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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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INTRODUCTION

Most of the fixed point theorems have been derived in a topological space provided with one topology. Usually either the strong or the weak topology has been used. M. Švec in [21], V. M. Sehgal and S. P. Singh in [19] and A. Haščák in [9] worked with two or even three topologies although they did not stress this fact. Here we shall derive fixed point theorems with two or more topologies. The advantage of the method developed consists in dealing with sequences rather than with nets. The results in the paper extend theorems from [21], [9], [17] and [19] to multifunctions acting in metrizable locally convex spaces, and a surjectivity theorem proved by G. Conti and P. Nistri in [3]. The results are also closely related to those of J. Himmelberg in [10], S. Reich in [14] and F. E. Browder in [1].

1. PRELIMINARIES

First we introduce the notation and give the results which will be needed in the sequel. By a *locally convex space* X we shall always understand a locally convex Hausdorff topological vector space. 2^X will denote the family of all nonempty subsets of the space X.

The following definitions are taken from [20] and [5]. If K is a nonempty subset of a locally convex space X, then the multifunction $F: K \to 2^X$ will be said to be *upper semicontinuous in* K, u. sc. for short, iff F is u. sc. at each point $x \in K$. The last statement means that for an arbitrary neighbourhood V of the set-image $F(x)$ there exists such a neighbourhood U of the point x that $F(U(x) \cap K) \subset V$, where $F(U(x) \cap K) = \bigcup_{z \in U(x) \cap K} F(z)$.

F will be said to be *sequentially upper semicompact in* K, s. u. sco. for short, iff F is s. u. sco. at each point $x \in K$. This means that the assumptions $x_n \in K$, $x_n \to x \in K$ as $n \to \infty$, $y_n \in F(x_n)$ imply that there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$, convergent to some $y \in F(x)$.

F will be said to be *sequentially lower semicontinuous in* K, s. l. sc. for short, iff F is s. l. sc. at each point $x \in K$. This means that the assumptions $\{x_n\} \subset K$,

$x_n \to x \in K$ as $n \to \infty$, $y \in F(x)$ imply that there exists a sequence $\{y_n\}$ such that $y_n \to y$ as $n \to \infty$, $y_n \in F(x_n)$.

The relation between sequentially upper semicompactness and upper semicontinuity is given by the following lemma.

Lemma 1 ([18], pp. 1126–1127). *Let X be a metrizable locally convex space, let $\emptyset \neq K \subset X$ be a set and $F: K \to 2^X$ a multifunction. Then F is s. u. sco. in K iff F is u. sc. in K and the set $F(x)$ is compact for each $x \in K$.*

Compactness of the closed convex hull cl co (K) of a compact set is ensured by the Krejn theorem.

Lemma 2 ([13], p. 241). *If K is a compact set of a quasicomplete locally convex space, then the closed convex hull* cl co (K) *of K is compact.*

We recall that a topological vector space is quasi-complete if each bounded, closed subset of that space is complete ([15], p. 41). Since each Cauchy sequence is bounded, a quasi-complete metrizable topological vector space is complete.

We shall use the Himmelberg fixed point theorem ([10], p. 205) which we denote as Lemma 3. Here we need the following notation. A subset $K \neq \emptyset$ of a locally convex space X is said to be *almost convex* if for any neighbourhood U of 0 and for any finite set $\{w_1, \ldots, w_n\}$ of points of K there exist $z_1, \ldots, z_n \in K$ such that $(z_i - w_i) \in U$ for all $i = 1, \ldots, n$, and the convex hull co $\{z_1, \ldots, z_n\} \subset K$.

Lemma 3. *Let L be a nonvoid convex compact subset of a locally convex space X, and let $G: L \to 2^L$ be an u. sc. multifunction such that $G(x)$ is closed for all $x \in L$ and convex for all x in some dense almost convex subset K of L. Then G has a fixed point in L,* i.e., *there is a $u \in L$ such that $u \in G(u)$.*

Another important lemma is the Ky Fan Intersection Lemma.

Lemma 4 ([8], pp. 189–190, [11], p. 354). Let $K \neq \emptyset$ be a compact convex set in a real Hausdorff topological vector space X, or let $K \neq \emptyset$ be a compact set in a real quasi-complete locally convex space X. Let $F: K \to 2^K$ be a multifunction such that (i) $F(x)$ is closed for each $x \in K$, (ii) for any finite set $A = \{x_1, x_2, \ldots, x_n\} \subset K$, the convex hull co (A) of A is contained in $\bigcup_{i=1}^{n} F(x_i)$. Then $\bigcap_{x \in K} F(x) \neq \emptyset$.

2. FIXED POINT THEOREMS

First we will give a new proof of the Himmelberg fixed point theorem (Theorem 2 in [10], p. 206). In the original proof of this theorem there is a gap, since an u. sc. multifunction acting in a locally convex space X need not be u. sc. in the completion $\tilde{X}$ of that space. We recall that if $F: X \to 2^Y$ is a multifunction, X and Y are Hausdorff topological spaces, Y is a compact space and each value of F is closed, then F is u. ac. iff F has a closed graph, i. e., F is a closed subset of $X \times Y$ (see e. g. [10], p. 205).

Let (X, τ) be a locally convex space. Then by Theorem 1.5 and Theorem 4.1 in [16], p. 29 and p. 66, respectively, there exists a complete locally convex space $\tilde{X}$ containing X as a dense subspace. When $\{V_\alpha: \alpha \in A\}$ is the base of the closed, convex, balanced and absorbing neighbourhoods of 0 in X and q_α is the Minkowski functional of V_α for each $\alpha \in A$, then the seminorms $\{q_\alpha: \alpha \in A\}$ have a unique continuous extension $\tilde{q}_\alpha$ to $\tilde{X}$ and the topology $\tilde{\tau}$ of $\tilde{X}$ is determined by the family $\{\tilde{q}_\alpha: \alpha \in A\}$. This implies the following statements:

1. The topology in X induced by $\tilde{\tau}$ coincides with the original topology τ, i.e. $(X, \tilde{\tau}) = (X, \tau)$.

2. If (X, τ) is metrizable, then $(\tilde{X}, \tilde{\tau})$ is metrizable, too, and thus it is a Fréchet space.

3. If K is a compact set in (X, τ), then it is compact in the completion $(\tilde{X}, \tilde{\tau})$ of the space (X, τ).

