Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 38 (1988), No. 4, 705–712

Persistent URL: http://dml.cz/dmlcz/102266

Czechoslovak Mathematical Journal, 38 (113) 1988, Praha

EMBEDDING m-QUASISTARS INTO n-CUBES

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(Received November 13, 1986)

In the present paper the letters i, j, k, m, n and p denote integers. By a graph we mean a graph in the sense of [1]; $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. We shall say that graphs G_1 and G_2 are vertex-disjoint if $V(G_1) \cap V(G_2) = \emptyset$.

A graph which is homeomorphic to the star $K(1, m)$, where $m \geqq 3$, will be referred to as an m-quasistar. We say that an m-quasistar T of order p is balanced if p is even and there exists a 2-coloring of T with $p/2$ blue vertices and $p/2$ yellow ones. I. Havel [2] conjectured that

if $3 \leqq m \leqq n$, then every balanced m-quasistar of order 2^n can be embedded into the n-cube.

The conjecture has been proved for $m = 3$ by Havel [2], for $m = 4$ and 5 by the present author [4], and for $m = 6$ by N. B. Limaye [3]. In the present paper the conjecture will be proved for every $m \geqq 5$.

Let P be a nontrivial path. Then P is a graph homeomorphic to K_2. If u is a vertex of degree one in P, then we say that P is a u-path. If P is a u-path, then the only vertex of degree one in P hich is different from u will be denoted by $\varepsilon(P, u)$.

Let G be an n-cube, $n \geqq 1$. If u_1 and u_2 are adjacent vertices in G, P_1 and P_2 are vertex-disjoint nontrivial paths in G such that P_1 is a u_1-path and P_2 is a u_2-path, then we denote by $P_1 + u_1u_2 + P_2$ the path in G induced by $E(P_1) \cup \{u_1u_2\} \cup \cup E(P_2)$. Since G is an n-cube, where $n \geqq 1$, it is clear that there exist vertex-disjoint $(n - 1)$-cubes G' and G'' such that $V(G') \cup V(G'') = V(G)$ and $E(G') \cup E(G'') \subseteq \subseteq E(G)$; the set $\{G', G''\}$ will be referred to as a canonical partition of G. If $\{G', G''\}$ is a canonical partition of G and $u \in V(G')$, then the only vertex of G'' which is adjacent to u in G will be denoted by u/G''.

The proof of Havel's conjecture (for $m \geqq 5$) will be divided into two lemmas and two theorems.

Lemma 1. *Let $m \geqq 1$, let G be an m-cube, let $u \in V(G)$, and let $W \subseteq V(G)$ such that $|W| \leqq m - 1$. Then there exists a hamiltonian u-path P in G such that $\varepsilon(P, u) \notin W$.*

Proof. Obviously, there exists a 2-coloring of G with 2^{m-1} blue vertices and 2^{m-1}

yellow ones. Without loss of generality, let u be blue. Havel [2] has shown that for each yellow vertex v of G, there exists a hamiltonian path P in G such that $\varepsilon(P,u) = v$. Since $m - 1 < 2^{m-1}$, the assertion of the lemma follows.

Lemma 2. *Let $m \geqq 2$, let G be an m-cube, let u, v_1, v_2 be distinct vertices of G such that $v_1v_2 \in E(G)$, and let $W \subseteq V(G - v_1 - v_2)$ such that $|W| \leqq m - 2$. Then there exists a hamiltonian u-path P in $G - v_1 - v_2$ such that $\varepsilon(P, u) \notin W$.*

Proof. We proceed by induction on m. The case when $m = 2, 3$ is obvious. Let $m \geqq 4$. Assume that the lemma is proved for $m - 1$. It is clear that there exists a canonical partition $\{G', G''\}$ of G such that

$$|W \cap V(G'')| \leqq m - 3 \quad \text{and} \quad v_1, v_2 \in V(G'').$$

We distinguish two cases.

