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ALGEBRAS WITH TOLERANCE EXTENSION PROPERTY IN 0

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A tolerance on an algebra A is a reflexive and symmetric binary relation T on A which is compatible with all operators of A (i.e., it is a subalgebra of the direct product $A \times A$). The set of all tolerances on A (denoted by $LT(A)$) forms an algebraic lattice with respect to set inclusion. Therefore, for each two elements a, b of A there exists the least tolerance containing the pair $\langle a, b\rangle$; we denote it by $T_A(a, b)$ and call it the *principal tolerance* (generated by $\langle a, b\rangle$).

Definition 1. A class C of algebras closed under subalgebras is said to possess the (*Principal*) *Tolerance Extension Property* if for each $A \in C$ and every subalgebra B of A, every (principal) tolerance on B is the restriction of a tolerance on A.

We abbreviate the Principal Tolerance Extension Property by **PTEP** and the Tolerance Extension Property by TEP.

It was proved in [1] that the class of all semilattices does not possess TEP but every class of tree-semilattices closed under subsemilattices possesses TEP. Moreover, see Proposition of [1], the class K of lattices closed under sublattices possesses TEP if and only if K is the class of chains.

Conversely, every variety of unary algebras possesses TEP (see Example 5 in [2]), and the variety of all distributive lattices possesses PTEP (see [2]); however, every principal tolerance on a distributive lattice is a congruence (see [3]). Further, the non-trivial variety of semilattices does not possess PTEP (Example 2 in [2]).

Hence we can see than PTEP or TEP occur in classes of algebras only rarely. The aim of this note is to modify the definition of PTEP to occur in other varieties of algebras known from applications.

Definition 2. Let A be an algebra with a constant 0. A possesses a Tolerance Extension Property in 0 (briefly 0-TEP) if for every subalgebra B of A and each x of B, $T_B(0, x)$ is the restriction of $T_A(0, x)$, i.e. if

$$T_A(0, x) \mid B = T_B(0, x)\,,$$

where $T_A(0, x) \mid B$ means $T_A(0, x) \cap (B \times B)$. A class C of algebras with a constant 0 closed under subalgebras possesses 0-TEP if each $A \in C$ has this property.

Clearly $T_A(0, x) \mid B \supseteq T_B(0, x)$ is trivially satisfied, thus 0-TEP is equivalent to

$$T_A(0, x) \mid B \subseteq T_B(0, x)\,.$$

First, we will investigate 0-TEP for a semigroup with zero.

Let S be a semigroup. For x, y of S, by $[x, y]$ we denote the subsemigroup of S generated by x and y. Put $[x] = [x, x]$. Let U be a subsemigroup of S. The notation U^1 stands for U if U has an identity, otherwise it stands for U with an identity adjoined.

Let $a \neq 0$ be an element of a semigroup S with the zero 0. It is easy to show that for $x, y \in S$, $x \neq y$, we have

$$(1) \qquad \langle x, y \rangle \in T_S(0, a) \text{ if and only if } x = 0, \quad y \in S^1 a S^1 \text{ or } x \in S^1 a S^1, \quad y = 0.$$

Theorem 1. *Let S be a semigroup with the zero* 0. *Then S possesses* 0-TEP *if and only if for every pair of elements $x, y \in S$ we have*

$$(2) \qquad x \in S^1 y S^1 \Rightarrow x \in [x, y]^1 \, y[x, y]^1 .$$

Proof. Let S be a semigroup with the zero 0.

Assume that S possesses 0-TEP. Let $x, y \in S$ and $x \in S^1 y S^1$. We can suppose that $x \neq 0$. Put $U = [x, y] \cup \{0\}$. We have $\langle 0, x \rangle \in T_S(0, y) \mid U \subseteq T_U(0, y)$ and so, by (1), we obtain $x \in U^1 y U^1$. Therefore $x \in [x, y]^1 \, y[x, y]^1$.

Let the implication (2) hold. Let U be a subsemigroup of S containing 0 and $a \neq 0$. Suppose that $\langle u, v \rangle \in T_S(0, a) \mid U$ and $u \neq v$. According to (1), we have $u = 0$, $v \in S^1 a S^1$ or $u \in S^1 a S^1$, $v = 0$. Without loss of generality we can assume that $v \in S^1 a S^1$ and so, by (2), we have $v \in [v, a]^1 \, a[v, a]^1 \subseteq U^1 a U^1$. It follows from (1) that $\langle u, v \rangle \in T_U(0, a)$. Hence $T_S(0, a) \mid U \subseteq T_U(0, a)$ and so S possesses 0-TEP.