4. If K is closed in (X, τ), then it need not be closed in $(\tilde{X}, \tilde{\tau})$. For example, if the whole space (X, τ) is not complete, then it is not closed in $(\tilde{X}, \tilde{\tau})$, since the $\tilde{\tau}$-closure of X is $\tilde{X}$. (The $\tilde{\tau}$-closure means the closure with respect to $\tilde{\tau}$-topology. Similarly the τ-compactness will mean the compactness with respect to τ-topology).

Lemma 5. *Let (X, τ) be a locally convex space, and let $(\tilde{X}, \tilde{\tau})$ be its completion. Let K be a nonvoid, convex and τ-closed subset of X. Let $F: K \to 2^K$ be an u. sc. multifunction in the topology τ such that $F(x)$ is a convex and τ-closed subset of K for each $x \in K$ and $\mathrm{cl}\, F(K)_\tau$ (the τ-closure) is τ-compact. Then there exists a multifunction $\tilde{F}: \mathrm{cl}\, K_{\tilde{\tau}} \to 2^{\mathrm{cl}\, F(K)_\tau}$ which is u. sc. in the topology $\tilde{\tau}$, $\tilde{F}(x) = F(x)$ for each $x \in K$ and $\tilde{F}(x)$ is $\tilde{\tau}$-closed for each $x \in \mathrm{cl}\, K_{\tilde{\tau}}$.*

Proof. By the assumptions on F, F is a $\tau \times \tau$-closed subset of $K \times \mathrm{cl}\, F(K)_\tau$. $\tau \times \tau$ denotes the product topology in the cartesian product $X \times X$. The topology $\tilde{\tau} \times \tilde{\tau}$ has a similar meaning. Let $\tilde{F}$ be the $\tilde{\tau} \times \tilde{\tau}$-closure of F, i.e. $\tilde{F} = \mathrm{cl}\, F_{\tilde{\tau} \times \tilde{\tau}}$. Then $\tilde{F} \subset \mathrm{cl}\,(K \times \mathrm{cl}\, F(K)_\tau)_{\tilde{\tau} \times \tilde{\tau}} = \mathrm{cl}\, K_{\tilde{\tau}} \times \mathrm{cl}\,(\mathrm{cl}\, F(K)_\tau)_{\tilde{\tau}} = \mathrm{cl}\, K_{\tilde{\tau}} \times \mathrm{cl}\, F(K)_\tau$, since $\mathrm{cl}\, F(K)_\tau$ is compact in both τ and $\tilde{\tau}$ topologies. Hence for any net $x_\alpha \in K$ converging to an $x \in \mathrm{cl}\, K_{\tilde{\tau}}$ and any $y_\alpha \in F(x_\alpha)$ there exists a subnet $y_{\alpha\beta}$ which converges to an element $y \in \tilde{F}(x)$. Thus $\tilde{F}: \mathrm{cl}\, K_{\tilde{\tau}} \to 2^{\mathrm{cl}\, F(K)_\tau}$ and $\tilde{F}$ is u.sc. in the topology $\tilde{\tau}$. Further, $(x, y) \in \tilde{F}$ iff there is a net $(x_\alpha, y_\alpha) \in F$ such that x_α converges to x, $y_\alpha \in F(x_\alpha)$ and y_α converges to y (both in the topology $\tilde{\tau}$). Hence, if $x \in K$, then $\tilde{F}(x) \supset \mathrm{cl}\, F(x)_\tau = F(x)$, since $F(x)$ is a τ-closed subset of a τ-compact set $\mathrm{cl}\, F(K)_\tau$, and thus $F(x)$ is also τ-compact as well as $\tilde{\tau}$-compact which gives that $F(x)$ is a $\tilde{\tau}$-closed set. If $y \notin F(x)$, then by Theorem 7 in [12], p. 190, there would exist disjoint neighbourhoods $U(F(x))$ and $U(y)$ of the set $F(x)$ and of the point y, respectively. As F is u. sc. at the point x, all values $y_\alpha \in F(x_\alpha)$ for all x_α sufficiently close to x lie in $U(F(x))$ and not in $U(y)$. Thus $y \notin \tilde{F}(x)$ and hence $\tilde{F}(x) = F(x)$ for each $x \in K$. Finally, it follows from the $\tilde{\tau} \times \tilde{\tau}$-closedness of $\tilde{F}$ that the set $\tilde{F}(x)$ is $\tilde{\tau}$-closed for each $x \in \mathrm{cl}\, K_{\tilde{\tau}}$.

Using this lemma we prove the following Himmelberg theorem.

Theorem 1 (See [10], p. 206). *Let K be a nonvoid, convex and closed subset of a locally convex space X. Let $F: K \to 2^K$ be an u. sc. multifunction such that $F(x)$ is closed and convex for all $x \in K$, and* cl $F(K)$ *is compact. Then F has a fixed point.*

Proof. Denote by τ the topology of the locally convex space X and let $(\tilde{X}, \tilde{\tau})$ be the completion of the space (X, τ). By Lemma 5, there exists an u. sc. extension $\tilde{F}$ of the multifunction F which is defined in $\mathrm{cl}\, K_{\tilde{\tau}}$. Put $L = \mathrm{cl}\, \mathrm{co}_{\tilde{\tau}}(\mathrm{cl}\, F(K)_{\tilde{\tau}})$. Here in the middle term both closures are taken with respect to the topology $\tilde{\tau}$. By Lemma 2, L is $\tilde{\tau}$-compact and $L \subset \mathrm{cl}\, K_{\tilde{\tau}}$. Therefore

$$\tilde{F}(L) \subset \tilde{F}(\mathrm{cl}\, K_{\tilde{\tau}}) \subset \mathrm{cl}\, \mathrm{co}_{\tilde{\tau}}(\mathrm{cl}\, F(K)_{\tilde{\tau}}) = L \subset \mathrm{cl}\, K_{\tilde{\tau}} .$$

Thus we see that all assumptions of Lemma 3 are fulfilled with $G = \tilde{F}$ and with $\tilde{X}$ instead of the space X. By this lemma, $\tilde{F}$ has a fixed point $u \in L$. But $u \in \tilde{F}(u) \subset \subset \mathrm{cl}\, F(K)_\tau \subset K$, which gives that $u \in K$ and $u \in F(u)$.

Corollary 1 (The Reich fixed point theorem, [14], p. 193). *Let K be a closed convex subset with nonempty interior of a locally convex space X. Let $F: K \to 2^X$ be an u. sc. multifunction such that $F(x)$ is closed and convex for all $x \in K$, and $F(K)$ is compact. Let F satisfy the Leray-Schauder condition on K:*

There is a point w in the interior of K such that

(L$-$S) *for every $x \in \partial K$ (the boundary of K) and every $y \in F(x)$,*

$$y - w \neq m(x - w) \quad \text{for all} \quad m > 1 .$$

Then F has a fixed point.

