Czechoslovak Mathematical Journal

Vojtěch Bartík; Martin Markl

On bijectivity of the canonical transformation $[\beta_G X; Y]_G \to [X; Y]_G$

Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 4, 682–700

Persistent URL: `http://dml.cz/dmlcz/102264`

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

ON BIJECTIVITY OF THE CANONICAL TRANSFORMATION $[\beta_G X; Y]_G \to [X; Y]_G$

VOJTĚCH BARTÍK and MARTIN MARKL, Praha

(Received November 10, 1986)

0. INTRODUCTION

In [2] (see also [6]), necessary and sufficient conditions on a CW-complex Y were found in order that the canonical transformation

$$[\beta X; Y] \to [X; Y], \tag{0.1}$$

induced by the inclusion map of a completely regular space X into its Čech-Stone compactification βX, be bijective for all completely regular spaces X satisfying the condition $\dim \beta X < \infty$. By [16], the Čech-Stone compactification has an analog in the category of completely regular G-spaces, where G is a compact Lie group. It is therefore natural to ask whether a similar result holds for the canonical transformation

$$[\beta_G X; Y]_G \to [X; Y]_G \tag{0.2}$$

between the sets of G-homotopy classes of G-maps, where G is a compact Lie group, $\beta_G X$ is the Čech-Stone G-compactification, in the sense of [16], of a completely regular G-space X, and Y is a G-CW-complex in the sense of [11]. The aim of this paper is to prove that (0.2) is bijective for all completely regular G-spaces X satisfying the condition $\dim \beta(X/G) < \infty$ if and only if Y satisfies certain simple homotopy conditions, which represent a very natural generalization of the conditions mentioned above and reduce to them in the case of G a trivial group.

The paper is organized as follows. Our main result, Theorem 1.3, some simple corollaries to it and its more compact equivalent formulation, Theorem 1.8, are stated in Section 1. In Sections 2–4, necessary technical tools are developed, and in Section 5, the proof of Theorem 1.3 is accomplished. Finally, in Section 6, a result from dimension theory is proved, which is needed in Section 4 and which we have not been able to find in literature although it is likely to be well-known.

Our proof of Theorem 1.3 is based on some recent results of M. Murayama [15], S. Warner [18] and the second author [9] but also on results of T. Matumoto [11] and the first author [2]. It follows, however, [6] and [7] rather than [2] because the more direct method of [2] proved to be less suitable for a translation into the category of G-spaces and G-maps than the method of A. Calder and J. Siegel based on the notions of the relative compressibility property and the bounded lifting property. Although the idea of the proof is the same as in the non-equivariant case, its technical realization is far from being a straightforward copy of the proof of the corresponding theorem on the transformation (0.1). Difficulties that one meets when trying to carry over this proof into the category of G-spaces and G-maps are due to the fact that some results and tools used in it fail to have obvious analogs in this category if G is not finite. This concerns e.g. the bridge mapping theorem (see [1] or [8, Appendix]) and the fact that a bridge of a map, i.e. the geometric nerve of a suitable covering, has a canonical *CW*-complex structure, which both play an important role in the proof of surjectivity of (0.1) for X satisfying $\dim \beta X < \infty$. The work of M. Murayama [15] suggested how to define a bridge and a bridge mapping for a G-map, but still a difficulty has remained caused by the fact that Murayama's geometric nerve of a *TN* G-covering fails to have a canonical G-*CW*-complex structure which behave reasonably with respect to its skeletons.

The first version of Theorem 1.3, dealing only with paracompact G-spaces, was proved as early as in 1983. In the present form, the results of this paper were announced at 4th International Conference "Topology and its Applications", Dubrovnik, Sept. 30 – Oct. 5, 1985 [3].

In terminology and notation we closely follow [16], [11] and [15]. All spaces are supposed to be Hausdorff and G always denotes a compact Lie group.

1. MAIN RESULTS

1.1. Let $\mathscr{S}(G)$ denote the set of the conjugacy classes of all closed subgroups of G, and let (H) denote the conjugacy class of a closed subgroup H.

For a subset $\mathscr{S}$ of $\mathscr{S}(G)$ we shall denote by $\mathscr{S}^{\sim}$ the set of all conjugacy classes of the form $(\bigcap\{H_i \mid i \in I\})$ where $\{H_i \mid i \in I\}$ is an arbitrary family of closed subgroups of G such that $(H_i) \in \mathscr{S}$ for all $i \in I$. Clearly $\mathscr{S} \subset \mathscr{S}^{\sim}$, and we shall say that $\mathscr{S}$ is closed with respect to intersections if $\mathscr{S} = \mathscr{S}^{\sim}$.

Since G is a compact Lie group, $\mathscr{S}$ is obviously closed with respect to intersections if and only if it is closed with respect to finite intersections, i.e. if and only if $(\bigcap\{H_i \mid i \in I\}) \in \mathscr{S}$ for every finite family $\{H_i \mid i \in I\}$ of closed subgroups of G such that $(H_i) \in \mathscr{S}$ for all $i \in I$.

Moreover, using [4, Chap. VII, Theorems 1.1 and 2.1] one can easily show that the finiteness of $\mathscr{S}$ implies the finiteness of $\mathscr{S}^{\sim}$. (The proof can be found e.g. in [9, p. 542].)

Finally, given $\mathscr{S} \subset \mathscr{S}(G)$ and a closed subgroup H of G contained in some closed subgroup H' of G with $(H') \in \mathscr{S}$, we shall denote by $H(\mathscr{S})$ the intersection of all such H'.

1.2. By [16, Theorem 1.5.4], for every completely regular G-space X there exists a compact G-space X^* with the following properties:

(a) X is a dense G-subspace of X^*,

(b) every G-map of X into a compact G-space Y extends to a G-map of X^* into Y.

Any such space X^* is denoted by $\beta_G X$ and called the Čech-Stone G-compactification of X. It is clear that $\beta_G X$ is uniquely determined by X up to a canonical G-homeomorphism identical on X, and that $\beta_G X$ coincides, as a topological space, with the usual Čech-Stone compactification of X if the group G is finite.

Now we are ready to state our main results on the canonical transformation

$$j_X^*\colon [\beta_G X; Y]_G \to [X; Y]_G \tag{1.1}$$

induced by the inclusion map $j_X\colon X \hookrightarrow \beta_G X$.

1.3. Theorem. *Let Y be a G-space having the G-homotopy type of a G-CW-complex in the sense of* [11]. *Then the canonical transformation* (1.1) *is bijective for all completely regular G-spaces X with* $\dim \beta(X/G) < \infty$ *if and only if Y satisfies the following two conditions*:

(a) *The set $\pi_0(Y^H, y_H)$ and the group $\pi_1(Y^H, y_H)$ are finite and the groups $\pi_i(Y^H, y_H)$ $(i = 2, 3, \ldots)$ are finitely generated for each closed subgroup H of G with $Y^H \neq \emptyset$ and for each base point $y_H \in Y^H$.*

(b) *There exist finite subsets $\mathscr{S}_0, \mathscr{S}_1, \mathscr{S}_2, \ldots$ of $\mathscr{S}(G)$ closed with respect to intersections and such that for each closed subgroup H of G with $Y^H \neq \emptyset$ and for each $n = 0, 1, 2, \ldots$ the group $H(\mathscr{S}_n)$ is defined* (see 1.1) *and the inclusion map $Y^{H(\mathscr{S}_n)} \hookrightarrow Y^H$ is an n-equivalence* (*in the usual sense, see* [17, p. 404]).

The proof of this theorem will be given in Section 5.

1.4. Remarks. (a) By [16, Proposition 1.1.8], the orbit space X/G of a completely regular G-space X is also completely regular, and so it makes sense to speak of the Čech-Stone compactification $\beta(X/G)$.

(b) If X is a completely regular space then $\dim \beta X$ coincides with the "normal covering dimension" $\dim_N X$ defined by means of finite normal coverings of X, which was studied by K. Morita in [14]. We recall that by one of his results [14, Theorem 1.3] $\dim_N X \leqq n$ if and only if every normal covering of X has a normal refinement of dimension at most n. (We will make use of this result in Section 2.)

We now show that in many interesting cases the conditions (a) and (b) of Theorem 1.3 can be considerably simplified.

1.5. Corollary. *Let us suppose that Y is as in Theorem* 1.3. *If Y has only finitely many orbit types then the transformation* (1.1) *is bijective for every completely regular G-space X with* $\dim_N X/G < \infty$ *if and only if Y satisfies the condition* (a) *of Theorem* 1.3.

Proof. It suffices to show that Y satisfies the condition (b) of Theorem 1.3. Let $\mathscr{S}$ denote the set of the conjugacy classes of all isotropy groups G_y, $y \in Y$. $\mathscr{S}$ is finite by our assumption, and if H is any closed subgroup of G then clearly $Y^H = Y^{H(\mathscr{S})} =$

$= Y^{H(\mathscr{S}^\sim)}$ provided $Y^H \neq \emptyset$. Since $\mathscr{S}^\sim$ is finite by 1.1 and closed with respect to intersections, we see that the required condition is trivially satisfied for $\mathscr{S}_0 = = \mathscr{S}_1 = \ldots = \mathscr{S}^\sim$.

