z_i +_1 g_1, g_2 +_1 g_1, t_j +_1 g_1]$. According to 5.5, $z_i +_1 g_1 = z_i + g_1$ and $t_j +_1 g_1 = t_j + g_1$. Thus

$$[z_i + g_1, g_2 +_1 g_1, t_j + g_1]$$

is valid for each $z_i \in Z$ and each $t_j \in T$. Therefore in view of (1) we obtain $g_2 +_1 g_1 = g_2 + g_1$.

5.7. Lemma. *Let $G_1 \in \mathscr{C}(G)$, $g_1 \in G_1 \cap D(G_0)$. Then $-_1 g_1 \in D(G_0)$.*

Proof. There exist $g_2, g_3 \in G_0$ such that $g_2 < g_3$ and $[g_2, g_1, g_3]$. Thus $-g_3 < -g_2$. Clearly $-g_3 = -_1 g_3$ and $-g_2 = -_1 g_2$. Next, we have $[-_1 g_3, -_1 g_1, -_1 g_2]$, hence $[-g_3, -_1 g_1, -g_2]$. Now by analogous reasoning as in the last part of the proof of 5.5 (i.e., by using 4.9) we obtain that $-_1 g_1 \in D(G_0)$.

5.8. Lemma. *Let g_1 be as in 5.7. Then $-_1 g_1$ is the inverse of the element x_1 in the semigroup $D(G_0)$.*

Proof. This is a consequence of 5.7 and 5.6.

From 5.7 and 5.8 we infer:

5.9. Proposition. *Let $G_1 \in \mathscr{C}(G)$, $h \in D(G_0)$. If h has no inverse in $D(G_0)$, then h does not belong to G_1. Let us remark that for $g_1, g_2 \in G$ we have*

$$0 < g_2 - g_1 \in G_0 \Leftrightarrow 0 < -g_1 + g_2 \in G_0 .$$

5.10. Lemma. *Let h be a cut of type G_0 in G and let $G_1 \in \mathscr{C}(G)$. If $h \in G_1$, then there exists $g_1 \in G$ such that $-g_1 +_1 h$ belongs to $m(G_0)$.*

Proof. Suppose that $h \in G_1$. There are elements $g_1, g_2 \in G$ such that $[g_1, h, g_2]$, $-g_1 + g_2 \in G_0$ and $0 < -g_1 + g_2$ in G_0. Hence $[0, -g_1 +_1 h, -g_1 +_1 g_2]$, whence $-g_1 +_1 h \in D(G_0)$ (because of $-g_1 +_1 g_2 = -g_1 + g_2$). Since $-g_1 +_1 +_1 h \in G_1$, in view of 5.9 we infer that $-g_1 +_1 h$ belongs to $m(G_0)$.

6. THE SUBGROUP G_1^*

Suppose that $G_0 \neq \{0\}$.

Let $g \in G$. We introduce a linear order $<$ on the set $g + G_0$ as follows. For $g_1, g_2 \in g + G_0$ we put $g_1 \leqq g_2$ if $-g + g_1 \leqq -g + g_2$ holds in G_0. The linear order $<$ on $g + G_0$ is independent of the choice of the element g of $g + G_0$ and the mapping $\varphi(t) = g + t$ is an isomorphism of the linearly ordered set G_0 onto the linearly ordered set $g + G_0$.

We denote by $T(g)$ the set of those cuts h of type G_0 in G for which there are $g_1, g_2 \in g + G_0$ with $g_1 < g_2$, $[g_1, h, g_2]$. The mapping φ induces uniquely an extension φ' which maps isomorphically the cyclically ordered set $D(G_0)$ onto the cyclically ordered set $T(g)$. For $h_1, h_2 \in T(g)$ we put $h_1 \leqq h_2$ if $(\varphi')^{-1}(h_1) \leqq (\varphi')^{-1}(h_2)$ is valid in $D(G_0)$. We obtain a linear order on $T(g)$ extending the linear order on $g + G_0$.

Let $D_1(G)$ be the set of all cuts of type G_0 in G. Let $h \in D_1(G)$. There exists $g \in G$ such that $h \in T(g)$. Put

$$u(h) = \{x \in T(G) : x \geqq h\}, \quad l(h) = \{x \in T(G) : x \leqq h\}.$$

If $h_1 \in T(g_1)$, $h_2 \in T(g_2)$, $x_1 \in l(h_1)$ and $x_2 \in l(h_2)$ (or $x_1 \in u(h_1)$ and $x_2 \in u(h_2)$), then $x_1 + x_2$ belongs to $T(g_1 + g_2)$. Denote

$$h_1 +^* h_2 = \sup(l(h_1) + l(h_2))$$

(this supremum clearly does exist in $T(g_1 + g_2)$). The verification of the following lemma is a routine.

6.1. Lemma. (i) $h_1 +^* h_2 = \inf(u(h_1) + u(h_2))$.

(ii) *The operation* $+^*$ *on* $D_1(G)$ *is associative.*

(iii) *If* $h_1, h_2, h_3 \in D_1(G)$, $h \in D_1(G)$ *and* $[h_1, h_2, h_3]$, *then* $[h_1 +^* h, h_2 +^* h, h_3 +^* h]$ *and* $[h +^* h_1, h +^* h_2, h +^* h_3]$.