An equivalent formulation of the (L$-$S) condition is:

(L$-$S)$'$ If $m(x - w) \in F(x) - w$ for some $x \in \partial K$, then $m \leqq 1$.

The next theorem extends a result by H. Schaeffer in [17] dealing with the method of a priori estimate in the Leray-Schauder theory.

Theorem 2. *Let X be a locally convex space, let $F: X \to 2^X$ be an u. sc. multifunction such that $F(x)$ is closed and convex for all $x \in X$ and there exists a closed, convex, balanced and absorbing neighbourhood U of* 0 *with the property that the sets* cl $F(nU)$ *are compact for all natural n.*

Then either for any $\lambda \in \langle 0, 1 \rangle$ there exists an x such that

$$x \in \lambda F(x) , \tag{1}$$

or the set $\{x \in X: x \in \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded.

Proof. Denote by p the Minkowski functional of the set U. Since 0 is the inner point of U and U is closed, we have that p is a continuous seminorm and $U = \{x \in X: p(x) \leqq 1\}$ (Theorem 3.41-C in [22], p. 137). Clearly 0 is the unique solution of (1) for $\lambda = 0$. If for a $\lambda_0 \in (0, 1\rangle$ there is no solution of (1) for $\lambda = \lambda_0$, we consider the u. sc. multifunction G which is defined by $G(x) = \lambda_0 F(x)$ for all $x \in X$ and we shall show that for any natural n there exists a $y_n \in \mu_n F(y_n)$ with $0 < \mu_n < 1$ and $p(y_n) = n$. This will complete the proof of the theorem.

Let n be a natural number. Define a continuous retraction $r_n: X \to nU$ by $r_n(x) = x$

for all $x \in nU$ and $r_n(x) = n\, x/p(x)$ for all x such that $p(x) > n$. Consider the composition $H_n = G \circ r_n$. Then H_n is u. sc. and satisfies all assumptions of Theorem 1 for $K = X$. By this theorem H_n has a fixed point x_n in X, i.e. $x_n \in H_n(x_n) = G[r_n(x_n)] = \lambda_0\, F[r_n(x_n)]$. The case $p(x_n) \leqq n$ cannot occur, otherwise we would have $x_n \in \lambda_0\, F(x_n)$ which contradicts our assumption. Hence $p(x_n) > n$ and thus $r_n(x_n)\, p(x_n)/n \in \lambda_0.\ F(r_n(x_n))$. This gives that $y_n \in \mu_n\, F(y_n)$ with $y_n = r_n(x_n)$, $\mu_n = n\, \lambda_0/p(x_n) < 1$ and $p(y_n) = n$.

Now we will give a condition under which the multifunction $I - F$ is onto, where I is the identity on X. This means that for each $y \in X$ there exists a solution x of the inclusion $y \in x - F(x)$. This inclusion is equivalent to the inclusion

$$x \in F(x) + y\,. \tag{2}$$

Let $F\colon X \to 2^X$. Similarly as in [3], we say that a real number μ is an eigenvalue of F if there exists an $x \in X$, $x \neq 0$, such that $sx \in F(x)$ for some $s \in R$ and

$$\mu = \sup \{s \in R\colon sx \in F(x)\}\,.$$

Then x is said to be an *eigenvector of* F belonging to μ. Hence if there is an $x \neq 0$ such that $sx \in F(x)$, then either $s \leqq \mu < \infty$ or there is a sequence $\{s_n\} \to \infty$ such that $s_n x \in F(x)$. The latter case cannot occur if $F(x)$ is a bounded set.

Let p be a continuous seminorm in the locally convex space X and let $r > 0$. Consider the extended real number

$b_p(r, F) = \sup \{\mu \geqq 0\colon \mu$ is an eigenvalue of F with an eigenvector x such that $p(x) = r\}$ if there exists an eigenvector with this property, and

$b_p(r, F) = 0$ otherwise.

For any $y \in X$ we define

$(F)_{p,0} = \inf \{b_p(r, F)\colon r > 0\}$;

$(F)_{p,y} = (F + y)_{p,0}$;

$(F_p) = \sup \{(F)_{p,y}\colon y \in X\}$.

Corollary 1 implies the following theorem which is closely related to Theorem 1 in [3], p. 197.

Theorem 3. *Let X be a locally convex space, let $F\colon X \to 2^X$ be an u. sc. multifunction such that $F(x)$ is closed and convex for all $x \in X$ and there exists a closed, convex, balanced and absorbing neighbourhood U of* 0 *with the property that the sets* cl $F(nU)$ *are compact for all natural n. Let p be the Minkowski functional of the set U. Then following statements hold*:

(i) *If $(F)_{p,0} < 1$, then F has a fixed point.*

(ii) *If $(F)_{p,y} < 1$, then the equation* (2) *has a solution.*

(iii) *If $(F)_p < 1$, then $I - F$ is surjective* (*i.e., the equation* (2) *has a solution for each* $y \in X$).

Proof. Since the statement (i) implies (ii) and this in turn implies the statement (iii), it suffices to prove (i). By $(F)_{p,0} < 1$ there exists an $r > 0$ such that $b_p(r, F) < 1$, which means that all eigenvalues corresponding to the eigenvectors x with $p(x) = r$ are smaller than 1. Let $K = \{x \in X: p(x) \leqq r\}$. Since $\mathrm{cl}\, F(K) \subset \mathrm{cl}\, F(L)$ provided $L = \{x \in X: p(x) \leqq n_0\}$ with $r < n_0$, $\mathrm{cl}\, F(K)$ is compact. Hence the condition $(\mathrm{L-S})'$ is satisfied on the boundary of K. Thus by Corollary 1, F has at least one fixed point.