1. Let $u \in V(G')$. Recall that $m - 1 \geqq 3$. According to Lemma 1 there exists a hamiltonian u-path P' in G' such that $\varepsilon(P', u) \notin \{v_1/G', v_2/G'\}$. Denote $u' = \varepsilon(P', u)$ and $u'' = u'/G''$. According to the induction hypothesis, there exists a hamiltonian u''-path P'' in $G'' - v_1 - v_2$ such that $\varepsilon(P'', u'') \notin W \cap V(G'')$. Clearly,

(1) $P' + u'u'' + P''$ is a hamiltonian u-path in $G - v_1 - v_2$ such that $\varepsilon(P' + u'u'' + P'', u) \notin W$.

2. Let $u \in V(G'')$. According to the induction hypothesis, there exists a hamiltonian u-path P'' in $G'' - v_1 - v_2$. Denote $u'' = \varepsilon(P'', u)$ and $u' = u''/G'$. According to Lemma 1, there exists a hamiltonian u'-path P' in G' such that $\varepsilon(P', u') \notin W \cap V(G')$. Clearly, (1). Thus the proof is complete.

The following theorem is the main step in our proof of Havel's conjecture.

Theorem 1. *Let k and m be integers such that*

$$1 \leqq k \leqq m \quad \textit{if} \quad 1 \leqq m \leqq 3 \quad \textit{and}$$
$$1 \leqq k < m \quad \textit{if} \quad m \geqq 4.$$

Then $Q(k, m)$, where $Q(k, m)$ is the statement as follows:
for any $G, u_1, \ldots, u_k, a_1, \ldots, a_k, W_1, \ldots, W_k$ such that

(2) *G is an m-cube,*

(3) *$u_1, \ldots, u_k$ are distinct vertices of G,*

(4) *$a_1, \ldots, a_k$ are positive even integers with $a_1 + \ldots + a_k = 2^m$,*

(5) *$W_1, \ldots, W_k$ are subsets of $V(G)$ fulfilling*
$|W_1| \leqq m - k, \ldots, |W_k| \leqq m - k$,

there exist vertex-disjoint paths $P_{(1)}, \ldots, P_{(k)}$ in G such that

(6) *$P_{(i)}$ is a u_i-path of order a_i such that $\varepsilon(P_{(i)}, u_i) \notin W_i$, for each i, $1 \leqq i \leqq k$.*

Proof. It is easy to prove $Q(1, 1)$, $Q(2, 2)$ and $Q(3, 3)$ by an immediate inspection. Thus, we shall prove that if $m \geqq 2$ then $Q(k, m)$, for each k, $1 \leqq k \leqq m - 1$. We proceed by induction on m. The case $m = 2$ is obvious. Let $m \geqq 3$. Assume that we have proved $Q(k^*, m-1)$ for each k^*, $1 \leqq k^* \leqq m-2$.

Let $1 \leqq k \leqq m-1$. Consider G, $u_1, \ldots, u_k$, $a_1, \ldots, a_k$, $W_1, \ldots, W_k$ such that (2)–(5). For any canonical partition $\{G_1, G_2\}$ of G and any $f \in \{1, 2\}$, we define

$$\begin{aligned} I(G_f) &= \{i;\ 1 \leqq i \leqq k \text{ and } u_i \in V(G_f)\}\,, \\ k(G_f) &= |I(G_f)|\,, \\ U(G_f) &= \{u_i;\ i \in I(G_f)\}\,, \quad \text{and} \\ A(G_f) &= \sum_{i \in I(G_f)} a_i\,. \end{aligned}$$

We distinguish several cases and subcases.

1. Assume that there exists a canonical partition $\{G_1, G_2\}$ of G such that $A(G_1) = A(G_2)$.

Consider $f \in \{1, 2\}$. Obviously, $A(G_f) = 2^{m-1}$ and $1 \leqq k(G_f) \leqq k - 1 < m - 1$. Denote

$$\begin{aligned} &I_f = I(G_f)\,, \\ &u_{if} = u_i,\ a_{if} = a_i \quad \text{and} \quad W_{if} = W_i \cap V(G_f) \quad \text{for each} \quad i \in I_f\,. \end{aligned}$$

It is clear that

$(7)_f$ $\qquad u_{if}$ $(i \in I_f)$ are distinct vertices of G_f,

and

$(8)_f$ $\qquad a_{if}$ $(i \in I_f)$ are even positive integers such that $\sum_{i \in I_f} a_{if} = 2^{m-1}$.