In the next corollary we shall need the notion of a G-CW-complex of G-finite type. We recall that a G-CW-complex Y is said to be of G-finite type if all its skeletons $Y^0, Y^1, Y^2, \ldots$ are G-finite G-CW-complexes in the sense of [11, Definition (1.4)]. Clearly, Y is of G-finite type if and only if the CW-complex Y/G is of finite type in the usual sense.

1.6. Corollary. *Let us suppose that a G-space Y satisfies one of the following two conditions:*

(a) *Y is G-homotopy equivalent to a G-CW-complex of G-finite type;*

(b) *Y is compact and G-homotopy equivalent to a G-CW-complex.*

Then (1.1) is bijective for every completely regular G-space X with $\dim_N X/G < \infty$ if and only if the group $\pi_1(Y^H, y_H)$ is finite for all closed subgroups H of G with $Y^H \neq \emptyset$ and for all base points $y_H \in Y^H$.

Proof. We need to show that in both cases Y satisfies the conditions (a) and (b) of Theorem 1.3. Let H denote a closed subgroup of G such that $Y^H \neq \emptyset$, and let n be a natural number.

First let us consider the case (a). We may clearly suppose that Y itself is a G-CW-complex of G-finite type. Let us denote by $\mathscr{S}_n$ the set of the conjugacy classes of all isotropy groups G_y for y belonging to the n-skeleton Y^n of Y. We shall show that the sets $\mathscr{S}_0^\sim, \mathscr{S}_1^\sim, \mathscr{S}_2^\sim, \ldots$ have the properties required in the condition (b) of Theorem 1.3. This is, however, easy. The finiteness of $\mathscr{S}_n^\sim$ follows immediately from the G-finiteness of Y^n and from 1.1, and the assertion that the inclusion map $Y^{H(\mathscr{S}_n^\sim)} \hookrightarrow Y^H$ is an n-equivalence follows from the observation that $\emptyset \neq (Y^n)^H = (Y^n)^{H(\mathscr{S}_n)} = (Y^n)^{H(\mathscr{S}_n^\sim)}$ and from [11, Proposition (4.3)], which implies that the inclusion maps $(Y^n)^H \hookrightarrow Y^H$, $(Y^n)^{H(\mathscr{S}_n^\sim)} \hookrightarrow Y^{H(\mathscr{S}_n^\sim)}$ are n-equivalences. It remains to prove that the sets $\pi_0(Y^H, y_H)$ are finite and the groups $\pi_i(Y^H, y_H)$ $(i = 2, 3, \ldots)$ are finitely generated for all $y_H \in Y^H$. Since the inclusion map $(Y^n)^H \hookrightarrow Y^H$ is an n-equivalence, it suffices to prove instead that the sets $\pi_0((Y^n)^H, y_H)$ are finite and the groups $\pi_i((Y^n)^H, y_H)$ $(i = 2, 3, \ldots)$ are finitely generated for all $y_H \in (Y^n)^H$ and all $n \geqq 2$. This follows, however, from the compactness of $(Y^n)^H$ and from [2, Proposition 2.1], because by [18, Corollary 4.13] each space $(Y^n)^H$ is homotopy equivalent to a CW-complex.

In the case (b) there is a diagram

$$Y \xrightarrow{h} K \xrightarrow{g} Y',$$

where Y' is a G-CW-complex, K is a G-finite G-CW-subcomplex of Y', h is a G-map, g is the inclusion map, and the composition $f = g \circ h$ is a G-homotopy equivalence. The induced diagram

$$Y^H \xrightarrow{h^H} K^H \xrightarrow{g^H} (Y')^H$$

has similar properties: the space K^H is compact, the composition $f^H = g^H \circ h^H$ is a homotopy equivalence, and [18, Corollary 4.13] implies that K^H and $(Y')^H$ are homotopy equivalent to CW-complexes. These properties clearly imply that the set $\pi_0(Y^H, y_H)$ is finite. Moreover, they mean that we may apply [2, Proposition 2.1] to $(Y')^H$, which immediately yields that the groups $\pi_i(Y^H, y_H) \approx \pi_i((Y')^H, f(y_H))$ $(i = 2, 3, \ldots)$ are finitely generated for all $y_H \in Y^H$. In order to verify that Y satisfies also the condition (b) of Theorem 1.3, let us denote by $\mathcal{S}$ the set of the conjugacy classes of all isotropy groups G_y $(y \in K)$ and consider the diagram

$$\begin{array}{ccccc} Y^{H(\mathcal{S}^\sim)} & \xrightarrow{h^{H(\mathcal{S}^\sim)}} & K^{H(\mathcal{S}^\sim)} & \xrightarrow{g^{H(\mathcal{S}^\sim)}} & (Y')^{H(\mathcal{S}^\sim)} \\ \cap\!\downarrow & & \cap\!\downarrow & & \cap\!\downarrow \\ Y^H & \xrightarrow{h^H} & K^H & \xrightarrow{g^H} & (Y')^H \end{array}$$

Since the compositions $f^{H(\mathcal{S}^\sim)} = g^{H(\mathcal{S}^\sim)} \circ h^{H(\mathcal{S}^\sim)}$ and $f^H = g^H \circ h^H$ are homotopy equivalences and, obviously, $K^{H(\mathcal{S}^\sim)} = K^H$, this diagram implies that the inclusion map $Y^{H(\mathcal{S}^\sim)} \hookrightarrow Y^H$ is a weak homotopy equivalence. Since $\mathcal{S}^\sim$ is finite by the G-finiteness of K and by 1.1, we are done.

1.7. Remark. In fact, if G and Y are as in Theorem 1.3 and Y satisfies both its conditions (a) and (b), then conversely, Y is G-homotopy equivalent to a G-CW-complex of G-finite type. This follows immediately from [9, Theorem 2.3] and from the equivariant version of J. H. C. Whitehead's theorem [11, Theorem (5.3)], and will be used in the proof of Theorem 1.3 in Section 5.

It follows from Corollary 1.6 and Remark 1.7 that Theorem 1.3 can be reformulated in the following equivalent way.

1.8. Theorem. (An equivalent version of 1.3). *Let Y be a G-space which is G-homotopy equivalent to a G-CW-complex. Then* (1.1) *is bijective for every completely regular G-space X satisfying the condition* $\dim_N X/G < \infty$ *if and only if Y has the G-homotopy type of a G-CW-complex of G-finite type and the group $\pi_1(Y^H, y_H)$ is finite for each closed subgroup H of G and for each point $y_H \in Y^H$.*

1.9. Corollary. *If Y is a compact differentiable G-space in the sense of* [16, Definition 1.1.19], *then* (1.1) *is bijective for all completely regular G-spaces X with* $\dim_N X/G < \infty$ *if and only if the group $\pi_1(Y^H, y_H)$ is finite for each closed subgroup H of G and for each point $y_H \in Y^H$.*

Proof. This is a special case of 1.6 because it is known, see e.g. [10, Proposition (4.4)], that Y has a G-CW-complex structure.

2. A WEAK EQUIVARIANT VERSION OF THE BRIDGE-MAPPING THEOREM

We start with a brief review of some definitions and results of M. Murayama concerning TN G-coverings and their nerves. If not otherwise stated, we use the terminology of [15, Sections 1 and 2].

2.1. Let Y be a G-space and $O = Gy \subset Y$ a G-orbit. A G-tube about O is defined to be a pair $T = (T, r)$ consisting of an open G-neighbourhood T of O and a G-retraction $r\colon T \to O$. The orbit O is then called the *central orbit* of T.

When V is an open subset of O, the open subset $S = r^{-1}(V)$ of Y is called the *open tube segment of* T generated by V. Then, for each $g \in G$, gS is an open tube segment of T generated by gV, and $gS \cap 0 = gV$.

Clearly, any open G-neighbourhood T' of O in T is also a tube about O with the G-retraction $r' = r \mid T'$. If Y is completely regular, then by the Mostow theorem (see [5, Chap. II, Theorem 5.4] or [16, Corollary 1.7.19]) the open tube segments form an open base for the topology of Y.

Modifying slightly the original definition of Murayama, we now define an open tube-segmental G-covering of Y as a pair

$$\mathscr{S} = (\{S_\lambda \mid \lambda \in \Lambda\},\ \{(T_\alpha, r_\alpha, O_\alpha) \mid \alpha \in \Lambda/G\}) \tag{2.1}$$

where $\{S_\lambda \mid \lambda \in \Lambda\}$ is an open G-covering of Y, $T_\alpha = \bigcup\{S_\lambda \mid \lambda \in \alpha\}$, (T_α, r_α) is a G-tube with the central orbit O_α and S_λ is an open tube segment of T_α for all $\alpha \in \Lambda/G$ and $\lambda \in \alpha$. If, moreover, there exists a G-invariant locally finite partition of unity $\{p_\alpha \mid \alpha \in \Lambda/G\}$ on Y such that $p_\alpha^{-1}((0, 1]) \subset T_\alpha$ for all $\alpha \in \Lambda/G$, then $\mathscr{S}$ is called a *TN* G-covering (tubular numerable G-covering).