3. FIXED POINT THEOREMS IN METRIZABLE LOCALLY CONVEX SPACES

Let (X, ϱ), (X, σ) and (X, τ) be three locally convex spaces defined on the same vector space X and let the topology ϱ be weaker or equal to the topology σ and the topology σ weaker or equal to the topology τ. We will also assume that (X, τ) is a metrizable space. Denote by $x_n \to_\varrho x$, $x_n \to_\sigma x$ and $x_n \to_\tau x$ the convergence of the sequence $\{x_n\}$ to x as $n \to \infty$ in the space (X, ϱ), in the space (X, σ) and in the space (X, τ), respectively. Clearly

$$(3) \qquad x_n \to_\tau x \quad \text{implies} \quad x_n \to_\sigma x \quad \text{and} \quad x_n \to_\sigma x \quad \text{implies} \quad x_n \to_\varrho x\,.$$

Further, the closedness and the compactness with respect to one of these three topologies will be denoted by the corresponding prefix, e.g. a τ-closed set will mean a closed set with respect to topology τ, and a σ-compact set will denote a set which is compact in the space (X, σ). Similary, $\mathrm{cl}\, A_\tau$ and ∂A_τ will mean the closure of A and the boundary of A, respectively, in the space (X, τ). We shall use the following two implications:

If a subset K of X is τ-compact, then K is σ-compact and the σ-compactness of K implies the ϱ-compactness of K.

Suppose $\emptyset \neq K \subset X$. Given a multifunction $F: K \to 2^X$, F will be said to be $\varrho\tau$ *sequentially lower semicontinuous in* K, $\varrho\tau$ s. l. sc. for short, iff for each point $x \in K$, arbitrary $y \in F(x)$ and arbitrary $\{x_n\} \subset K$ such that $x_n \to_\varrho x$ there exists a sequence $\{y_n\}$, $y_n \in F(x_n)$, such that $y_n \to_\tau y$.

F will be said to be $\varrho\tau$ *sequentially upper semicompact in* K, $\varrho\tau$ s. u. sco. for short, iff for each point $x \in K$, arbitrary $x_n \in K$ such that $x_n \to_\varrho x$ and arbitrary $y_n \in F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ and a $y \in F(x)$ with $y_{n_k} \to_\tau y$ (as $k \to \infty$).

The last property can be strengthened.

F will be said to be *strictly* $\varrho\tau$ *sequentially upper semicompact in* K iff for each $x \in \mathrm{cl}\, K_\varrho$, arbitrary $x_n \in K$ such that $x_n \to_\varrho x$ and arbitrary $y_n \in F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ and a $y \in X$ such that $y_{n_k} \to_\tau y$ where $y \in F(x)$ if $x \in K$.

Comparing the last two definitions we see that the following implication is true: if F is strictly $\varrho\tau$ s. u. sco. in K, then it is also $\varrho\tau$ s. u. sco. in K. Further, we see that both definitions are equivalent if $K = X$.

The following lemmas deal with the properties of the s. u. sco. multifunctions. Some of them generalize the results from [9].

In view of Lemma 1 and (3) we have the following lemma.

Lemma 6. *If $F: K \to 2^X$ is $\varrho\tau$ s. u. sco. in K, then in the topology τ, F is u. sc. in K and the set $F(x)$ is τ-compact for each $x \in K$.*

Proof. By (3) we get that F is s. u. sco. in K with respect to the topology τ. Then the result follows from Lemma 1.

Lemma 7. *If $F: K \to 2^X$ is $\varrho\tau$ s. u. sco. in K and K is ϱ-sequentially compact, then the set $F(K)$ is τ-compact.*

Proof. Let $\{y_n\}$ be a sequence from $F(K)$ and let $\{x_n\}$ be such a sequence that $x_n \in K$ and $y_n \in F(x_n)$, $n = 1, 2, \ldots$. Since K is a ϱ-sequentially compact set, there is a subsequence $\{x_{1n}\}$ of the sequence $\{x_n\}$ and an $x \in K$ such that $x_{1n} \to_\varrho x$. As F is $\varrho\tau$ s. u. sco. in K, there is a subsequence $\{y_{2n}\}$ of the sequence $\{y_{1n}\}$ such that $y_{2n} \to_\tau y \in F(x) \subset F(K)$ and the proof of the lemma is complete.

An alternative to the last lemma is the following lemma. Its proof can be done by modifying the proof of Lemma 7.

Lemma 7'. *If $F: K \to 2^X$ is strictly $\varrho\tau$ s. u. sco. in K, and K is ϱ-sequentially relatively compact, then the set $\operatorname{cl} F(K)_\tau$ is τ-compact.*

The next lemma follows directly from the definition.

Lemma 8. *Let the multifunctions $F_i: K \to 2^X$, $i = 1, 2$, be $\varrho\tau$ s. u. sco. Then the mappings $-F_1, F_1 + F_2$ are also $\varrho\tau$ s. u. sco.*

A more complicated situation arises when we consider the composite multifunction. Suppose we have two multifunctions $F: K \to 2^X$ and $G: L \to 2^X$, where $F(K) \subset \subset L \subset X, K \subset X$. Then the composite multifunction $G \circ F$ is defined by $(G \circ F)(x) = = \{z \in X:$ there is a $y \in F(x)$ such that $z \in G(y)\}$. Suppose, further, that F is $\varrho_F\tau_F$ s. u. sco. and G is $\varrho_G\tau_G$ s. u. sco. This means that in general we can have four (or more) topologies in X, namely $\varrho_F, \tau_F, \varrho_G, \tau_G$ from which ϱ_F is weaker or equal to τ_F and ϱ_G is weaker or equal to τ_G. Here τ_F need not be metrizable, but τ_G should be.

Lemma 9. *If F is $\varrho_F\tau_F$ s. u. sco., G is $\varrho_G\tau_G$ s. u. sco. and the topology ϱ_G is weaker or equal to the topology τ_F, then the composite multifunction $G \circ F$ is $\varrho_F\tau_G$ s. u. sco.*

Proof. Suppose that $x_n \to_{\varrho_F} x$, x_n, $x \in K$, $y_n \in F(x_n)$, $z_n \in G(y_n)$, are arbitrary sequences. Then there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to_{\tau_F} y$ for some $y \in F(x)$. Since the topology ϱ_G is weaker or equal to τ_F, we also have $y_{n_k} \to_{\varrho_G} y$. But G is $\varrho_G\tau_G$ s. u. sco. and therefore there is a subsequence $\{z_{1n_k}\}$ of the sequence $\{z_{n_k}\}$ and a $z \in G(y)$ such that $z_{1n_k} \to_{\tau_G} z$. The relations $z \in G(y)$, $y \in F(x)$ mean that $z \in (G \circ F)(x)$ and the proof is complete.

Under the above assumptions, Theorem 1 has the following modifications.

Theorem 4. *Let K be a nonvoid, convex, τ-closed and ϱ-sequentially compact subset of X. Let $F: K \to 2^K$ be a $\varrho\tau$ s. u. sco. multifunction such that $F(x)$ is convex for each $x \in K$. Then F has a fixed point.*

Proof. We shall work in the space (X, τ). By Lemma 6, F is u. sc. in K and $F(x)$

is τ-compact for each $x \in K$. Further, Lemma 7 gives that $F(K)$ is τ-compact and Theorem 4 follows directly from Theorem 1.