Obviously, $|W_{if}| \leqq |W_i| \leqq m - k$ for $i \in I_f$. Since $m - k \leqq (m-1) - |I_f|$,

$(9)_f$ $\qquad |W_{if}| \leqq (m-1) - |I_f|$, for each $i \in I_f$.

According to $Q(k(G_f), m-1)$, there exists a set of $|I_f|$ vertex-disjoint paths P_{if} $(i \in I_f)$ in G_f such that

$(10)_f$ $\qquad P_{if}$ is a u_{if}-path of order a_{if} with the property that $\varepsilon(P_{if}, u_{if}) \notin W_{if}$ for each $i \in I_f$.

Denote

$$P_{(i)} = P_{i1} \text{ if } i \in I_1\,, \quad \text{and} \quad P_{(i)} = P_{i2} \text{ if } i \in I_2\,.$$

Clearly, $P_{(1)}, \ldots, P_{(k)}$ are vertex-disjoint paths in G such that (6).

2. Assume that $A(G^*) \neq A(G^{**})$ for any canonical partition $\{G^*, G^{**}\}$ of G.

2.1. Let $k = 1$. Then $a_1 = 2^m$. Lemma 1 implies that there exists a path $P_{(1)}$ in G such that (6).

2.2. Let $k = 2$. Clearly, $a_1 \neq a_2$. Without loss of generality we assume that $a_1 > a_2$.

2.2.1. Let $a_2 = 2$. Since $|W_2| \leqq m - 2$, there exists $u_2^* \in V(G) - (\{u_1\} \cup W_2)$ such that $u_2 u_2^* \in E(G)$. We denote by $P_{(2)}$ the path in G induced by $\{u_2 u_2^*\}$. Since $|W_1| \leqq m - 2$, it follows from Lemma 2 that there exists a hamiltonian u_1-path $P_{(1)}$

in $G - u_2 - u_2^*$ such that $\varepsilon(P_{(1)}, u_1) \notin W_1$. Hence, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in G such that (6).

2.2.2. Let $a_2 \geqq 4$. Since $a_1 > a_2$, $m \geqq 4$. Clearly, there exists a canonical partition $\{G_1, G_2\}$ of G such that

$$|W_1 \cap V(G_f)| \leqq m - 3 \quad \text{for} \quad f = 1 \quad \text{and} \quad 2\,. \tag{11}$$

Without loss of generality we assume that $u_1 \in V(G_1)$.

2.2.2.1. Let $u_2 \in V(G_1)$ and $W_2 \cap V(G_1) = \emptyset$. Denote

$$I_1 = \{1, 2\}\,, \quad u_{11} = u_1\,, \quad u_{21} = u_2\,, \quad a_{11} = 2^{n-1} - a_2\,,$$
$$a_{21} = a_2\,, \quad W_{11} = \emptyset = W_{21}\,.$$

It is clear that $(7)_1-(9)_1$. According to Q$(2, m - 1)$, there exist vertex-disjoint paths P_{11} and P_{21} in G_1 such that $(10)_1$. Denote $v = \varepsilon(P_{11}, u_{11})$ and $u_{12} = v/G_2$. As follows from (11) and Lemma 1, there exists a hamiltonian u_{12}-path P_{12} in G_2 such that $\varepsilon(P_{12}, u_{12}) \notin W_1 \cap V(G_2)$. Define $P_{(1)} = P_{11} + vu_{12} + P_{12}$ and $P_{(2)} = P_{21}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in G such that (6).