2.2. Lemma ([15, Proposition 2.3]). *For every open G-covering $\mathscr{V}$ of a paracompact G-space Y there exists a TN G-covering* (2.1) *of Y with $\{S_\lambda \mid \lambda \in \Lambda\}$ a star-refinement of $\mathscr{V}$.*

2.3. We shall now modify M. Murayama's definition of the simplicial G-nerve and the (geometric) G-nerve of a *TN* G-covering.

Let $f\colon X \to Y$ be a G-map, (2.1) an open tube-segmental G-covering of Y and $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda/G\}$ an open G-invariant covering of X such that $f(U_\alpha) \subset T_\alpha$ for all $\alpha \in \Lambda/G$. Let us denote by $N = N(\mathscr{U})$ the nerve of $\mathscr{U}$ and by $N_n = N_n(\mathscr{U})$ the set of n-simplexes of N, and assume that Λ/G is partially ordered in such a way that the induced order on the set of the vertices of each simplex of N is linear.

For each n-simplex σ of N with vertices $\alpha_0 < \ldots < \alpha_n$ let $K_\sigma = K_\sigma(\mathscr{U}, f, \mathscr{S})$ denote the open G-subspace of $O_{\alpha_0} \ldots \times O_{\alpha_n}$ defined by

$$K_\sigma = \bigcup\{O_{\lambda_0} \times \ldots \times O_{\lambda_n} \mid \lambda_i \in \alpha_i,\ i = 0, \ldots, n,\ \bigcap_{i=0}^{n} S_{\lambda_i} \neq \emptyset\}$$

where O_λ denotes the open subset $r_\alpha(S_\lambda) = S_\lambda \cap O_\alpha$ of O_α for each $\lambda \in \alpha$ and $\alpha \in \Lambda/G$.

We now define a simplicial G-space (without degeneracy operators) $K_* = K_*(\mathscr{U}, f, \mathscr{S})$ as follows: The n-th space K_n of K_* $(n \geqq 0)$ is given by

$$K_n = \coprod_{\sigma \in N_n} K_\sigma$$

where $\coprod$ denotes disjoint union, and the i-th face operator $\partial_i\colon K_n \to K_{n-1}$ $(n = 1, 2, 3, \ldots;\ i = 0, 1, \ldots, n)$ acts by omitting the i-th term.

The geometric realization $|K_*(\mathscr{U}, f, \mathscr{S})| = \coprod_{n=0}^{\infty} K_n(\mathscr{U}, f, \mathscr{S}) \times \Delta^n/\underset{n}{\sim}$ of the simplicial G-space $K_*(\mathscr{U}, f, \mathscr{S})$ is denoted by $K(\mathscr{U}, f, \mathscr{S})$. The image of $\coprod_{i=0}^{n} K_i(\mathscr{U}, f, \mathscr{S}) \times \Delta^i$ in $K(\mathscr{U}, f, \mathscr{S})$ is denoted by $K^n(\mathscr{U}, f, \mathscr{S})$ and called the *n-skeleton of* $K(\mathscr{U}, f, \mathscr{S})$. If $\dim N(\mathscr{U}) \leqq n$, then clearly $K^i(\mathscr{U}, f, \mathscr{S}) = K(\mathscr{U}, f, \mathscr{S})$ for $i \geqq n$.

Since $K_*(\mathscr{U}, f, \mathscr{S})$ has no degeneracy operators, the inclusions $K^n(\mathscr{U}, f, \mathscr{S}) \subsetneq K^{n+1}(\mathscr{U}, f, \mathscr{S})$ are G-cofibrations and $K(\mathscr{U}, f, \mathscr{S})$ is a Hausdorff G-space.

The image of $(x, t) \in K_n \times \Delta^n$ in $K(\mathscr{U}, f, \mathscr{S})$ is denoted by $|x, t|$ and also by $|x_{\alpha_0}, \ldots, x_{\alpha_n};\ t_0, \ldots, t_n|$ if $x = (x_{\alpha_0}, \ldots, x_{\alpha_n}) \in K_\sigma \subset O_{\alpha_0} \times \ldots \times O_{\alpha_n}$ and $t_0, \ldots, t_n$ are the barycentric coordinates of $t \in \Delta^n$.

2.4. It is easy to see that the construction just described in 2.3 has the following property: If $g: X' \to X$ is a G-map and $\mathscr{U}' = \{U'_\alpha \mid \alpha \in \Lambda/G\}$ an open G-invariant covering of X' such that $g(U'_\alpha) \subset U_\alpha$ for all $\alpha \in \Lambda/G$, then the obvious canonical map

$$K(\mathscr{U}', f \circ g, \mathscr{S}) \to K(\mathscr{U}, f, \mathscr{S})$$

is a G-homeomorphism onto a closed G-subspace of $K(\mathscr{U}, f, \mathscr{S})$ and a G-cofibration.

Consequently, we can identify $K(\mathscr{U}', f \circ g, \mathscr{S})$ with a G-subspace of the G-space $K(\mathscr{U}, f, \mathscr{S})$, and, in particular, $K(\mathscr{U}, f, \mathscr{S})$ can be regarded as a G-subspace of the G-nerve $K(\mathscr{S})$ of $\mathscr{S}$ in the sense of M. Murayama.

Using this identification and denoting the covering $\{T_\alpha \mid \alpha \in \Lambda/G\}$ by $\mathscr{T}$, one can also easily show that $K(\mathscr{U}, f, \mathscr{S})$ coincides with $p^{-1}(|N(\mathscr{U})|)$, where $p: K(\mathscr{S}) \to |N(\mathscr{T})|$ is the canonical projection and $|N(\mathscr{U})|$ is identified in a canonical way with a subspace of $|N(\mathscr{T})|$.

2.5. Lemma. *Let $f, X, Y, \mathscr{S}$ and $\mathscr{U}$ be as in* 2.3, *and let us suppose that $\mathscr{U}$ is numerable.*

(a) *If $\{p_\alpha \mid \alpha \in \Lambda/G\}$ is a G-invariant locally finite partition of unity on X, which is strictly subordinated to $\mathscr{U}$ in the sense that $p_\alpha^{-1}((0, 1]) \subset U_\alpha$ for all $\alpha \in \Lambda/G$, then the formula*

$$P(x) = |r_{\alpha_0}(f(x)), \ldots, r_{\alpha_n}(f(x));\ p_{\alpha_0}(x), \ldots, p_{\alpha_0}(x)|\,,$$

where $x \in X$, $\{\alpha_0, \ldots, \alpha_n\} = \{\alpha \in \Lambda/G \mid p_\alpha(x) \neq 0\}$ and $\alpha_0 < \ldots < \alpha_n$, defines a G-map $P: X \to K(\mathscr{U}, f, \mathscr{S})$ such that $P(X') \subset K(\mathscr{U}', f', \mathscr{S})$ for every G-subspace X' of X, $\mathscr{U}' = \mathscr{U} \cap X' = \{U_\alpha \cap X' \mid \alpha \in \Lambda/G\}$ and $f' = f|X'$.

We shall say that P is a canonical G-map associated to $\{p_\alpha \mid \alpha \in \Lambda/G\}$.

(b) *If $\{p'_\alpha \mid \alpha \in \Lambda/G\}$ is another G-invariant locally finite partition of unity on X strictly subordinated to $\mathscr{U}$ and if P' is the canonical G-map associated to it, then the formula*

$$H(x, t) = |r_{\alpha_0}(f(x)), \ldots, r_{\alpha_n} f(x));\ (1 - t)\, p_{\alpha_0}(x) + t\, p'_{\alpha_0}(x), \ldots$$
$$\ldots, (1 - t)\, p_{\alpha_n}(x) + t\, p'_{\alpha_n}(x)|\,,$$

where x and $\alpha_0, \ldots, \alpha_n$ are as in (a) *and $t \in [0, 1]$, defines a G-homotopy $H: X \times$*

$\times [0, 1] \to K(\mathscr{U}, f, \mathscr{S})$ *from* P *to* P' *such that* $H(X' \times [0, 1]) \subset K(\mathscr{U}, f', \mathscr{S})$ *for every G-subspace* X' *of* X, $\mathscr{U}' = \mathscr{U} \cap X'$ *and* $f' = f \mid X'$.

Proof. See the proof of Proposition 2.4 in [15].