(iv) *If* $G_1 \in \mathscr{C}(G)$ *and* $h_1, h_2 \in G_1$, *then* $h_1 +^* h_2 = h_1 +_1 h_2$.

(v) *If* $h_1, h_2 \in D(G_0)$, *then* $h_1 +^* h_2 = h_1 + h_2$.

The zero element of G is clearly a neutral element of the semigroup $D_1(G)$. The set G_1^* of all elements of $D_1(G)$ having inverses is a cyclically ordered group. From (iv) we infer that for each $G_1 \in \mathscr{C}(G)$ we have $G_1 \leqq G_1^*$. Hence we obtain

6.2. Theorem. *Let* $G_0 \neq \{0\}$. *Then* G_1^* *is a completion of the cyclically ordered group* G.

6.3. Proposition. *Let* $G_0 \neq \{0\}$. *Let* h *be a regular cut in* G. *Then the following conditions are equivalent:*

(i) *h belongs to the completion of the cyclically ordered group G.*

(ii) *h is a cut of type G_0 and there exists $g \in G$ such that $g +^* h$ belongs to the completion of the linearly ordered group G_0.*

Proof. Let (i) be valid. According to 3.5, h is of type G_0. Hence there is $g_1 \in G$ such that $h \in T(g_1)$. Thus there is $h_1 \in D(G_0)$ with $h = g_1 +^* h_1$. In view of 6.2 we have $h \in G_1^*$; clearly g_1 and $-g_1$ belong to G_1^*. Put $g = -g_1$. Then $g +^* h \in G_1^*$. Since $g +^* g_1 = g + g_1$, we obtain $g +^* g_1 = 0$, whence

$$g +^* h = g +^* (g_1 +^* h_1) = (g +^* g_1) +^* h_1 = h_1.$$

Therefore $h_1 \in G_1^*$ and thus the element h_1 has an inverse in G_1^*, hence it has an inverse in $D(G_0)$ (cf. Lemma 5.7 and Lemma 5.8 with $G_1 = G_1^*$) and so it belongs to the completion of the linearly ordered group G_0.

Conversely, suppose that (ii) holds. Hence $g +^* h = h_1$, where $h_1 \in m(G_0)$. Then we have $h = -g +^* h_1$. Put $h' = (-h_1) +^* g$. We obtain $h +^* h' = 0$, whence $h \in G_1^*$. In view of 6.2, (i) is valid.

In the analogous way as in the proof of 6.3 we obtain:

6.4. Proposition. *Let* $a(G_0) \neq \{0\}$. *Let* h *be a regular cut in* G. *Then the following conditions are equivalent:*

(i) *h belongs to the completion of the cyclically ordered group G.*

(ii) *h is of type $a(G_0)$ and there is $g \in G$ such that $g +^* h$ belongs to the completion of the linearly ordered group $a(G_0)$.*

7. COMPLETE CYCLICALLY ORDERED GROUPS

First let us suppose that $G_0 = \{0\}$; this assumption will be applied in 7.1–7.4. Hence (in the notation as in Section 1) the projection of G into L is $\{0\}$. Thus G is isomorphic to a subgroup of K. Therefore without loss of generality we may suppose that G is a subgroup of K (with the inherited cyclic order).

First assume that G is finite. Then clearly $C(G) = G$ and hence we have (recall that if G is finite then $G_0 = \{0\}$).

7.1. Proposition. *Let G be a finite cyclically ordered group. Then $d_1(G) = G$.*

Now suppose that G is infinite. If g_1, g_2 are any distinct elements of G, then there is $g_3 \in G$ with $[g_1, g_3, g_2]$.

Let $h_1 = <_1$ be a cut in G. There exists a uniquely determined real r with $0 \leqq r < 1$ such that for any distinct elements g_1, g_2 of G the relation $g_1 <_1 g_2$ holds if and only if some of the following conditions is fulfilled:

(i) $g_2 < r \leqq g_1$,

(ii) $g_1 < g_2 < r$,

(iii) $r \leqq g_1 < g_2$.

The mapping ψ: $h_1 \to r$ is an isomorphism of the cyclically ordered set $C(G)$ onto the cyclically ordered set K. If $g \in G$, then $\psi(g) = g$. Thus $C(G)$ can be identified with K.

Because G is a subgroup of K we infer that K is an element of $\mathscr{C}(G)$. The symbol $+$ will denote the group operation in both G and K. From 4.6 we infer:

7.2. Lemma. *Let $G_1 \in \mathscr{C}(G)$. Then G_1 is a subgroup of K.*

As a corollary we obtain:

7.3. Theorem. *Let G be an infinite cyclically ordered group. Suppose that $G_0 = \{0\}$. Then $d_1(G)$ is isomorphic to K.*

7.4. Lemma. *Let G be as in 7.3. Assume that G is isomorphic to K. Then $d_1(G) = G$.*

Proof. Since G is isomorphic to K, each regular cut of G belongs to G; hence $d_1(G) = G$.