Corollary 1. *Let S be a commutative semigroup with the zero* 0. *Then S possesses* 0-TEP *if and only if for every pair of elements $x, y \in S$ we have $xy \in [xy]^1 \, [y]$.*

Corollary 2. *Let S be a commutative regular periodic semigroup (particularly a semilattice) with the zero* 0. *Then S possesses* 0-TEP.

(See Example 1.)

Theorem 2. *Let S be a commutative semigroup with the identity* 1. *Then S possesses* 1-TEP.

(For semilattices, see also Example 2.)

Proof. Let S be a commutative semigroup with the identity 1. Let $a \neq 1$ be an element of a subsemigroup U of S with $1 \in U$. It is easy to verify that for $x, y \in U$, $x \neq y$, we have $\langle x, y \rangle \in T_U(1, a)$ if and only if $x = ya^n$ or $y = xa^n$ for some positive integer n. This implies that $T_S(1, a) \mid U \subseteq T_U(1, a)$. Therefore S possesses 1-TEP.

Now, we can generalize our investigations to varieties of algebras having a nullary operation 0. We can characterize varieties possessing 0-TEP by a polynomial condition and, in the case of semilattices, we show this condition is satisfied.

For the sake of brevity, denote the sequence $y_1, y_2, \ldots, y_n$ shortly by $\mathbf{y}_i$.

Theorem 3. *Let V be a variety with a nullary operation* 0. *The following conditions are equivalent*:

(i) *V possesses* 0-TEP;

(ii) *For every* $(2 + n)$*-ary polynomial t there exists a* 5*-ary polynomial s such that*

$$t(0, x, \mathbf{y}_i) = s(0, x, x, t(0, x, \mathbf{y}_i),\ t(x, 0, \mathbf{y}_i)),$$
$$t(x, 0, \mathbf{y}_i) = s(x, 0, x, t(0, x, \mathbf{y}_i),\ t(x, 0, \mathbf{y}_i)).$$

Proof. (i) $\Rightarrow$ (ii). Let $A = F_V(x, y_1, \ldots, y_n)$ be a free algebra of V with the set of free generators $\{x, y_1, \ldots, y_n\}$. Let t be a $(2 + n)$-ary polynomial in V. Put

$$a = t(0, x, y_1, \ldots, y_n), \quad b = t(x, 0, y_1, \ldots, y_n).$$

Let B be an algebra generated by the generators a, b, x. Then B is a subalgebra of A and, evidently,

$$\langle a, b\rangle \in T_A(0, x) \mid B.$$

By Definition 2, $\langle a, b\rangle \in T_B(0, x)$. By Lemma 2 in [4], there exists a binary algebraic function φ over B such that

$$a = \varphi(0, x), \quad b = \varphi(x, 0).$$

Since B has only three generators x, a, b, it means that

$$a = s(0, x, x, a, b), \quad b = s(x, 0, x, a, b)$$

for some 5-ary polynomial s. Hence (ii) is evident.

(ii) $\Rightarrow$ (i). Let $A \in V$, B be a subalgebra of A and $x \in B$. Suppose $\langle a, b\rangle \in T_A(0, x) \mid B$. Then $a, b \in B$ and, by Lemma 2 in [4], there exist a $(2 + n)$-ary polynomial t and elements $c_1, \ldots, c_n \in A$ such that

$$a = t(0, x, c_1, \ldots, c_n), \quad b = t(x, 0, c_1, \ldots, c_n).$$

By (ii), there exists a 5-ary polynomial s with

$$a = s(0, x, x, a, b), \quad b = s(x, 0, x, a, b).$$

Since $x, a, b \in B$, we conclude $\langle a, b\rangle \in T_B(0, x)$, completing the proof.

Example 1. *Every $\wedge$-semilattice with the least element* 0 *possesses* 0-TEP.