2.6. Definition. Let $f, X, Y, \mathscr{S}$ and $\mathscr{U}$ be as in 2.3. The G-space $K(\mathscr{U}, f, \mathscr{S})$ is called a *G-bridge for* f if the covering $\mathscr{U}$ is numerable and there exists a G-map $F: K(\mathscr{U}, f, \mathscr{S}) \to Y$ such that the G-maps f and $F \circ P$ are G-homotopic for any canonical G-map $P: X \to K(\mathscr{U}, f, \mathscr{S})$. Every G-map F with this property is called a *G-bridge mapping for* f.

2.7. Proposition. *Let* $f: X \to Y$ *be a G-map, let* $X' \subset X$ *and* $Y' \subset Y$ *be G-subspaces such that* $f(X') \subset Y'$, *and let us suppose that* Y *and* Y' *are G-ANR's and that* Y' *is closed in* Y. *Then there exist a G-bridge* $K(\mathscr{U}, f, \mathscr{S})$, *where* $\mathscr{S}$ *is of the form* (2.1) *and* $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda/G\}$, *and a G-bridge mapping* $F: K(\mathscr{U}, f, \mathscr{S}) \to Y$ *for* f *with the following properties*:

(a) *If* $\mathscr{U}' = \mathscr{U} \cap X' = \{U_\alpha \cap X' \mid \alpha \in \Lambda/G\}$ and $f' = f \mid X'$, *then* $F(K(\mathscr{U}', f', \mathscr{S})) \subset Y'$.

(b) *If* P *is any canonical G-map from* X *into* $K(\mathscr{U}, f, \mathscr{S})$, *then* $P(X')$ *is contained in* $K(\mathscr{U}', f', \mathscr{S})$ *and there is a G-homotopy* $H: X \times [0, 1] \to Y$ *from* t *to* $F \circ P$ *such that* $H(X' \times [0, 1]) \subset Y'$.

(c) *If* $\mathscr{V} = \{V_\alpha \mid \alpha \in \Lambda/G\}$ *is any numerable G-invariant open covering of* X *such that* $V_\alpha \subset U_\alpha$ *for all* $\alpha \in \Lambda/G$, *then the space* $K(\mathscr{V}, f, \mathscr{S})$ *and the map* $F \mid K(\mathscr{V}, f, \mathscr{S})$ *also have the properties* (a) *and* (b).

Proof. By the equivariant version [15, Theorem 6.2] of the Wojdysławski theorem, we may suppose that Y is a G-invariant closed subspace of a G-invariant convex subspace C of a Banach G-space. Since Y is a G-ANR, it follows that there exists a G-retraction $r: W \to Y$, where W is a G-invariant open neighbourhood of Y in C.

Since Y and Y' are ANR's and Y' is closed in Y, there exist a G-homotopy $h: Y \times [0, 1] \to Y$ and a G-invariant open neighbourhood V of Y' in Y such that $h(y, 0) = y$ for all $y \in Y$, $h(y', t) = y'$ for all $y' \in Y'$ and $t \in [0, 1]$, and $h(V \times 1) \subset Y'$. W being an open G-invariant subset of a convex set C, there exists a G-covering $\mathscr{W} = \{W_\mu \mid \mu \in M\}$ of W consisting of open convex subsets of C and refining the covering $\{r^{-1}(V), W - r^{-1}(Y')\}$. By Lemma 2.2, there exists a TN G-covering $\mathscr{S}$ of Y of the form (2.1) such that $\{\mathscr{S}_\lambda \mid \lambda \in \Lambda\}$ is a star-refinement of $\mathscr{W} \cap Y$, and we can clearly find a numerable G-invariant open covering $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda/G\}$ of X such that $f(U_\alpha) \subset T_\alpha$ for all $\alpha \in \Lambda/G$.

Let σ be an n-simplex of $N(\mathscr{U})$ with vertices $\alpha_0 < \ldots < \alpha_n$, $x \in K_\sigma = K_\sigma(\mathscr{U}, f, \mathscr{S})$ and $t \in \Delta^n$. Let $x = (x_{\alpha_0}, \ldots, x_{\alpha_n})$ and let $t_0, \ldots, t_n$ be the barycentric coordinates of t. By the definition of K_σ, there are $\lambda_0, \ldots, \lambda_n \in \alpha_n$ such that $x_{\alpha_i} \in O_{\lambda_i} = r_{\alpha_i}(S_{\lambda_i})$ for $i = 0, 1, \ldots, n$, and $\bigcap_{i=0}^{n} S_{\lambda_i} \neq \emptyset$. Since $\{S_\lambda \mid \lambda \in \Lambda\}$ is a star-refinement of $\mathscr{W} \cap Y$, this implies that $S_{\lambda_0} \cup \ldots \cup S_{\lambda_n} \subset W_\mu$ for some $\mu \in M$. Consequently, $\sum_{i=0}^{n} t_i x_i \in W_\mu$

and we can put

$$F(|x, t|) = h(r(\sum_{i=0}^{n} t_i y_{\alpha_i}), 1) .$$

It is easily verified that this formula defines a G-map $F: K(\mathscr{U}, f, \mathscr{S}) \to Y$.

Similarly, if $P: X \to K(\mathscr{U}, f, \mathscr{S})$ is the canonical G-map associated to a G-invariant locally finite partition of unity $\{p_\alpha \mid \alpha \in \Lambda/G\}$ on X, the formulae

$$H(x, s) = h(f(x), 2s) , \quad 0 \leqq s \leqq \tfrac{1}{2} .$$

$$H(x, s) = h(r((2 - 2s) f(x) + (2s - 1) \sum_{i=0}^{n} p_{\alpha_i}(x)\, r_{\alpha_i}(f(x))), 1) , \quad \tfrac{1}{2} \leqq s \leqq 1 ,$$

where $x \in X$, $\{\alpha_0, \ldots, \alpha_n\} = \{\alpha \in \Lambda/G \mid p_\alpha(x) \neq 0\}$ and $\alpha_0 < \ldots < \alpha_n$, define a G-homotopy $H: X \times [0, 1] \to Y$ from f to $F \circ P$.

It remains to verify that $K(\mathscr{U}, f, \mathscr{S})$, F and H have the required properties. This is, however, not difficult and may be left to the reader.

2.8. Definition. Let $f, X, Y, \mathscr{S}$ and $\mathscr{U}$ be as in 2.3, and let $F: K(\mathscr{U}, f, \mathscr{S}) \to L$ be a G-map, where L is a G-CW-complex. We shall say that F is *quasi-cellular* if $F(K^n(\mathscr{U}, f, \mathscr{S})) \subset L^{\omega(n)}$ for $n = 0, 1, 2, \ldots$, where $\omega(n) = n + (n + 1) \dim G$ and L^m denotes the m-skeleton of L.

2.9. Lemma. *Let $f, X, Y, \mathscr{S}$ and $\mathscr{U}$ be as in 2.3, let $X' \subset X$ be a G-subspace, $\mathscr{U}' = \mathscr{U} \cap X'$, $f' = f \mid X'$ and let (L, L') a G-CW-pair. Let us put $K = K(\mathscr{U}, f, \mathscr{S})$ and $K' = K(\mathscr{U}', f', \mathscr{S})$, and let $F: K \to L$ be a G-map such that $F(K') \subset L'$. Then there exist a quasi-cellular G-map $\Phi: K \to L$ and a G-homotopy $H: K \times [0, 1] \to L$ from F to Φ such that $\Phi(K') \subset L'$ and $H(K' \times [0, 1]) \subset L'$.*

Proof. Let $\sigma \in N = N(\mathscr{U})$. By definition, K_σ is an open G-invariant subspace of the differentiable G-manifold $O_{\alpha_0} \times \ldots \times O_{\alpha_n}$, where $\alpha_0 < \ldots < \alpha_n$ are all the vertices of σ, and thus also a differentiable G-manifold. Applying [10, Proposition (4.4)] we obtain that K_σ has a G-CW-complex structure, and it is obvious that the dimension of K_σ is at most $(n + 1) \dim G$. Consequently, each of the G-pairs $(\coprod_{\sigma \in N_n'} K_\sigma \times \Delta^n, \coprod_{\sigma \in N_n'} K_\sigma \times \dot{\Delta}^n)$ and $(\coprod_{\sigma \in N_n - N_n'} K_\sigma \times \Delta^n, \coprod_{\sigma \in N_n - N_n'} K_\sigma \times \dot{\Delta}^n)$ has a G-CW-pair structure of dimension at most $n + (n + 1) \dim G$.