Remark. If we apply Lemma 7′ instead of Lemma 7, we get another modification of Theorem 1. In the same way we can obtain modifications of the other results.

Theorem 4′. *Let K be a nonvoid, convex, τ-closed and ϱ-sequentially relatively compact subset of X. Let $F\colon K \to 2^K$ be a strictly $\varrho\tau$ s. u. sco. multifunction in K such that $F(x)$ is convex for each $x \in K$. Then F has a fixed point.*

While the last theorem extends Theorem 8 from [9], the next theorem generalizes Theorem 9 from the same paper.

Theorem 5. *Let (X_1, σ) be a Fréchet space such that (i) (X, τ) is continuously embedded into (X_1, σ) and hence $X \subset X_1$; (ii) for each sequence $\{x_n\} \subset X$ which is τ-bounded and $x_n \to_\sigma x$ we have $x \in X$. Let K be a nonvoid, convex and τ-closed subset of X. Let $F\colon K \to 2^K$ be a strictly $\varrho\tau$ s. u. sco. multifunction in K such that $F(x)$ is convex for each $x \in K$. Let $F(K)$ be τ-bounded and σ-relatively compact. Then F has a fixed point.*

Proof. Consider the set $K_0 = \mathrm{cl}\,(\mathrm{co}\,(F(K))_\tau$. Then K_0 is a nonvoid, convex, τ-closed and τ-bounded set. On the other hand, $\mathrm{co}\,(F(K))$ is a σ-precompact set and thus $K_1 = \mathrm{cl}\,(\mathrm{co}\,(F(K)))_\sigma$ is also a σ-precompact set. Since the topology σ in X is weaker or equal to the topology τ, we have $K_0 \subset K_1$. Therefore K_0 is σ-precompact, too.

Let $\{x_n\} \subset K_0$ be an arbitrary sequence. Let $\{U_m\}$ be a countable base of closed, convex, balanced and absorbing neighbourhoods of 0 in (X_1, σ). By the σ-precompactness of K_0, for each $m = 1, 2, \ldots$ there exists a subsequence $\{x_{m,n}\}$ of the sequence $\{x_{m-1,n}\}$ with $x_{0,n} = x_n$, $n = 1, 2, \ldots$, and a point $a_m \in X_1$ such that $x_{m,n} \in a_m + U_m$, $n = 1, 2, \ldots$. Hence the diagonal sequence $\{x_{n,n}\}$ enjoys the property $x_{n,n} \in a_m + U_m$ for each $n \geqq m$ and thus we can write $x_{n,n} = a_m + y_{m,n}$ with $y_{m,n} \in U_m$. Therefore $x_{n,n} - x_{k,k} = y_{m,n} - y_{m,k} \in U_m - U_m$ for each $n > k \geqq m$. This implies that $\{x_{n,n}\}$ is a σ-Cauchy sequence which is τ-bounded. Therefore there exists a subsequence $\{x_{n(k),n(k)}\}$ such that $x_{n(k),n(k)} \to_\sigma x$ for some $x \in X_1$ and by (ii), $x \in X$. Hence K_0 is also σ- as well as ϱ-sequentially relatively compact.

As $F(K) \subset K$, on the basis of the properties of K we also have $K_0 \subset K$. Then $F(K_0) \subset F(K) \subset K_0 \subset K$. The result now follows from Theorem 4′, where instead of K we work with K_0. By this theorem F has a fixed point in K_0.

Corollary 1 can be modified as follows.

Corollary 2. *Let K be a convex, τ-closed subset of X with a nonempty τ-interior. Let $F\colon K \to 2^X$ be a $\varrho\tau$ u. sco. multifunction such that $F(x)$ is convex for every $x \in K$. Let K be ϱ-sequentially compact. Let there exist a point w in the interior of K which satisfies the $(\mathrm{L-S})'$ condition. Then F has a fixed point.*

Proof. Similarly as in the proof of Theorem 4 the assumptions of the corollary imply that F is u. sc. in K with respect to the τ-topology, and $F(x)$ for every $x \in K$

as well as $F(K)$ are τ-compact. Since both conditions $(L-S)$ and $(L-S)'$ are equivalent, the result now follows from Corollary 1.

Theorem 2 can be given the following form.

Theorem 6. *Let $F\colon X \to 2^X$ be a $\varrho\tau$ u. sco. multifunction in X such that $F(x)$ is convex for all $x \in X$. Let there exist a τ-closed, convex, balanced and absorbing neighbourhood U of* 0 *which is ϱ-sequentially compact. Then either the inclusion* (1) *has a solution for each $\lambda \in \langle 0, 1 \rangle$ or the set $\{x \in X\colon x \in \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded in the topology τ.*

Proof. In view of Lemmas 6 and 7, the assumptions of the theorem imply that F is u. sc. in (X, τ) and the set $F(x)$ is τ-compact for each $x \in K$. Since the sets nU, $n = 1, 2, \ldots$, are all τ-closed, convex, balanced, absorbing and ϱ-sequentially compact, all $F(nU)$ are τ-compact. Thus Theorem 2 can be applied. By this theorem the alternative in Theorem 6 holds.

Finally, we modify Theorem 3.

Theorem 7. *Let all assumptions of Theorem* 6 *be satisfied. Let p be the Minkowski functional of the set U in the space (X, τ). Then the statements* (i),(ii), (iii) *from Theorem* 3 *hold.*

Proof. By the proof of Theorem 6 we see that all assumptions of Theorem 3 are satisfied. Hence the statement of Theorem 7 follows from Theorem 3.

4. FIXED POINT THEOREMS INVOLVING WEAK TOPOLOGY

In the theorems of the previous section the weak topology can be used instead of the ϱ- or σ-topology, especially when (X, τ) is a reflexive locally convex space ([14], p. 110). Let us recall some facts about the weak topology. When X' is the dual space of (X, τ), the weak $\sigma(X, X')$ topology will be denoted by τ_w. (X, τ_w) is again a locally convex space and the topology τ_w is weaker or equal to τ. Suppose that $K \neq \emptyset$ is a subset of X. If K is closed and convex, then it is weakly closed. Hence, in a reflexive locally convex space X a closed, convex and bounded set K is weakly compact (see [14], p. 110). Since X is metrizable, by the Shmulyan theorem ([7], p. 751) a weakly compact set is also sequentially weakly compact. In the case that (X, τ) is a Banach space the two notions of weak compactness and of sequentially weak compactness are equivalent ([4], p. 92).