2.2.2.2. Let $u_2 \in V(G_1)$ and $W_2 \cap V(G_1) \neq \emptyset$. Hence,

$$|W_2 \cap V(G_2)| \leqq m - 3\,. \tag{12}$$

Denote

$$I_1 = \{1, 2\}\,, \quad u_{11} = u_1\,, \quad u_{21} = u_2\,, \quad a_{11} = 2^{m-1} - 2\,, \quad a_{21} = 2\,,$$
$$W_{11} = \emptyset = W_{21}\,.$$

It is clear that $(7)_1-(9)_1$. According to Q$(2, m - 1)$, there exist vertex-disjoint paths P_{11} and P_{21} in G_1 such that $(10)_1$. Denote

$$I_2 = \{1, 2\}\,, \quad v_1 = \varepsilon(P_{11}, u_{11})\,, \quad v_2 = \varepsilon(P_{21}, u_{21})\,, \quad u_{12} = v_1/G_2\,,$$
$$u_{22} = v_2/G_2\,, \quad a_{12} = a_1 + 2 - 2^{m-1}\,, \quad a_{22} = a_2 - 2\,,$$
$$W_{12} = W_1 \cap V(G_2)\,, \quad W_{22} = W_2 \cap V(G_2)\,.$$

It is clear that $(7)_2$ and $(8)_2$. It follows from (11) and (12) that $(9)_2$. According to Q$(2, m - 1)$, there exist vertex-disjoint paths P_{12} and P_{22} in G_2 such that $(10)_2$. Define $P_{(1)} = P_{11} + v_1u_{12} + P_{12}$ and $P_{(2)} = P_{21} + v_2u_{22} + P_{21}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in G such that (6).

2.2.2.3. Let $u_2 \in V(G_2)$ and $W_2 \cap V(G_2) = \emptyset$. According to Lemma 1 there exists a hamiltonian u_1-path P_{11} in G_1 such that $\varepsilon(P_{11}, u_1) \neq u_2/G_1$. Denote

$$v_1 = \varepsilon(P_{11}, u_1)\,, \quad I_2 = \{1, 2\}\,, \quad u_{12} = v_1/G_2\,, \quad u_{22} = u_2\,,$$
$$a_{12} = a_1 - 2^{m-1}\,, \quad a_{22} = a_2\,, \quad W_{12} = W_1 \cap V(G_2)\,, \quad W_{22} = W_2 \cap V(G_2)\,.$$

It is clear that $(7)_2-(9)_2$. According to Q$(2, m - 1)$, there exist vertex-disjoint paths P_{12} and P_{22} in G_2 such that $(10)_2$. Define $P_{(1)} = P_{11} + v_1u_{12} + P_{12}$ and $P_{(2)} = P_{22}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in G such that (6).

2.2.2.4. Let $u_2 \in V(G_2)$ and $V(G_2) \cap W_2 \neq \emptyset$. Hence,

$$|W_2 \cap V(G_1)| \leqq m - 3\,. \tag{13}$$

There exists $v_2 \in V(G_2 - u_2)$ such that v_2 is adjacent to u_2 in G_2 and $v_2 \neq u_1/G_2$. We denote by P_{22} the path in G_2 induced by $\{u_2v_2\}$. Denote

$$I_1 = \{1, 2\}\,, \quad u_{11} = u_1\,, \quad u_{21} = v_2/G_1\,, \quad a_{11} = 2^{n-1} + 2 - a_2\,,$$
$$a_{21} = a_2 - 2\,, \quad W_{11} = \{u_2/G_1\} \quad \text{and} \quad W_{21} = W_2 \cap V(G_1)\,.$$

It is clear that $(7)_1$ and $(8)_1$. Since $m - 1 \geqq 3$, (13) implies that $(9)_1$. As follows from $Q(2, m - 1)$, there exist vertex-disjoint paths P_{11} and P_{21} such that $(10)_1$. Denote $v_1 = \varepsilon(P_{11}, u_{11})$ and $u_{12} = v_1/G_2$. It is easy to see that $u_{12} \notin \{u_2, v_2\}$. It follows from Lemma 2 and (11) that there exists a hamiltonian u_{12}-path P_{12} in $G_2 - u_2 - v_2$ such that $\varepsilon(P_{12}, u_{12}) \notin W_1 \cap V(G_2)$. Define $P_{(1)} = P_{11} + v_1u_{12} + P_{12}$ and $P_{(2)} = P_{22} + v_2u_{21} + P_{21}$. Obviously, $P_{(1)}$ and $P_{(2)}$ are vertex-disjoint paths in G such that (6).