It follows from the construction of the space K that the G-pairs (K_n', K_{n-1}') and $(K_n, K_{n-1} \cup K_n')$ can be identified with the G-pairs

$$(K_{n-1}' \cup_{\varphi'} \coprod_{\sigma \in N_n'} K_\sigma \times \Delta^n, K_{n-1}') \quad \text{and}$$

$$(K_{n-1} \cup K_n' \cup_\varphi \coprod_{\sigma \in N_n - N_n'} K_\sigma \times \Delta^n, K_{n-1} \cup K_n') ,$$

respectively, where $\varphi': \coprod_{\sigma \in N_n'} K_\sigma \times \dot{\Delta}^n \to K_{n-1}'$ and $\varphi: \coprod_{\sigma \in N_n - N_n'} K_\sigma \times \dot{\Delta}^n \to K_{n-1} \cup K_n'$ are obvious canonical G-maps. In view of the preceding observation this clearly implies that each of the G-pairs (K_n', K_{n-1}') and $(K_n, K_{n-1} \cup K_n')$ has the structure of a relative G-CW-complex of dimension at most $n + (n + 1) \dim G$.

The conclusion of the lemma now follows from an obvious generalization of [11, Proposition (3.3)] and from [11, Proposition (4.3)] in a similar way as the G-cellular approximation theorem.

Combining Proposition 2.7 and Lemma 2.9 we easily obtain the following proposition, which is the main result of this section and plays an important role in Section 4.

2.10. Proposition. *Let $f: X \to Y$ be a G-map, let $X' \subset X$ and $Y' \subset Y$ be G-subspaces such that $f(X') \subset Y'$, and let $n = \dim_N X/G$* (see Remark 1.4 (b)). *If (Y, Y') is a G-CW-pair and Y is a G-ANR (so that, by* [15, Theorem 6.4 and Theorem 12.5], *Y' is a G-ANR, too) then there exists a G-homotopy $H: X \times [0, 1] \to Y$ such that $H(-, 0) = f$, $H(X' \times [0, 1]) \subset Y'$ and $H(X \times \{1\}) \subset Y^{\omega(n)}$, where $\omega(n) = n + (n + 1) \dim G$.*

3. LOCALLY G-HOMOTOPICALLY TRIVIAL G-FIBRATIONS

By a G-fibration or, equivalently, a G-space over a G-space B we simply mean a G-map $p: E \to B$.

3.1. Definition. Let $p: E \to B$ be a G-fibration.

(a) We shall say that p is *G-homotopically trivial* (*with a fibre* (F, H)) if H is a closed subgroup of G, F is an H-space and there is an H-slice S in B (in the sense of [16, Definition 1.7.1]) such that $B = GS$ and p is G-homotopy equivalent over B to the G-fibration $q: G \times_H (S \times F) \to B$ defined in the obvious canonical way.

(b) We shall say that p is *G-homotopically trivial* (*with a fibre* (F, H)) *over a G-subspace $B' \subset B$* if the induced G-fibration $p_{B'}: p^{-1}(B') \to B'$ is G-homotopically trivial (with a fibre (F, H)).

(c) We shall say that p is *locally G-homotopically trivial* (with a family of fibres $\{(F_i, H_i) \mid i \in I\}$) if there is a G-invariant open covering $\{U_i \mid i \in I\}$ of B such that p is G-homotopically trivial (with a fibre (F_i, H_i)) over U_i for each $i \in I$.

3.2. Remark. If, in the situation of Definition 3.1 (a), S' is an H'-slice in S, then S' is also an H'-slice in B, so that $B' = GS'$ is an open G-subspace of B, and $p_{B'}$ is easily seen to be G-homotopy equivalent over B' to the G-fibration $G \times_{H'} (S' \times F) \to B'$.

This observation and the Mostow theorem on the existence of slices (see e.g. [5, Chap. II, Theorem 5.4]) further imply that a G-fibration p over a completely regular G-space B is locally G-homotopically trivial if and only if for each point $b \in B$ there exists a G-tube $T = (T, r)$ about an orbit $O = G.b$ such that the G-fibration p_T: $p^{-1}(T) \to T$ is G-homotopy equivalent over B to the G-fibration $G \times_H (S \times F) \to T$ where $H = G_b$, $S = r^{-1}(b)$ and $F = p^{-1}(b)$.

The following useful lemma, which can also serve as a motivation for Definition 3.1, is an easy consequence of [16, Corollary 1.7.8 and Theorem 1.7.10].

3.3. Lemma. *Let $p_i\colon E_i \to B$ $(i = 1, 2)$ be G-fibrations, $H \subset G$ a closed subgroup and $S \subset B$ an H-slice in B such that $B = GS$.*

(a) *There is a one-to-one correspondence between the G-maps $E_1 \to E_2$ over B and the H-maps $p_1^{-1}(S) \to p_2^{-1}(S)$ over S, which is given by the operation of restriction.*

(b) *A G-map $\varphi\colon E_1 \to E_2$ over B is a G-homotopy equivalence over B if and only if its restriction $\varphi_S\colon p_1^{-1}(S) \to p_2^{-1}(S)$ is an H-homotopy equivalence over S.*

3.4. Proposition. *Let Y be a G-ANR, $E = Y^{[0,1]}$ and $B = Y \times Y$. Then the G-map $p\colon E \to B$ defined by $p(\omega) = (\omega(0), \omega(1))$ is a locally G-homotopically trivial G-fibration.*

Proof. Let $b = (b_1, b_2)$ be any point of B and $O = G \, . \, b$. Applying successively [15, Proposition 6.8, Theorem 8.8 and Proposition 9.6] we obtain that (B, O) is a G-NDR-pair, which means among others that there are an open G-neighbourhood T of O and a G-homotopy $h\colon B \times [0, 1] \to B$ such that $h(y, t) = y$ for $(y, t) \in B \times \{0\} \cup O \times [0, 1]$ and $h(T \times \{1\}) = O$. Consequently, the formula $r(y) = h(y, 1)$ defines a G-retraction $r\colon T \to O$, and $T = (T, r)$ is a G-tube about O.

Let $S = r^{-1}(b)$, $F = p^{-1}(b)$ and $H = G_b$, and let us define G_b-maps

$$\chi\colon p^{-1}(S) \to F\,, \quad \psi\colon p^{-1}(S) \to S \times F\,, \quad \psi'\colon S \times F \to p^{-1}(S)$$

by the formulae

$$\chi(\omega)(t) = \begin{cases} p_1 \circ h(p(\omega), 1 - 3t) & \text{for} \quad 0 \leqq t \leqq 1/3\,, \\ \omega(3t - 1) & \text{for} \quad 1/3 \leqq t \leqq 2/3\,, \\ p_2 \circ h(p(\omega), 3t - 2) & \text{for} \quad 2/3 \leqq t \leqq 1\,, \end{cases}$$

$$\psi(\omega) = (p(\omega), \chi(\omega))\,,$$

$$\psi'(y, \omega)(t) = \begin{cases} p_1 \circ h(y, 3t) & \text{for} \quad 0 \leqq t \leqq 1/3\,, \\ \omega(3t - 1) & \text{for} \quad 1/3 \leqq t \leqq 2/3\,, \\ p_2 \circ h(y, 3 - 3t) & \text{for} \quad 2/3 \leqq t \leqq 1\,, \end{cases}$$

in which p_1 and p_2 denote the canonical projections of $B = Y \times Y$ onto the first and second factor, respectively. The maps ψ and ψ' are easily verified to be H-homotopy equivalences over S, which in view of Lemma 3.3 completes the proof.

4. RELATIVE BOUNDED *G*-LIFTING PROPERTY

In this section, some definitions and results of [7] are carried over (with slight modifications and generalizations) to the category of G-spaces and G-maps.

4.1. Definition. Let Y be a G-space. We shall say that Y *has the relative G-compressibility property with respect to a G-space X* (briefly $RCP_G(X)$) if, for every

G-subspace $A \subset X$ and for every G-map $f: X \to Y$ such that, for some halo (= functional neighbourhood) U around A, $\overline{f(U)}$ is compact, there exists a G-map $g: X \to Y$ with $\overline{g(X)}$ compact, which is G-homotopic to f relative to A.

4.2. Remark. It is easy to show that, for two G-spaces Y and Y' of the same G-homotopy type, Y has $RCP_G(X)$ if and only if Y' has $RCP_G(x)$.

4.3. Proposition. *Let Y be a G-CW-complex of G-finite type. Then Y has $RCP_G(X)$ for every G-space X with* $\dim_N X/G < \infty$.

Proof. Let us consider the G-CW-complex

$$\operatorname{tel}(Y) = \bigcup_{n=0}^{\infty} Y^n \times [n, n+1] \subset Y \times [0, +\infty),$$

which is sometimes called *the telescope of Y*. It is well known (and can be easily derived e.g. from the results of [11]) that the canonical projection $\operatorname{tel}(Y) \to Y$ is a G-homotopy equivalence. Consequently, by Remark 4.2 we may assume that Y is G-locally finite and therefore a G-ANR by [15, Theorem 12.5].

