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SMALL DIRECTED GRAPHS AS NEIGHBOURHOOD GRAPHS

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Local properties of graphs were studied by many authors. The first hint to their study was given by a problem of A. A. Zykov and B. A. Trahtenbrot at the Symposium on Graph Theory in Smolenice [1] in 1963. Survey papers on these properties were written by J. Sedláček [2], [3].

Most results on local properties of graphs concern undirected graphs. Local properties of directed graphs were studied in [4] and [5]; this paper continues this study.

Let G be a directed graph, let v be its vertex. By $N_G(v)$ we denote the subgraph of G induced by the set of all terminal vertices of edges of G whose initial vertex is v. The graph $N_G(v)$ will be called the neighbourhood graph of v in G.

Let $\mathscr{H}$ be the class of all digraphs H with the property that there exists a digraph G such that $N_G(v) \cong H$ for each vertex v of G. The problem to determine $\mathscr{H}$ is the digraph variant of the mentioned problem from Smolenice. We shall study digraphs which belong to $\mathscr{H}$ and have at most three vertices.

Before formulating a theorem we introduce an auxiliary concept. Let m, n be positive integers. By $V(m, n)$ we denote the set of all n-dimensional vectors $(v_1, \ldots, v_n)$ where $v_i \in \{0, 1, \ldots, m - 1\}$ for $i = 1, \ldots, n$. If we perform additions or subtractions with coordinates of these vectors, we consider them modulo m.

Theorem. *Let H be a directed graph with at most three vertices. Then $H \in \mathscr{H}$ if and only if the number of double edges of H is not* 2.

Remark. By a double we mean a pair of edges which join the same pair of vertices and are directed oppositely to each other.

Proof. In Fig. 1 we see all possible directed graphs with at most three vertices

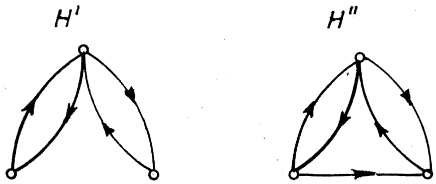

Fig. 1.

and with exactly two double edges. Suppose that $H' \in \mathscr{H}$; let G' be the corresponding graph from the definition of H. Then G' contains at least one induced subgraph isomorphic to H'; let its centre be u_1, its other vertices u_2, u_3. The graph $N_{G'}(u_1)$ contains u_2 and u_3. As $N_{G'}(u_1) \cong H'$, there exists a vertex u_4 in $N_{G'}(u_1)$ joined by double edges with u_2 and u_3. The graph $N_{G'}(u_2)$ contains the vertices u_1, u_4; they are joined by an edge. As H contains only double edges, there must be a double edge joining u_1 and u_4. Now $N_{G'}(u_2)$ must contain the third vertex v joined by a double edge with exactly one of the vertices u_1, u_4. It cannot be u_3, because it is joined by double edges with both u_1, u_4. Thus v is different from all u_1, u_2, u_3, u_4 and is joined by a double edge with u_1 and u_4. But then u_1 or u_4 has the degree at least 4 and its neighbourhood graph has at least 4 vertices, which is a contradiction.

Now suppose that $H'' \in \mathscr{H}$ and let G'' be the corresponding graph. The graph G'' contains an induced subgraph isomorphic to H''; let the common end vertex of two double edges in it be u_1, the initial vertex of the simple edge u_2 and its terminal vertex u_3. Again the graph $N_{G''}(u_1)$ contains a vertex u_4 joined by double edges with u_2 and u_3. The graph $N_{G''}(u_2)$ contains u_1, u_3, u_4 and double edges between u_1 and u_3 and between u_3 and u_4. Thus u_1 and u_4 must be joined by a simple edge. The graph $N_{G''}(u_3)$ contains the vertices u_1, u_4 joined by a simple edge; therefore there must exist a vertex v in $N_{G''}(u_3)$ joined by double edges with u_1 and u_4. It cannot be u_2, because then u_2 and u_3 would be joined by a double edge. Thus v is different from all u_1, u_2, u_3, u_4 and is joined by double edges with u_1 and u_4. But then u_1 has the degree at least 4, which is a contradiction.