2.3. Let $k \geqq 3$. Then $m \geqq 4$. Recall that $A(G^*) \neq A(G^{**})$ for any canonical partition $\{G^*, G^{**}\}$ of G. We first prove that

(14) there exists a canonical parition $\{G_1, G_2\}$ of G such that $A(G_1) > A(G_2)$ and $1 \leqq k(G_2) \leqq k - 2$.

To the contrary, let us assume that

$(\overline{14})$ for any canonical partition $\{G^*, G^{**}\}$ of G, if $A(G^*) > A(G^{**})$ and $1 \leqq \leqq k(G^{**})$, then $k(G^{**}) = k - 1$.

Since $k \geqq 3$, there exists a canonical partition $\{G_{11}, G_{12}\}$ of G such that $A(G_{11}) > > A(G_{12})$ and $k(G_{12}) \geqq 1$. According to $(\overline{14})$, $k(G_{12}) = k - 1$, and therefore $k(G_{11}) = 1$. Obviously, there exists i, $1 \leqq i \leqq k$, such that $U(G_{11}) = \{u_i\}$. Since $A(G_{11}) > A(G_{12})$, $a_i > 2^{m-1}$.

Since $k(G_{12}) = k - 1 \geqq 2$, there exists a canonical partition $\{G_{21}, G_{22}\}$ of G such that

$$U(G_{12}) \cap V(G_{21}) \neq \emptyset \neq U(G_{12}) \cap V(G_{22})\,.$$

Without loss of generality we assume that $A(G_{21}) > A(G_{22})$. Since $U(G_{12}) \cap \cap V(G_{22}) \neq \emptyset$, $k(G_{22}) \geqq 1$. According to $(\overline{14})$, $k(G_{22}) = k - 1$, and therefore $k(G_{21}) = 1$. There exists j, $1 \leqq j \leqq k$, such that $U(G_{21}) = \{u_j\}$. Since $A(G_{21}) > > A(G_{22})$, $a_j > 2^{m-1}$. Since $U(G_{12}) \cap V(G_{21}) \neq \emptyset$ and $U(G_{21}) = \{u_j\}$, we can see that $u_j \in V(G_{12})$. Hence $i \neq j$. As follows from (4), $a_i + a_j < 2^m$, which is a contradiction. Thus, we have proved (14).

Denote

$$a = \min_{i \in I(G_1)} a_i\,.$$

We shall prove that

$$a \leqq 2^{m-1} - 2(k(G_1) - 1)\,. \tag{15}$$

To the contrary, let

$$a > 2^{m-1} - 2(k(G_1) - 1)\,.$$

Since a is even, we have that

$$(\overline{15}) \qquad a \geqq 2^{m-1} - 2(k(G_1) - 2).$$

Since $k(G_2) \geqq 1$, $A(G_2) \geqq 2$. Hence,

$$(16) \qquad a \leqq \frac{2^m - 2}{k(G_1)}.$$

If $k(G_1) = 2$, then – combining $(\overline{15})$ and (16) – we get that $2^{m-1} - 1 \geqq 2^{m-1}$, which is a contradiction. Let $k(G_1) \geqq 3$. Obviously, $m - 2 \geqq k(G_1)$. Thus – according to $(\overline{15})$ and (16) – we get that

$$\frac{2^m - 2}{3} \geqq \frac{2^m - 2}{k(G_1)} \geqq 2^{m-1} - 2(k(G_1) - 2) \geqq 2^{m-1} - 2(m - 4).$$

Hence, $6m - 26 \geqq 2^{m-1}$, which is a contradiction. Thus, we have proved (15).