Now let f, A and U be as in Definition 4.1. There is a G-finite G-CW-subcomplex B of Y containing $f(U)$. Since $f(G \,.\, U) \subset B$ and B is also a G-ANR, we may apply Proposition 2.10 and obtain a G-homotopy $H: X \times [0, 1] \to Y$ such that $H(-, 0) = f$, $H(U \times [0, 1]) \subset B$ and $H(X \times \{1\}) \subset Y^{\omega(n)}$. Let $\varphi: X \to [0, 1]$ be any G-invariant continuous function such that $\varphi(A) \subset \{0\}$ and $\varphi(X - U) \subset \{1\}$, and let us define a G-map $g: X \to Y$ by $g(x) = H(x, \varphi(x))$. Then $g(x) = f(x)$ for $x \in A$, $g(x)$ is contained in the compact subspace $B \cup Y^{\omega(n)}$ for each $x \in X$, and $(x, t) \mapsto H(x, \varphi(x)\, t)$ is a G-homotopy from f to g relative to A, which proves the proposition.

4.4. Definition. A G-map $p: E \to B$ is said to *have the relative bounded G-lifting property with respect to a G-space X* (briefly $RBLP_G(X)$) if, for every G-subspace $A \subset X$ and for every G-map $f: X \to E$ such that, for some halo U around A, the spaces $\overline{f(U)}$ and $\overline{p \circ f(X)}$ are compact, there exists a G-map $g: X \to E$ with the following properties: g is G-homotopic to f over B relatively to A, and $\overline{g(X)}$ is compact.

4.5. Remark. Similarly to Remark 4.2, for G-fibrations $p_i: E_i \to B$ $(i = 1, 2)$ of the same G-homotopy type over B it is true that p_1 has $RBLP_G(X)$ if and only if p_2 has $RBLP_G(X)$.

It follows from the definitions that if B is a one-point space then a G-map $p: E \to B$ has $RBLP_G(X)$ if and only if E has $RCP_G(X)$. This suggests that there could be a connection between the $RBLP_G$ of E and the RCP_{G_b} of the fibers of p, and the aim of the rest of this section is to show that under certain additional conditions such a connection really exists.

We start with two simple lemmas, the proofs of which are left to the reader. For the basic properties of slices that are needed in the first (and also in the proof of Proposition 4.9 below), see e.g. [16, Section 1.7].

4.6. Lemma. *Let $p: E \to B$ be a G-fibration, H a closed subgroup of G, S an H-slice in B such that $GS = B$, and X a G-space. Then p has $RBLP_G(X)$ if and only if, for every H-slice S' in X such that $GS' = X$, the induced H-fibration $p_S: p^{-1}(S) \to \to S$ has $RBLP_H(S')$.*

4.7. Lemma. *Let $E = B \times F$ be the direct product of G-spaces B and F, and let $p: E \to B$ be the canonical projection. If F has $RCP_G(X)$, then p has $RBLP_G(X)$.*

4.8. Remark. If there exists at least one G-map from X to B, then the sufficient condition in Lemma 4.7 is clearly also necessary. A G-map from X to B exists e.g. when $G = G_b$ for some $b \in B$.

4.9. Proposition. *Let $p: E \to B$ be a locally G-homotopically trivial G-fibration with a family of fibres $\{(F_i, H_i) \mid i \in I\}$ and let X be a G-space. If, for each $i \in I$, the fibre F_i has $RCP_{H_i}(S)$ for every H_i-slice S in X such that GS is a cozero set in X, then p has $RBLP_G(X)$.*

Proof. (Compare with the proof of [7, Lemma 1].)

I. By Definition 3.1, for each $i \in I$ there is an H_i-slice S_i in B such that $\{B_i \mid i \in I\}$, where $B_i = GS_i$, is an open G-invariant covering of B, and each induced G-fibration $p_i: p^{-1}(B_i) \to B_i$ is G-homotopy equivalent over B_i to the canonical G-fibration $G \times_{H_i} (S_i \times F_i) \to B_i$. By Remark 4.5, Lemma 4.6 and Lemma 4.7, each G-fibration p_i has $RBLP_G(GS)$ for every H_i-slice S in X such that GS is a cozero set in X.

II. Now let A, U and f be as in Definition 4.4. We may clearly suppose without loss of generality that U is G-invariant. Since the space $B' = \overline{p \circ f(X)}$ is compact, there are finitely many indices $i_1, \dots, i_n \in I$ such that $B' \subset \bigcup_{k=1}^{n} B_{i_k}$. To simplify the notation, let us write from now on H_k, S_k, B_k and p_k instead of H_{i_k}, S_{i_k}, B_{i_k} and p_{i_k} for $k = 1, 2, \dots, n$.

Since B' is normal, the group G is compact and the subsets B_i are G-invariant, there are G-invariant continuous functions $\varphi_i: B' \to [0, 1]$, $i = 1, \dots, n$, such that the support of φ_i is contained in B_i and $\sum_{i=1}^{n} \varphi_i = 1$. For $i = 1, 2, \dots, n$ let us put

$$X_i = \{x \in X \mid \varphi_i \circ p \circ f(x) > 0\} ,$$

$$A_i = \{x \in X \mid \varphi_i \circ p \circ f(x) \geqq 1/n\} ,$$

$$S'_i = \{x \in X_i \mid p \circ f(x) \in S_i\} .$$

Then S'_i is an H_i-slice in X and $X_i = GS'_i$ is a cozero set in X, so that, by part I of the proof, p_i has $RBLP_G(X_i)$ for each $i = 1, 2, \dots, n$. Moreover, X_i is an open G-invariant halo around A_i, and $\bigcup_{i=1}^{n} A_i = X$.

III. We shall now prove that there are G-maps $g_1, \dots, g_n: X \to E$ and G-invariant subsets $U_1, \dots, U_n$ of X satisfying (with $g_0 = f$, $U_0 = U$), for each $i = 1, 2, \dots, n$:

(i) g_i is G-homotopic to g_{i-1} over B relatively to $A \cup A_1 \cup \ldots \cup A_{i-1} \cup (X - X_i)$,

(ii) U_i is a halo around $A \cup \ldots \cup A_i$ and the space $\overline{g_i(U_i)}$ is compact.

This will prove the proposition because the map $g = g_n$ will clearly have all the properties required in Definition 4.4.

The proof proceeds by induction. Let us suppose that, for some integer k satisfying $1 \leqq k \leqq n$, we have constructed G-maps $g_1, \ldots, g_{k-1}: X \to E$ and G-invariant subsets $U_1, \ldots, U_{k-1}$ of X in such a way that the assertions (i) and (ii) are true for $i = 1, 2, \ldots, k-1$, and consider the restriction $g'_{k-1}: X_k \to p^{-1}(B_k)$ of g_{k-1}. Since U_{k-1} is a G-invariant halo around $A \cup A_1 \cup \ldots \cup A_{k-1}$, there is a G-invariant open subset U'_{k-1} of U_{k-1} such that U_{k-1} is a halo around U'_{k-1} and U'_{k-1} is a halo around $A \cup A_1 \cup \ldots \cup A_{k-1}$. Then $U_{k-1} \cap X_k$ is a halo around $U'_{k-1} \cap X_k$, in X_k, and the closure of $g'_{k-1}(U_{k-1} \cap X_k)$ in $p^{-1}(B_k)$ and the closure of $p_k \circ g'_{k-1}(X_k) = p \circ f(X_k)$ in B_k are easily verified to be compact. Since p_k has $RBLP_G(X_k)$, we conclude that there exists a G-map $g'_k: X_k \to p^{-1}(B_k)$ such that the closure of $g'_k(X_k)$ in $p^{-1}(B_k)$ is compact, and a G-homotopy $h: X_k \times [0, 1] \to p^{-1}(B_k)$ over B_k relatively to $U'_{k-1} \cap X_k$ from g'_{k-1} to g'_k. Since U'_{k-1} is a G-invariant halo around $A \cup A_1 \cup \ldots \ldots \cup A_{k-1}$ and X_k is a G-invariant halo around A_k, $U'_{k-1} \cup X_k$ is a G-invariant halo around $A \cup A_1 \cup \ldots \cup A_k$. Consequently, there is a G-invariant continuous function $\varphi: X \to [0, 1]$ such that the support of φ is contained in $U'_{k-1} \cup X_k$ and $\varphi(U_k) \subset \{1\}$ for some halo U_k around $A \cup A_1 \cup \ldots \cup A_k$. Putting now

$$g_k(x) = \begin{cases} h(x, \varphi(x)) & \text{for } x \in X_k\,, \\ g_{k-1}(x) & \text{for } x \notin X_k\,, \end{cases}$$

we obtain a well-defined G-map $g_k: X \to E$ which is easily verified to satisfy (i) and (ii) for $i = k$. This completes the proof.

Now we can prove the main results of this section.

4.10. Proposition. *Let $p: E \to B$ be a locally G-homotopically trivial G-fibration over a completely regular G-space B. If for each $b \in B$ the fibre $F_b = p^{-1}(b)$ has the G_b-homotopy type of a G_b-CW-complex of G_b-finite type, then p has $RBLP_G(X)$ for every G-space X such that* $\dim_N X/G < \infty$.