Proof. Let t be a $(2 + n)$-ary semilattice polynomial. If t depends neither on the first nor on the second variable, then

$$t(0, x, y_1, \ldots, y_n) = t(x, 0, y_1, \ldots, y_n)$$

and we can put $s(a, b, c, d, e) = d$. If t depends on both the first and the second variable, the argument is the same. Suppose that t depends on the first variable but not on the second. Then

$$t(0, x, \mathbf{y}_i) = 0 \wedge f(\mathbf{y}_i) = 0,$$
$$t(x, 0, \mathbf{y}_i) = x \wedge f(\mathbf{y}_i)$$

for some n-ary semilattice polynomial f. Put $s(a, b, c, d, e) = a \wedge e$. Then

$$s(0, x, x, t(0, x, \mathbf{y}_i), t(x, 0, \mathbf{y}_i)) = 0 \wedge t(x, 0, \mathbf{y}_i) = 0 = t(0, x, \mathbf{y}_i)$$

and

$$s(x, 0, x, t(0, x, y_i), t(x, 0, y_i)) = x \wedge t(x, 0, y_i) =$$
$$= x \wedge (x \wedge f(y_i)) = x \wedge f(y_i) = t(x, 0, y_i)\,.$$

Thus, by Theorem 3, the variety of all $\wedge$-semilattices with 0 possesses 0-TEP (the case when t does not depend on the first but depends on the second variable is similar).

Example 2. *Every* $\vee$*-semilattice with the least element* 0 *possesses* 0-TEP.

Proof. Let t be a $(2 + n)$-ary semilattice polynomial. Analogously as in the proof of the above Example 1, we can investigate only the case when t depends on the first variable but does not depend on the second. Then

$$t(0, x, y_i) = 0 \vee g(y_i) = g(y_i)\,,$$
$$t(x, 0, y_i) = x \vee g(y_i)$$

for some n-ary semilattice polynomial g. Put $s(a, b, c, d, e) = a \vee d$. Then

$$s(0, x, x, t(0, x, y_i), t(x, 0, y_i)) = 0 \vee t(0, x, y_i) = t(0, x, y_i)$$

and

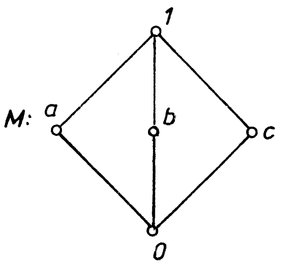
$$s(x, 0, x, t(0, x, y_i), t(x, 0, y_i)) = x \vee t(0, x, y_i) =$$
$$= x \vee (x \vee g(y_i)) = x \vee g(y_i) = t(x, 0, y_i)\,.$$

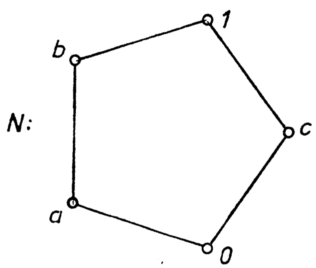
By Theorem 3, the variety of all $\vee$-semilattices with 0 posseses 0-TEP.

Finally, we shall characterize varieties of lattices having 0-TEP.

Theorem 4. *Let V be a variety of lattices with the least element* 0. *Then V has* 0-TEP *if and only if every lattice from V is distributive.*

To prove Theorem 4 it suffices to show that lattices M and N having the following diagrams:





do not possess 0-TEP.

Let $R = \{0, a, b, 1\}$ be a sublattice of M. Clearly $T_R(0, a) = \{\langle a, 0\rangle, \langle 0, a\rangle, \langle b, 1\rangle, \langle 1, b\rangle\} \cup \mathrm{id}_R$ contrary to $T_M(0, a) = M \times M$. Thus $T_M(0, a) \mid R = R \times \times R \neq T_R(0, a)$.

Let $S = \{0, a, b\}$ be a sublattice of N. It is easy to show that $T_S(0, a) = \{\langle a, 0\rangle,$

$\langle 0, a\rangle\} \cup \mathrm{id}_S$ and $T_N(0, a) = S \times S \cup \{c, 1\} \times \{c, 1\}$. Therefore $T_N(0, a) \mid S = S \times S \neq T_S(0, a)$ and the theorem is proved.

Remark. Clearly, if V from Theorem 4 has 0-TEP, then it is either trivial or the variety of all distributive lattices with the least element 0.

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