Denote $I_1 = I(G_1)$. It follows from (15) that there exist disjoint nonempty subsets $I^\#$ and $I^\flat$ of I_1 and even positive integers a_{1i} (for each $i \in I_1$) satisfying

$$\begin{aligned} &I_2 = I^\# \cup I^\flat, \\ &a_{i1} = a_i, \quad \text{if} \quad i \in I^\#, \\ &a_{i1} \leqq a_i - 2, \quad \text{if} \quad i \in I^\flat, \quad \text{and} \\ &\sum_{i \in I_1} a_{i1} = 2^{m-1}. \end{aligned}$$

Denote

$$\begin{aligned} &u_{i1} = u_i \quad \text{if} \quad i \in I_1, \\ &W_{j1} = W_j \cap V(G_1) \quad \text{if} \quad j \in I^\#, \quad \text{and} \\ &W_{j1} = \{v;\, v/G_2 \in I(G_2)\} \quad \text{if} \quad j \in I^\flat. \end{aligned}$$

Since $k(G_2) = k - k(G_1) \leqq (m - 1) - k(G_1)$, we can see that $(9)_1$. According to $Q(k(G_1), m - 1)$, there exists a set of $|I_1|$ vertex-disjoint paths P_{i1} $(i \in I_1)$ in G_1 such that $(10)_1$. Denote

$$v_j = \varepsilon(P_j, u_{j1}) \quad \text{for each} \quad j \in I^\flat.$$

Moreover, denote

$$\begin{aligned} &I_2 = I^\flat \cup I(G_2), \\ &u_{i2} = u_i \quad \text{if} \quad i \in I(G_2), \quad u_{i2} = v_i/G_2 \quad \text{if} \quad i \in I^\flat, \\ &a_{i2} = a_i \quad \text{if} \quad i \in I(G_2), \\ &a_{i2} = a_i - a_{i1} \quad \text{if} \quad i \in I^\flat, \quad \text{and} \\ &W_{j2} = W_j \cap V(G_2) \quad \text{if} \quad j \in I_2. \end{aligned}$$

It is clear that $(7)_2$–$(9)_2$. As follows from $Q(|I_2|, m - 1)$, there exists a set of $|I_2|$ vertex-disjoint paths P_{i2} $(i \in I_2)$ such that $(10)_2$.

Define

$$\begin{aligned} &P_{(i)} = P_{i1} \quad \text{if} \quad i \in I^\#, \\ &P_{(i)} = P_{i1} + v_i u_{i2} + P_{i2} \quad \text{if} \quad i \in I^\flat, \quad \text{and} \\ &P_{(i)} = P_{i2} \quad \text{if} \quad i \in I(G_2). \end{aligned}$$

It is obvious that $P_{(1)}, \ldots, P_{(k)}$ are vertex disjoint paths in G such that (6).

Thus, the proof of the theorem is complete.

Remark 1. Let $k \geqq m \geqq 4$. Consider G, $u_1, \ldots, u_k$, $a_1, \ldots, a_k$, $W_1, \ldots, W_k$ such that (2)–(5), $a_1 \geqq 4, \ldots, a_k \geqq 4$, and $u_1 u, \ldots, u_k u \in E(G)$, where u is a vertex of G. Then (6) holds for no set of k vertex-disjoint paths $P_{(1)}, \ldots, P_{(k)}$ of G. This means that for $k \geqq m \geqq 4$, Q(k, m) does not hold. (It is also clear that Q(k, m) does not hold for $m \leqq 3$ and $k > m$.)

Remark 2. Let $2 \leqq k < m$. Consider G, $u_1, \ldots, u_k$, $a_1, \ldots, a_k$, $W_1, \ldots, W_k$ such that (2)–(4), $a_1 = 2$, and

$$|W_1| \geqq m - k + 1 .$$

Let $u_1, \ldots, u_k$ be chosen so that there exist $m - k + 1$ vertices of W_1, say vertices $w_1, \ldots, w_{m-k+1}$, such that $u_1 w_1, \ldots, u_1 w_{m-k+1} \in E(G)$, $u_1 u_2, \ldots, u_1 u_k \in E(G)$, and

$$\{u_2, \ldots, u_k\} \cap \{w_1, \ldots, w_{m-k+1}\} = \emptyset .$$

Hence, no set of k vertex-disjoint paths $P_{(1)}, \ldots, P_{(k)}$ in G satisfies (6). Let $j \geqq 1$. We can see that in Theorem 1 the inequalities

$$|W_1| \leqq m - k, \ldots, |W_k| \leqq m - k$$

cannot be replaced by the inequalities

$$|W_1| \leqq m - k + j, \ldots, |W_k| \leqq m - k + j .$$

We are now prepared to show that Havel's conjecture is true.