Proof. It suffices to show that p satisfies the conditions of Proposition 4.9. By Remark 3.2, p is a locally G-homotopically trivial G-fibration with a family of fibres $\{(F_b, G_b) \mid b \in B\}$, and by Remark 4.2 and Proposition 4.3 each fibre F_b has $RCP_{G_b}(X)$ for every G-space X with $\dim_N X/G < \infty$. Finally, for each G_b-slice S in a G-space X such that GS is a cozero set in X, GS/G is a cozero set in X/G, and thus by Proposition 6.1 we have $\dim_N (GS/G) \leqq \dim_N X/G$.

For an arbitrary space X and points $x_0, x_1 \in X$ let $P(X; x_0, x_1)$ denote the subspace of $X^{[0,1]}$ consisting of all paths starting at x_0 and ending at x_1.

4.11. Lemma. *If X is a G-ANR, $x_0, x_1 \in X$ and $H = G_{x_0} \cap G_{x_1}$, then the H-space $P(X; x_0, x_1)$ is an H-ANR and therefore has the H-homotopy type of an H-CW-complex.*

Proof. The first assertion is an easy consequence of [15, Theorem 6.4 and Proposition 8.1] while the second follows from [15, Theorem 13.3].

4.12. Remark. Using [18, Proposition 3.8 and Corollary 4.13] one can prove that $P(X; x_0, x_1)$ is H-homotopy equivalent to an H-CW-complex also in the case that X is a G-CW-complex.

4.13. Proposition. *Let Y be a G-space and let $p: E \to B$ be the G-fibration considered in Proposition 3.4. If*

(a) *Y is a G-ANR satisfying the conditions* (a) *and* (b) *of Theorem* 1.3, *or*

(b) *Y is a G-locally finite G-CW-complex of G-finite type, and the set $\pi_0(Y^H, y)$ and the group $\pi_1(Y^H, y)$ are finite for each closed subgroup H of G and for each point $y \in Y^H$,*

then p has $RBLP_G(X)$ for every G-space X with $\dim_N X/G < \infty$.

Proof.

It is sufficient to verify that p satisfies the assumptions of Proposition 4.10. By [15, Theorem 12.5] and the proof of Corollary 1.6 the case (b) is a part of the case (a). By Proposition 3.4 p is locally G-homotopically trivial, and by Lemma 4.11 each fibre F_b $(b \in B)$ has a G_b-homotopy type of a G_b-CW-complex. Let $b = (y_0, y_1)$ and let $H = G_b = G_{y_0} \cap G_{y_1}$. Clearly $(F_b)^H = P(Y; y_0, y_1)^H = P(Y^H; y_0, y_1)$ and the last space is homotopy equivalent to the loop space $\Omega(Y^H; y_0)$. The exact homotopy sequence of the path fibration over Y^H therefore implies that the fibre F_b satisfies the conditions of [9, Theorem 2.3]. Since we already know that F_b is of a G_b-CW-homotopy type, this theorem and the equivariant version of the Whitehead theorem [11, Theorem (5.3)] imply that F_b is G_b-homotopy equivalent to a G_b-CW-complex of G_b-finite type.

5. PROOF OF THEOREM 1.3

5.1. Necessity of the condition (a). Let H be a closed subgroup of G such that $Y^H \neq \emptyset$ and let X be a completely regular space.

It is an easy consequence of the definition of the Čech-Stone G-compactification that $G/H \times \beta X = \beta_G(G/H \times X)$. Consequently, we have the commutative diagtam

$$\begin{array}{ccc} [\beta_G(G/H \times X); Y]_G & \xrightarrow{j^*_{G/H\times X}} & [G/H \times X; Y]_G \\ \downarrow \approx & & \downarrow \approx \\ [\beta X; Y^H] & \xrightarrow{i^*_X} & [X; Y^H] \end{array}$$

in which the vertical arrows denote the obvious canonical bijections. In view of the identification $(G/H \times X)/G = X$ this diagram implies that the canonical transformation i^*_X is bijective if $\dim_N X < \infty$. Since by [18, Corollary 4.13] the space Y^H is homotopy equivalent to a CW-complex, we may apply [2, Theorem 1.3] and obtain that the set $\pi_0(Y^H, y_H)$ and the group $\pi_1(Y^H, y_H)$ are finite and the groups $\pi_i(Y^H, y_H)$ $(i = 2, 3, \ldots)$ are finitely generated for every point $y_H \in Y^H$.

5.2. Necessity of the condition (b). It is easy to see that we may suppose without loss of generality that Y is a G-CW-complex. Let n be a natural number.

The surjectivity of the transformation $j^*_{Y_n}$ implies that the inclusion map $\iota_n: Y^n \hookrightarrow Y$ is G-homotopy equivalent to a G-map $\iota'_n: Y^n \to Y$, whose image is contained in a G-finite G-CW-subcomplex K_n of Y. The set $\mathscr{S}'_n$ of the conjugacy classes of the isotropy groups of all points of K_n is clearly finite, and therefore the set $\mathscr{S}_n = \widetilde{\mathscr{S}'_n}$ is finite as well. We now show that $\mathscr{S}_n$ has the property described in the condition (b).

Let H be a closed subgroup of G such that $Y^H \neq \emptyset$. One can easily show that each path component of Y^H intersects Y^0 (this fact represents a very special case of [11, Proposition (4.3)]). This immediately implies that the group $H(\mathscr{S}_n)$ is defined, because for $y \in Y^H \cap Y^n$ we have $H \subset G_y \subset G_{\iota'_n(y)}$, and the conjugacy class of the last subgroup belongs to $\mathscr{S}_n$ by the definition of this set.

It remains to prove that for $H_n = H(\mathscr{S}_n)$ the inclusion map $Y^{H_n} \hookrightarrow Y^H$ is an n-equivalence, i.e., for each $y \in Y^{H_n}$ the induced homomorphism $\pi_i(Y^{H_n}, y) \to \pi_i(Y^H, y)$ is bijective for $i = 0, 1, \ldots, n-1$ and surjective for $i = n$. Since each path component of Y^{H_n} intersects Y^0, we may restrict ourselves to the case of $y \in Y^{H_n} \cap Y^n$. Let $\tau: Y^n \times [0, 1] \to Y$ be any G-homotopy from ι_n to ι'_n, and let us define a path $\omega: [0, 1] \to Y$ by $\omega(t) = \tau(y, t)$. Then we have the diagram

$$\begin{tikzcd}
\pi_i(Y^{H_n}, y') \arrow[rrr] \arrow[dr, "h_{[\omega]}", "\approx"'] & & & \pi_i(Y^H, y') \arrow[dl, "h_{[\omega]}"', "\approx"] \\
& \pi_i(Y^{H_n}, y) \arrow[r] & \pi_i(Y^H, y) & \\
& \pi_i((Y^n)^{H_n}, y) \arrow[r] \arrow[u, "(\iota_n^{H_n})_*"] \arrow[uul, "(\iota_n'^{H_n})_*"] \arrow[dl, "(\iota_n''^{H_n})_*"'] & \pi_i((Y^n)^H, y) \arrow[u, "(\iota_n^H)_*"'] \arrow[uur, "(\iota_n'^H)_*"'] \arrow[dr, "(\iota_n''^H)_*"] & \\
\pi_i(K_n^{H_n}, y') \arrow[uuu] \arrow[rrr, "\approx"] & & & \pi_i(K_n^H, y') \arrow[uuu]
\end{tikzcd}$$

in which ι''_n is induced by ι'_n, $h_{[\omega]}$'s are induced by ω in the usual way and all unnamed homomorphisms are induced by inclusion maps. The diagram is easily seen to be commutative, the bottom homomorphism is an identity for all i's because $K_n^{H_n} = K_n^H$ by the definition of H_n, and $(\iota_n^{H_n})_*$ and $(\iota_n^H)_*$ are bijective for $i = 0, 1, \ldots, n-1$ and surjective for $i = n$ by [11, Proposition (4.3)]. An easy diagram chasing using these facts shows that the inclusion map $Y^{H_n} \hookrightarrow Y^H$ is an n-equivalence.

5.3. Sufficiency of the conditions (a) **and** (b). Let X be a completely regular G-space with $\dim_N X/G < \infty$. By [9, Theorem 2.3] and [11, Theorem (5.3)] (an equivariant version of the Whitehead theorem) we may suppose that Y is a G-CW-complex of G-finite type and therefore by Proposition 4.3 it has $RCP_G(X)$. This, however, implies the surjectivity of j_X^* because each G-map $f\colon X \to Y$ with $\overline{f(X)}$ compact can be extended to a G-map from $\beta_G X$ to Y.

The injectivity of j_X^* follows in a similar way from Proposition 4.13 because as we have shown in the proof of Proposition 4.3 we may in addition assume that Y is G-locally finite.