Now consider the graphs without double edges. We shall determine the graphs G for particular graphs H. If H is an empty graph, then G is any graph without edges. If H consists of one vertex, then G is any (directed) cycle. If H has two vertices and no edge, then G is the graph whose vertex set is $V(m, 2)$ for $m \geqq 3$ and in which from each (v_1, v_2) edges go to $(v_1 + 1, v_2)$ and $(v_1, v_2 + 1)$. If H has two vertices

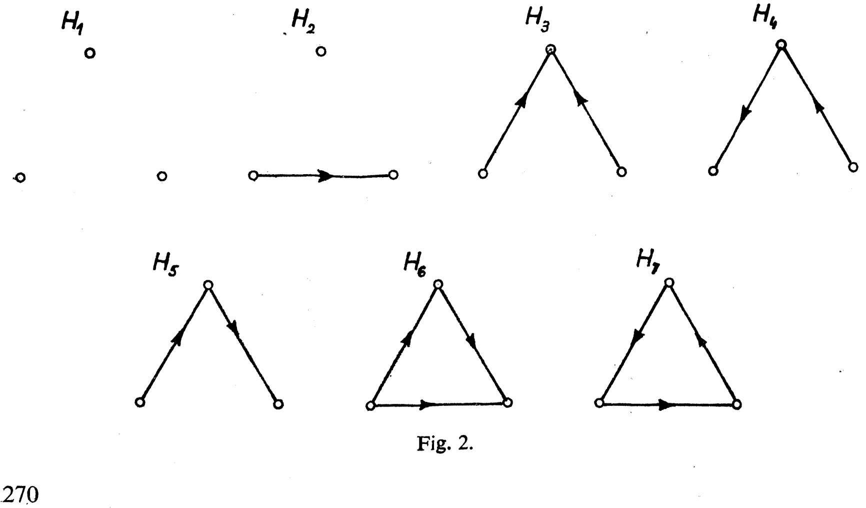

Fig. 2.

and one edge, then G is the graph whose vertex set is $V(m, 1)$ for $m \geqq 5$ and in which from each (v_1) edges go to $(v_1 + 1)$ and $(v_1 + 2)$. All graphs with three vertices without double edges are in Fig. 2. The graph G for H_i will be denoted by G_i for $i = 1, \ldots, 14$. The graph G_1 has the vertex $V(m, 3)$ for $m \geqq 5$ and from each vertex (v_1, v_2, v_3) edges go to $(v_1 + 1, v_2, v_3)$, $(v_1, v_2 + 1, v_3)$, $(v_1, v_2, v_3 + 1)$. The graphs G_2, G_3, G_4, G_5 have the vertex set $V(m, 2)$ for $m \geqq 5$, m even. In G_2 from each (v_1, v_2) edges go to $(v_1 + 1, v_2)$, $(v_1 + 2, v_2)$, $(v_1, v_2 + 1)$. In G_3 from (v_1, v_2) edges go to $(v_1 + 1, v_2)$, $(v_1, v_2 + 1)$, $(v_1 + 1, v_2 + 1)$. In G_4 we distinguish some cases. If both v_1, v_2 are even, then from (v_1, v_2) edges to go $(v_1 + 1, v_2)$, $(v_1, v_2 + 1)$, $(v_1 + 1, v_2 + 1)$. If both v_1, v_2 are odd, then from (v_1, v_2) edges go to $(v_1 - 1, v_2)$, $(v_1, v_2 - 1)$, $(v_1 - 1, v_2 - 1)$. If v_1 is even and v_2 is odd, then from (v_1, v_2) edges go to $(v_1 - 1, v_2)$, $(v_1, v_2 + 1)$ and $(v_1 - 1, v_2 + 1)$. If v_1 is odd and v_2 is even, then from (v_1, v_2) edges go to $(v_1 + 1, v_2)$, $(v_1, v_2 - 1)$, $(v_1 + 1, v_2 - 1)$. In G_5 from each (v_1, v_2) an edge goes to $(v_1, v_2 + 1)$. Further if v_2 is even, then edges go to $(v_1 + 1, v_2)$ and $(v_1 + 1, v_2 + 1)$, and if v_2 is odd, then to $(v_1 - 1, v_2)$ and $(v_1 - 1, v_2 + 1)$. The graphs H_6 and H_7 are tournaments and the assertion on them follows from the results in [4]. But we may say that G_6 has the vertex set $V(m, 1)$ for $m \geqq 7$ and from (v_1) edges go to $(v_1 + 1)$, $(v_1 + 2)$, $(v_1 + 3)$. The graph G_7 is a tournament on 7 vertices $u_1, \ldots, u_7$ in which from u_i edges go to $u_{i+2}, u_{i+4}, u_{i+6}$ for $i = 1, \ldots, 7$, the sums being taken modulo 7.