Let $F\colon K \to 2^X$ be a multifunction. We shall need the following definitions.

F is said to be *sequentially strongly lower semicontinuous in* K, s. s. l. sc. for short, iff for each point $x \in K$, arbitrary $y \in F(x)$ and arbitrary $\{x_n\} \subset K$ such that $x_n \to x$ weakly there exists a sequence $\{y_n\}$, $y_n \in F(x_n)$, such that $y_n \to y$. (Here and in the sequel the sign $\to$ will denote the strong convergence in X.) Comparing this notion with the above definitions, we see that F is $\tau_w\tau$ s. l. sc.

Similarly, F is said to be *sequentially weakly upper semicompact in* K, s. w. u. sco.

for short, iff for each point $x \in K$, arbitrary $x_n \in K$ such that $x_n \to x$ weakly and arbitrary $y_n \in F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ and $y \in F(x)$ with the property $y_{n_k} \to y$ weakly.

F is said to be *sequentially strongly upper semicompact in* K, s. s. u. sco. for short, iff for each point $x \in X$, arbitrary $x_n \in K$ such that $x_n \to x$ weakly and arbitrary $y_n \in F(x_n)$ there exists a subsequence $\{y_{n_k}\}$ of the sequence $\{y_n\}$ and a $y \in F(x)$ such that $y_{n_k} \to y$. This definition means that F is $\tau_w \tau$ s. u. sco.

Lemmas 6 and 7 can be applied to s. s. u. sco. multifunctions. For a s.w. u. sco. multifunction we have the following lemma.

Lemma 10. *If $F: K \to 2^X$ is a s. w. u. sco. multifunction in K and K is sequentially weakly compact, then*

(i) *the set $F(x)$ is sequentially weakly compact for each $x \in K$;*

(ii) *the set $F(K)$ is also sequentially weakly compact.*

Proof. The statement (i) follows from the definition of the s. w. u. sco. multifunction. If $\{y_n\} \subset F(K)$ is a sequence, then there exists a sequence $\{x_n\} \subset K$ such that $y_n \in F(x_n)$ for each natural n. Then, in view of the sequentially weak compactness of K, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ and a point $x \in K$ such that $x_{n_k} \to x$ weakly. Again by the definition of the s. w. u. sco. multifunction there is a subsequence $\{y_{1n_k}\}$ of the sequence $\{y_{n_k}\}$ and $y \in F(x)$ with the property $y_{1n_k} \to y$ weakly. This proves (ii).

Using the above definitions we can formulate the fixed point theorems from the preceding section in the special case when $\varrho = \tau_w$. For comparison we present here only Corollary 2 in this form.

Corollary 3. *Let K be a convex, closed subset of X with nonempty interior which is sequentially weakly compact (e.g., X is a reflexive Banach space and K is a closed, convex, bounded subset of X with nonempty interior). Let $F: K \to 2^X$ be a s. s. u. sco. multifunction in K such that $F(x)$ is convex for each $x \in K$. Let there exist a point w in the interior of K which satisfies the $(\mathrm{L-S})'$ condition. Then F has a fixed point.*

Now we will prove another fixed point theorem involving the weak topology. It generalizes the result from [19] to multifunctions. Here we will assume that the metrizable locally convex space (X, τ) is real.

Let P denote the family of all continuous seminorms in (X, τ). In addition to Lemma 4 we shall need the following two lemmas.

Lemma 11 ([19], p. 171). *Let $\{x_n\}$ be a sequence in X such that $\{x_n\}$ converges weakly to an $x \in X$. Then for each $p \in P$, we have $p(x) \leqq \liminf_{n \to \infty} p(x_n)$.*

Lemma 12 ([19], p. 173). *Let $K \neq \emptyset$ be a convex and weakly compact subset of X. Then for any $p \in P$ and $z \in X$ there exists $u \in K$ such that*

$$p(u - z) = \min \{p(y - z) : y \in K\} .$$

Theorem 8. *Let $K \neq \emptyset$ be a convex and weakly compact subset of a real metrizable locally convex space X. Let $F: K \to 2^X$ be a s. s. l. sc. and s. w. u. sco. multifunction in K such that $F(x)$ is convex and weakly compact for all $x \in K$. Then for each $p \in P$ there exists $u = u_p \in K$ and $y \in F(u)$ such that*

$$p(u - y) = \min \{p(x - v): x \in K,\ v \in F(u)\}.$$

Proof. Let $p \in P$. Define a multifunction $G: K \to 2^K$ by

$$G(x) = \{y \in K: \inf_{v \in F(y)} p(y - v) \leqq \inf_{v \in F(y)} p(x - v)\}. \tag{4}$$

Clearly $x \in G(x)$. Now we prove that $G(x)$ is weakly closed in X for each $x \in K$.

Let $\{x_\alpha: \alpha \in A\}$ be a net in $G(x)$ such that $x_\alpha \to y$ weakly. Since X is a metrizable locally convex space, $G(x) \subset K$ is a relatively weakly compact set and y belongs to the weak closure of $G(x)$, by Theorem 8. 12. 4. c in [7], p. 752, there exists a sequence $\{x_n\} \subset G(x)$ such that $x_n \to y$ weakly. Using Lemma 12 and (4), we get that there exist $v_n \in F(x_n)$ such that

$$p(x_n - v_n) = \inf_{v \in F(x_n)} p(x_n - v) \leqq \inf_{v \in F(x_n)} p(x - v).$$

Since F is s. w. sco. in K, there exists a subsequence $\{v_{n_k}\}$ of the sequence $\{v_n\}$ and $v \in F(y)$ such that $v_{n_k} \to v$ weakly as $k \to \infty$. On the basis of Lemma 11 we then have

$$p(y - v) \leqq \liminf_{k \to \infty} p(x_{n_k} - v_{n_k}) \leqq \liminf_{k \to \infty} \inf_{\hat{v} \in F(x_{n_k})} p(x - \hat{v}). \tag{5}$$

Let $\tilde{v} \in F(y)$ be an arbitrary element. As F is s. s. l. sc., there exists a sequence $\{\tilde{v}_{n_k}\}$ such that $\tilde{v}_{n_k} \to \tilde{v}$, $\tilde{v}_{n_k} \in F(x_{n_k})$. Hence we have

$$\liminf_{k \to \infty} \inf_{\hat{v} \in F(x_{n_k})} p(x - \hat{v}) \leqq \liminf_{k \to \infty} p(x - \tilde{v}_{n_k}) = \tag{6}$$

$$= \lim_{k \to \infty} p(x - \tilde{v}_{n_k}) = p(x - \tilde{v}).$$

The relations (5) and (6) imply that

$$p(y - v) \leqq \inf_{\tilde{v} \in F(y)} p(x - \tilde{v})$$

and hence $y \in G(x)$, which shows that $G(x)$ is weakly closed.