6. APPENDIX: ON $\dim_N$ OF COZERO SUBSETS

The aim of this section is to prove the following result from the dimension theory, which we have not been able to find in the literature although it is probably well-known.

6.1. Proposition. *If A is a cozero subset of a topological space X, then* $\dim_N A \leqq \dim_N X$.

6.2. Lemma. *Let $B \subset A \subset X$ and let us suppose that A is a halo around B. Then for each normal covering $\mathscr{U} = \{U_i \mid i \in I\}$ of the space A there exists a normal covering $\mathscr{V} = V_i \mid i \in I\}$ of A such that* $\dim N(\mathscr{V} \cap B) \leqq \dim_N X$ *and $V_i \subset U_i$ for each $i \in I$.*

Proof. Let $I_0 = I \cup \{\alpha\}$, where $\alpha \notin I$, and let us define a covering $\mathscr{U}^0 = \{U_i^0 \mid i \in I_0\}$ of X by $U_i^0 = U_i$ for $i \in I$ and $U_\alpha^0 = X - B$. The covering $\mathscr{U}^0$ is clearly normal, and therefore there is a normal covering $\mathscr{V}^0 = \{V_i^0 \mid i \in I_0\}$ of X such that $\dim N(\mathscr{V}^0) \leqq \dim_N X$ and $V_i^0 \subset U_i^0$ for all $i \in I_0$. Putting now $V_i = V_i^0 \cup (U_i - B)$ for $i \in I$, we obtain a covering $\mathscr{V} = \{V_i \mid i \in I\}$ of A which is easily checked to have all the required properties.

Let $\mathscr{U}$ be a normal covering of a space X and let $p\colon X \to |N(\mathscr{U})|$ be a canonical projection (canonical map). We shall say that p is locally finite if each point $x \in X$ has a neighbourhood which is mapped by p into a finite subcomplex of $|N(\mathscr{U})|$. Clearly, p is locally finite if and only if the corresponding partition of unity on X is locally finite. Consequently, locally finite canonical projections exist for every normal covering.

The following lemma is an immediate consequence of Lemma 6.2 and this remark.

6.3. Lemma. *Let $B \subset A \subset X$ and let us suppose that A is a halo around B. Then for each normal covering $\mathscr{U}$ of the space A there is a locally finite canonical projection $p\colon A \to |N(\mathscr{U})|$ such that $p(B) \subset |N(\mathscr{U})|^n$ where $n = \dim_N X$.*

6.4. Lemma. *Let $D \subset C \subset B \subset A \subset X$, let $\mathscr{U} = \{U_i \mid i \in I\}$ be a normal covering of A and let $p\colon A \to |N(\mathscr{U})|$ be locally finite canonical projection such that $p(C) \subset |N(\mathscr{U})|^n$ where $n = \dim_N X$. If C is a halo around D, B is a halo around C and A is a halo around B then there exists a locally finite canonical projection $q\colon A \to |N(\mathscr{U})|$ such that $q(B) \subset |N(\mathscr{U})|^n$ and $q|D = p|\,D$.*

Proof. Let $\Delta = \bigcup_{i \in I} \operatorname{Star}(v_i) \times \operatorname{Star}(v_i) \subset |N(\mathscr{U})| \times |N(\mathscr{U})|$, where v_i is the vertex of $|N(\mathscr{U})|$ corresponding to an index $i \in I$, and Star (v_i) is the open star neighbourhood of v_i. By [13, Proof of Lemma 2] there exists a not necessarily continuous map $\lambda: \Delta \times [0, 1] \to |N(\mathscr{U})|$ having the following properties:

(i) λ is continuous on each subspace of the form $((K \times K) \cap \Delta) \times [0, 1]$, where K is a finite subcomplex of $|N(\mathscr{U})|$;

(ii) $\lambda(\mathrm{Star}\,(v_i) \times \mathrm{Star}\,(v_i) \times [0, 1]) \subset \mathrm{Star}\,(v_i)$ for each $i \in I$;

(iii) $\lambda(((|N(\mathscr{U})|^m \times |N(\mathscr{U})|^m) \cap \Delta) \times [0, 1]) \subset |N(\mathscr{U})|^m$ for $m = 0, 1, 2, \ldots$;

(iv) $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$ for all $(x, y) \in \Delta$.

The lemma is now proved as follows. By Lemma 6.3 there is a locally finite canonical projection $q': A \to |N(\mathscr{U})|$ such that $q'(B) \subset |N(\mathscr{U})|^n$. Let $\varphi: X \to [0, 1]$ be a continuous function such that $\varphi(D) \subset \{0\}$ and $\varphi(X - C) \subset \{1\}$, and let us define a map $q: A \to |N(\mathscr{U})|$ by the formula $q(x) = \lambda(p(x), q'(x), \varphi(x))$ for each $x \in A$. Then q is a locally finite canonical projection with the required properties.

6.5. Proof of Proposition 6.1. Let $n = \dim_N X$ and let $\mathscr{U} = \{U_i \mid i \in I\}$ be a normal covering of the space A. Since A is a cozero subset of X, there is a sequence $A_0, A_1, A_2, \ldots$ of subsets of A such that $A = \bigcup_{k=0}^{\infty} A_k$ and A_k is a halo around A_{k-1} for $k = 1, 2, 3, \ldots$. By Lemma 6.3 and Lemma 6.4 there are locally finite canonical projections $p_k: A \to |N(\mathscr{U})|$ $(k = 1, 2, 3, \ldots)$ such that $p_k(A_k) \subset |N(\mathscr{U})|^n$ and $p_{k+1} \mid A_{k-1} = p_k \mid A_{k-1}$ for $k = 1, 2, 3, \ldots$. Let $p = \lim_{k \to \infty} p_k$. Then p is a locally finite canonical projection and $p(A) \subset |N(\mathscr{U})|^n$. Consequently, if we put $V_i = p^{-1}(\mathrm{Star}\,(v_i))$, where Star (v_i) has the same meaning as in the proof of Lemma 6.4, we get a normal covering $\mathscr{V} = \{V_i \mid i \in I\}$ of A such that $V_i \subset U_i$ for each $i \in I$ and $\dim N(\mathscr{V}) \leqq n$, which completes the proof.

References

[1] *Bartík V.:* General bridge-mapping theorem, Comment. Math. Univ. Carolinae *16*, 4 (1975), 693—698 (Russian).

[2] *Bartík V.:* On the bijectivity of the canonical transformation $[\beta X; Y] \to [X; Y]$, Quart. J. Math. Oxford (2), *29* (1978), 77—91.

[3] *Bartík V.:* On the bijectivity of the canonical transformation $[\beta_G X; Y]_G \to [X; Y]_G$, Abstracts of 4th International Conference „Topology and its Applications", Dubrovnik, Sept. 30—Oct. 5 1985, Zagreb 1985.

[4] *Borel A.:* Seminar on transformation groups, Annals of Math. Studies 46, Princeton University Press, 1960.

[5] *Bredon G. E.:* Introduction to compact transformation groups, New York, 1972.

[6] *Calder A.* and *Siegel J.:* Homotopy and uniform homotopy, Trans. Amer. Math. Soc. *235* (1978), 245—270.

[7] *Calder A.* and *Siegel J.:* Homotopy and uniform homotopy II, Proc. Amer. Math. Soc. *78* (1980), 288—290.

[8] *Dold A.:* Lectures on Algebraic Topology, Springer-Verlag 1972.

[9] *Markl M.:* On the G-spaces having an $\mathscr{S}$-G-CW-approximation by a G-CW-complex of finite G-type, Comment. Math. Univ. Carolinae 24, 3 (1983).

[10] *Matumoto T.:* Equivariant K-theory and Fredholm operators, J. Fac. Sci. Univ. Tokyo, Sect. IA, *18* (1971), 109—112.
[11] *Matumoto T.:* On G-CW-complexes and a theorem of J.H.C. Whitehead, J. Fac. Sci. Univ. Tokyo, Sect. I, *18* (1971), 363—74.
[12] *May J. P.:* The homotopical foundations of algebraic topology, Mimeographed notes, University of Chicago.
[13] *Milnor J.:* On space having the homotopy type of a CW-complex, Trans. Amer. Math. Soc. *90* (1959), 272—280.
[14] *Morita K.:* Čech cohomology and covering dimension for topological spaces, Fund. Math. *87* (1975), 31—52.
[15] *Murayama M.:* On G-ANR's and their G-homotopy types, Osaka J. Math. *20* (1983), 479—512.
[16] *Palais R. S.:* The classification of G-spaces, Memoirs of the Amer. Math. Soc., Number *36* (1960).
[17] *Spanier E. H.:* Algebraic Topology, Springer-Verlag.
[18] *Waner S.:* Equivariant homotopy theory and Milnor's theorem, Trans. Amer. Math. Soc. *258* (1980), 351—368.

Author's address: 115 67 Praha 1, Žitná 25, Czechoslovakia (Matematický ústav ČSAV).