7.5. Theorem. *Let G be a cyclically ordered group. Then G is complete if and only if some of the following conditions is satisfied:*

(i) *G is finite.*

(ii) *G is isomorphic to K.*

(iii) *$G_0 \neq \{0\}$ and $m(G_0) = G_0$.*

If $a(G_0) \neq \{0\}$, then G is complete if and only if G satisfies the condition

(iv) *$a(G_0)$ is isomorphic to Z or to R.*

Proof. Consider the condition

(α) $d_1(G) = G$.

In view of 7.1 and 7.4 we have (i) ⇒ (α) and (ii) ⇒ (α). If (iii) holds, then $G_1^* = G$ and hence according to 6.2, (α) is valid. Assume that $a(G_0) \neq \{0\}$. Let (iv) be fulfilled. Then $a(G_0)$ is a complete linearly ordered group. It was proved in [5] that in such a case we have $m(G_0) = G_0$ (in [5] it was assumed that the linearly ordered group under consideration was abelian, but the argument remains valid for the non-abelian case as well). Therefore (α) holds.

Conversely, suppose that (α) is valid. By way of contradiction, suppose that neither of the conditions (i)–(iii) is fulfilled. In particular, G is infinite and G fails to be isomorphic to K. Hence in view of 7.3 we have $G_0 \neq \{0\}$. Thus according to 6.2, $G_1^* = G$. In view of 6.3 we infer that $m(G_0) = G_0$, which is a contradiction. Thus some of the conditions (i), (ii) or (iii) holds. Suppose that, moreover, $a(G_0) \neq \{0\}$. Then $G_0 \neq \{0\}$ and hence (iii) must be valid. Now (α) and 6.4 imply that $a(G_0)$ is a complete linearly ordered group, whence (iv) holds.

7.6. Corollary. *Let G be a linearly ordered group. Then the following conditions are equivalent:*

(i) *G (considered as a linearly ordered group) is complete.*

(ii) *G (considered as a cyclically ordered group) is complete.*

Let L_1 and K_1 be as in Section 1. Let us consider the relations between the condition (α) above (cf. the proof of 7.5) and the conditions saying that L_1 or K_1, respectively, is complete. The following examples illustrate the situations which may occur.

7.7. Example. (G is complete, L_1 and K_1 are complete.) Let $L_1 = R$, $K_1 = K$, $G = L_1 \otimes K_1$.

7.8. Example. (G is complete, both L_1 and K_1 fail to be complete.) Let $G^* = R \otimes K$. Let G be the set of all $g \in G^*$, $g = (x, a)$ such that

(i) both x and a are rational numbers;

(ii) $x - a$ is an integer.

Then G is a subgroup of G^*. The largest linearly ordered subgroup G_0 of G consists of all elements $g = (x, a)$ of G such that $a = 0$ and x is an integer; hence G_0 is complete and thus in view of 7.5, G is complete. L_1 is isomorphic to the additive group of all rational numbers with the natural linear order. K_1 is the subgroup of K consisting of all rational numbers a with $0 \leqq a < 1$. Neither L_1 nor K_1 is complete.

7.9. Example. (G and L_1 are complete, K_1 fails to be complete.) Let $L_1 = R$ and let K_1 be as in 7.8. Put $G = L_1 \otimes K_1$. Then G_0 is isomorphic to R, hence in view of 7.5, G is complete.

7.10. Example. (G and K_1 are complete, L_1 fails to be complete.) Let L be the additive group of all rational numbers with the natural linear order. Put $G^* = L \otimes K$ and let G be the subset of G^* which consists of all $(x, a) \in G^*$ having the property such that there exist integers m, n with $mx + na \in Z$ (the multiplication na being performed as in R, i.e., it is not taken mod 1). Then G is a subgroup of G^* and G_0 is isomorphic to Z. In view of 7.5, G is complete. Moreover, L_1 is isomorphic to L (whence it is not complete), K_1 is isomorphic to K (whence it is complete).

7.11. Example. (G, L_1, K_1 fail to be complete.) Let Lbe as in 7.10 and K_1 as in 7.8, $G = L_1 \otimes K_1$.

7.12. Example. (G, L_1 fail to be complete, K_1 is complete.) Let L be as in 7.11, $G = L \otimes K$.

7.13. Example. (G is not complete, L_1 and K_1 are complete.) Let $G^* = R \otimes K$. Let G be the subset of G^* consisting of those $g = (x, a)$ for which $x + a$ is a rational number. Then G is a subgroup of G^*, and G_0 is isomorphic to L_1 from 7.8. Hence in view of 7.5, G is not complete. Both L_1 and K_1 are complete, since L_1 is isomorphic to R and K_1 is isomorphic to K.

7.14. Example. (G and K_1 fail to be complete, L_1 is complete.) Let K_1 be as in 7.8, $G = K_1$, hence $L_1 = \{0\}$.

The question whether it is possible for G and K_1 not to be complete, and for L_1 to be a nonzero complete linearly ordered group, remains open.

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