Now we turn to graphs with one or three double edges. If H is a graph with two vertices and one double edge, the corresponding graph G is the complete digraph with three vertices. The digraphs with three vertices and one or three double edges are in Fig. 3. The graph G_8 is obtained from the undirected graph of a trilateral prisma by replacing each undirected edge by a pair of oppositely directed edges. The graph G_9 has the vertex set $V(m, 1)$ for $m \geqq 5$ and in it from the vertex (v_1) edges go to $(v_1 - 1)$, $(v_1 + 1)$, $(v_1 + 2)$. The graph G_{10} has the vertex set $V(m, 1)$

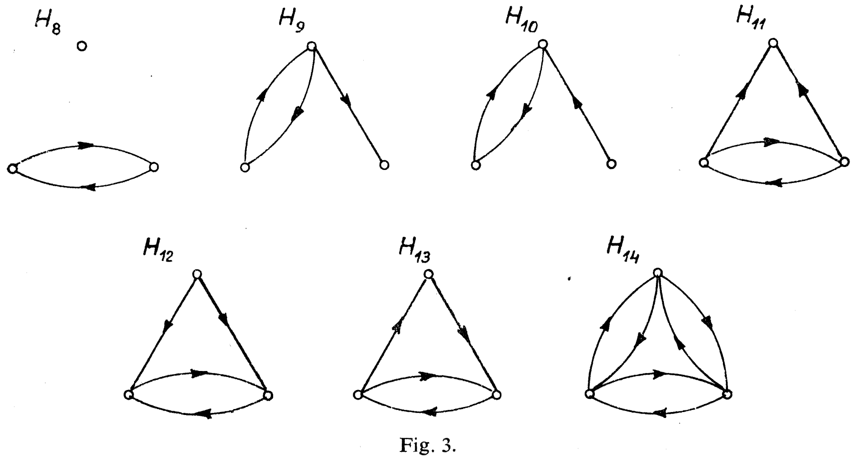

Fig. 3.

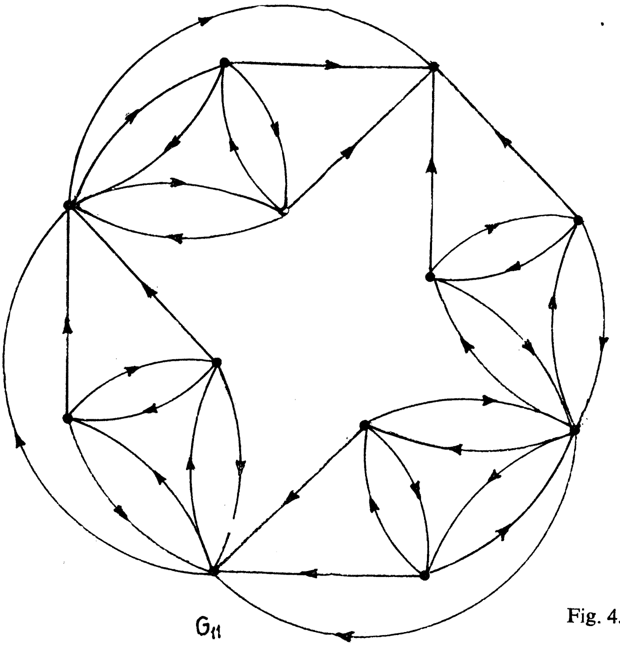

Fig. 4.

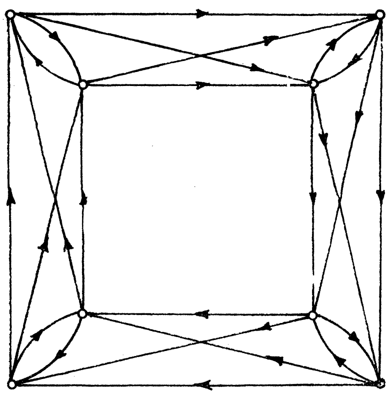

Fig. 5.

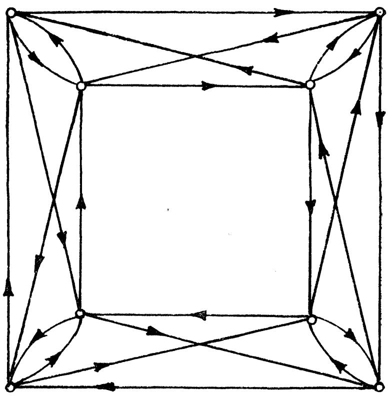

Fig. 6.

for $m \geqq 6$, m even. From (v_1) the edges go also to $(v_1 - 1)$ and $(v_1 + 1)$; further for v_1 even an edge goes to $(v_1 + 2)$ and for v_1 odd an edge goes to $(v_1 - 2)$. The graph G_{11} is in Fig. 4, the graph G_{12} is in Fig. 5, the graph G_{13} is in Fig. 6. The graph G_{14} is the complete directed graph with four vertices.

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