Now, let a finite subset $A = \{x_1, \ldots, x_n\}$ of K be given and let $z = \sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \geqq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$, be an arbitrary element of co (A). By Lemma 12, there exists $v_i \in F(z)$ such that $p(x_i - v_i) \leqq p(x_i - \tilde{v})$ for all $\tilde{v} \in F(z)$. Denote $v = \sum_{i=1}^{n} \alpha_i v_i$. Since $F(z)$ is convex, $v \in F(z)$. Then

$$\inf_{\tilde{v} \in F(z)} p(z - \tilde{v}) \leqq p(z - v) = p\Big(\sum_{i=1}^{n} \alpha_i (x_i - v_i)\Big) \leqq \sum_{i=1}^{n} \alpha_i\, p(x_i - v_i). \tag{7}$$

If $\inf_{\tilde{v} \in F(z)} p(z - \tilde{v}) > p(x_i - v_i)$ for all $i = 1, \ldots, n$, then $\inf_{\tilde{v} \in F(z)} p(z - \tilde{v}) > \sum_{i=1}^{n} \alpha_i\, p(x_i - v_i)$

which contradicts (7). Thus there exists $i_0 \in \{1, \ldots, n\}$ such that

$$\inf_{\tilde{v} \in F(z)} p(z - \tilde{v}) \leqq p(x_{i_0} - v_{i_0}) = \inf_{\tilde{v} \in F(z)} p(x_{i_0} - \tilde{v})$$

and, consequently, $z \in G(x_{i_0}) \subset \bigcup_{i=1}^{n} G(x_i)$. Hence, by Lemma 4, there exists $u \in \bigcap_{x \in K} G(x)$ which means that

$$\inf_{v \in F(u)} p(u - v) \leqq \inf_{v \in F(u)} p(x - v) \quad \text{for all} \quad x \in K,$$

and, in view of Lemma 12, there exists $y \in F(u)$ with the property

(8) $$p(u - y) \leqq p(x - v) \quad \text{for all} \quad x \in K \quad \text{and all} \quad v \in F(u).$$

Corollary 4. *Let all assumptions of Theorem* 8 *be satisfied. Then either*
(a) *there exist $u \in \partial K$ and $p \in P$ such that*

$$0 < \inf_{v \in F(u)} p(u - v) = \inf_{v \in F(u)} p(x - v) \quad \text{for all} \quad x \in K$$

or
(b) *F has a fixed point $u \in K$.*

Proof. By (8), for each $p \in P$ there exist $u = u_p \in K$ and $y_p \in F(u_p)$ with the property

(9) $$p(u_p - y_p) \leqq p(x - v) \quad \text{for all} \quad x \in K \quad \text{and for all} \quad v \in F(u_p).$$

Suppose that for some $p \in P$,

(10) $$p(u_p - y_p) > 0.$$

Then $y_p \notin K$, otherwise (9) for $x = y_p$ would imply $0 < p(u_p - y_p) \leqq p(y_p - y_p) = 0$, which is a contradiction. Further, $u_p \in \partial K$. In fact, K is weakly closed and hence a closed set. If $u_p \in \operatorname{int}(K)$ (the interior of K), then there would exist a real α, $0 < \alpha < 1$, such that $\alpha u_p + (1 - \alpha) y_p = z \in \partial K$, and by (9)

$$0 < p(u_p - y_p) \leqq p(z - y_p) = \alpha\, p(u_p - y_p) < p(u_p - y_p)$$

which is a contradiction. Hence $u_p \in \partial K$ and this, together with (10), proves (a).

The second case is that for each $p \in P$ there exist $u_p \in K$ and $y_p \in F(u_p)$ such that

(11) $$0 = p(u_p - y_p) \leqq p(x - v) \quad \text{for each} \quad x \in K \quad \text{and for each} \quad v \in F(u_p).$$

Let

$$A_p = \{u \in K\colon p(u - v) = 0 \text{ for some } v \in F(u)\} \quad \text{for each} \quad p \in P.$$

(11) implies that $u_p \in A_p$. We shall prove that A_p is weakly closed. If $\{x_\alpha\colon \alpha \in A\}$ is a net in A_p such that $x_\alpha \to y \in K$ weakly then, similarly as above, there exists a sequence $\{x_n\} \subset A_p$ such that $x_n \to y$ weakly. Since $x_n \in A_p$, there exists $v_n \in F(x_n)$ such that

(12) $$p(x_n - v_n) = 0 \quad \text{for each natural } n.$$

Since F is s. w. u. sco., there exists a subsequence $\{v_{n_k}\}$ of the sequence $\{v_n\}$ and $v \in F(y)$ such that $v_{n_k} \to v$ weakly as $k \to \infty$. Then on the basis of Lemma 11 and

(12) we have

$$p(y - v) \leqq \liminf_{k \to \infty} p(x_{n_k} - v_{n_k}) = 0$$

and thus $p(y - v) = 0$ which yields $y \in A_p$. Furthermore, for any finite subset $\{p_i: i = 1, 2, \ldots, n\} \subset P$ we have $\sum_{i=1}^{n} p_i \in P$ and therefore $\emptyset \neq A_{p_1 + p_2 + \ldots + p_n}$. But $A_{p_1 + p_2 + \ldots + p_n} \subset A_{p_1} \cap A_{p_2} \cap \ldots \cap A_{p_n}$ which implies that the family $\{A_p: p \in P\}$ has the finite intersection property. Consequently $\bigcap_{p \in P} A_p \neq \emptyset$ and there exists $u_0 \in K$ such that for each $p \in P$ there is $v_p \in F(u_0)$ satisfying $p(u_0 - v_p) = 0$.