Theorem 2. *If $3 \leqq m \leqq n$, then every balanced m-quasistar of order 2^n can be embedded into the n-cube.*

Proof. We proceed by induction on m. In our proof we make use of the fact that the case $m = 3$ has been proved in [2] and the case $m = 4$ has been proved in [4]. Let $m \geqq 5$. Assume that we have proved that for any j, $m - 1 \leqq j$, every balanced $(m - 1)$-quasistar of order 2^j can be embedded into the j-cube.

Let T be a balanced m-quasistar of order 2^n. Then T contains exactly one vertex of degree m, say a vertex s, and exactly m vertices of degree one, say vertices $t_1, \ldots, t_m$. We denote by b_i the distance between s and t_i in T for each i, $1 \leqq i \leqq m$. Without loss of generality we assume that $b_1 \geqq \ldots \geqq b_m$. Clearly, $b_1 + \ldots + b_m = 2^n - 1$. Since T is balanced, it is easy to see that there exists exactly one h, $1 \leqq h \leqq m$, such that b_h is odd.

We shall first prove that

$$b_1 + \ldots + b_{m-2} \geqq 2^{n-1} + 2(m - 4) + 1 . \tag{17}$$

To the contrary, let

$$b_1 + \ldots + b_{m-2} \leqq 2^{n-1} + 2(m - 4) . \tag{$\overline{17}$}$$

Since $b_1 + \ldots + b_m = 2^n - 1$ and $b_1 \geqq \ldots \geqq b_m$, it follows from $(\overline{17})$ that

$$2 \,.\, 2^{n-1} - 1 = 2^n - 1 \leqq m(2^{n-1} + 2m - 8)/(m - 2) ,$$

and thus

$$2(m-2) \cdot 2^{n-1} - (m-2) \leqq m\, 2^{n-1} + 2m^2 - 8m\,.$$

Since $m \leqq n$, we get that

$$(m-4)\, 2^{m-1} \leqq 2m^2 - 7m - 2\,.$$

Hence $m \leqq 4$, which is a contradiction. Thus, we have proved (17).

This means that there exist $I \subseteq \{1, \ldots, m-2\}$, even positive integers a_i for each $i \in I$, and exactly one $f \in I$ such that

$$\begin{aligned} &a_f = b_f\,, \\ &a_i < b_i \quad \text{for each} \quad i \in I - \{f\}\,, \quad \text{and} \\ &\sum_{i \in I} a_i = 2^{n-1}\,. \end{aligned}$$

For each $i \in I$ we denote by v_i and w_i the vertices which belong to the path connecting s and t_i in T and such that the distance between s and v_i equals $b_i - a_i$, and the distance between s and w_i equals $b_i - a_i + 1$. Obviously, the vertices v_i $(i \in I)$ are mutually distinct, and $v_f = s$. Denote

$$C = \{v_i w_i;\ i \in I\}\,.$$

Moreover, we denote by T' the component of $T - C$ which contains the vertex s. It is clear that T' is a balanced $(m-1)$-quasistar of order 2^{m-1}.

Let G be an n-cube, and let $\{G', G''\}$ be a canonical partition of G. According to the induction hypothesis, T' can be embedded into G'. Thus, we can assume that T' is a subgraph of G'. Denote

$$u_i = v_i / G'' \quad \text{for} \quad i \in I\,.$$

It follows from Theorem 1 that there exists a set of $|I|$ vertexdisjoint paths $P_{(i)}$ $(i \in I)$ in G'' such that $P_{(i)}$ is a u_i-path of order a_i for each $i \in I$. The subgraph of G induced by

$$E(T') \cup \{v_i u_i;\ i \in I\} \cup \bigcup_{i \in I} E(P_{(i)})$$

is isomorphic to T, which completes the proof of the theorem.

Acknowledgement. The author wishes to thank I. Havel and P. Liebl for their stimulating interest.

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