Let us now define

$$B_p = \{v \in F(u_0): p(u_0 - v) = 0\} \quad \text{for each} \quad p \in P .$$

Clearly $v_p \in B_p$. If $v_1, v_2 \in B_p$, then $p(\alpha u_0 + (1 - \alpha) u_0 - [\alpha v_1 + (1 - \alpha) v_2]) \leqq \leqq \alpha\, p(u_0 - v_1) + (1 - \alpha)\, p(u_0 - v_2) = 0$ for each α, $0 < \alpha < 1$, and hence $[\alpha v_1 + (1 - \alpha) v_2] \in B_p$ which implies that B_p is convex. Furthermore, B_p is strongly closed, since for the sequence $v_n \to v$, $v_n \in B_p$ we have $p(u_0 - v) = \lim_{n \to \infty} p(u_0 - v_n) = = 0$, and in view of the weak closedness of $F(u_0)$, $v \in F(u_0)$. Thus $v \in B_p$. By the above mentioned properties B_p is weakly closed, too. Every B_p is a subset of the weakly compact set $F(u_0)$. Similarly as in the case of the sets A_p we can prove that the system $\{B_p: p \in P\}$ has the finite intersection property. Hence there is $v_0 \in \bigcap_{p \in P} B_p$ and this implies that $p(u_0 - v_0) = 0$ for each $p \in P$. Therefore $u_0 = v_0 \in F(u_0)$.

Corollary 5. *Let all assumptions of Theorem 8 be satisfied. Further, let a multifunction F satisfy the boundary condition*

(13) *for each $u \in \partial K$ there exist $y \in F(u)$, $x \in K$ and $\lambda > 0$ such that*

$$y = u + \lambda(x - u) .$$

Then F has a fixed point.

Proof. Consider $u_p \in \partial K$, $y_p \in F(u_p)$ satisfying (9) and (10). The condition (13) implies that there exist $y \in F(u_p)$, $x \in K$ and $\lambda > 0$ such that

$$y = u_p + \lambda(x - u_p) .$$

Three cases may occur:

(i) $\lambda = 1$. Then $x = y$ and by (9), (10) we come to the contradictory inequality $0 = p(x - y) \geqq p(u_p - y_p) > 0$.

(ii) If $\lambda > 1$, then we can write $x = \alpha u_p + (1 - \alpha) y$ where $\alpha = (1 - \lambda^{-1}) \in (0, 1)$. Consider $v = \alpha y_p + (1 - \alpha) y$. Since y_p, $y \in F(u_p)$ and $F(u_p)$ is convex, $v \in F(u_p)$, too. Then $p(x - v) = \alpha\, p(u_p - y_p) < p(u_p - y_p)$ which contradicts (9), (10).

(iii) When $0 < \lambda < 1$, then $y = \lambda x + (1 - \lambda) u_p$ and since K is convex, $y \in K$. Again by (9), (10) we come to the contradictory inequality $0 = p(y - y) \geqq \geqq p(u_p - v_p) > 0$.

Remarks. 1. The last result is closely related to Corollary 2 in [19], p. 173 which deals with one-valued mappings, as well as to Theorem 3 in [1], p. 286.

2. For $\lambda = 1$ we get that the condition

(14) for each $u \in \partial K$ there exists $y \in F(u)$ such that $y \in K$ implies the condition (13).

Similarly as in the proof of Theorem 5 in [1], p. 288, we can apply the last corollary to the multifunction $G: K \to 2^X$ which is defined by

$$G(x) = 2x - F(x) \quad \text{for each} \quad x \in K .$$

It is clear that the set $G(x)$ is convex and weakly compact iff $F(x)$ shares this property. Further, if F is s. s. l. sc., $x_n \to x$ weakly in K and $z \in G(x)$, then $y = 2x - z \in F(x)$. This implies that there is a sequence $y_n \in F(x_n)$ such that $y_n \to y$. Hence the sequence z_n satisfies $z_n = 2x - y_n \to z = 2x - y$ and $z_n \in G(x_n)$. Therefore F is s. s. l. sc. iff G is s. s. l. sc.. Similarly, either both multifunctions F, G are s. w. u. sco. or none of them is. Hence if F satisfies all assumptions of Theorem 8, so does G. Similarly $u \in F(u) = 2u - G(u)$ iff $u \in G(u)$.

Now we can easily show that if F satisfies the boundary condition

(15) for each $u \in \partial K$ there exist $y \in F(u)$, $x \in K$ and $\lambda < 0$ such that $y = u + \lambda(x - u)$,

then G satisfies the condition (13).

In fact, $y \in F(u)$ iff $2u - y \in G(u)$ and $y = u + \lambda(x - u)$ iff $2u - y = u - \lambda(x - u)$. Thus Corollary 5 yields the following corollary.

Corollary 6. *Let all assumptions of Theorem 8 be satisfied. Further, let a multifunction F satisfy the boundary condition* (15). *Then F has a fixed point.*

Remark. Corollary 5 with the condition (14) instead of (13) is very close to Theorem 4, and Corollary 6 to Theorem 5 in in [1], p. 288.

5. EXAMPLE

Let $I \subset R$ be an arbitrary interval. Let X be the vector space of all continuous and bounded real functions having continuous and bounded derivatives to the m-th order on I. We shall consider four locally convex topologies τ, σ, ϱ and τ_w in this space.

(i) The topology τ is given by the norm

$$|f|_{B_m} = \max_{0 \leq i \leq m} \{\sup_{x \in I} |f^{(i)}(x)|\} , \quad f \in X .$$

Similarly as in [9], the space (X, τ) will be denoted by $B_m(I)$. It is a Banach space. The convergence $f_k \to_\tau f$ means the uniform convergence $f_k^{(j)} \rightrightarrows f^{(j)}$ on I for $j = 0, 1, \ldots, m$.

(ii) If I is compact, then the topology σ coincides with the topology τ. If I is a noncompact interval, the topology σ is given by the sequence of seminorms $\{p_n\}_{n=1}^\infty$

in the following way. If $K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$ is a sequence of compact subintervals of I such that $\bigcup_{n=1}^{\infty} K_n = I$ and $K_n \neq K_{n+1}$, then

$$p_n(f) = \max_{0 \leqq i \leqq m} \{ \sup_{x \in K_n} |f^{(i)}(x)| \} , \quad f \in X , \quad n = 1, 2, \dots .$$

The space (X, σ) will be then denoted by $B\,C_m(I)$. Its locally convex topology does not depend on the choice of $\{K_n\}$. The convergence $f_k \to_\sigma f$ means the uniform convergence $f_k^{(j)} \rightrightarrows f^{(j)}$ on each K_n for $j = 0, 1, \dots, m$, and hence the locally uniform convergence on I of all derivatives up to the order m. By the Ascoli lemma, a set $L \subset X$ is σ-relatively compact iff the derivatives $f^{(j)}, j = 0, 1, \dots, m$, of the functions f from the set L are uniformly bounded on each K_n while the functions $f^{(m)}$ are equicontinuous on each K_n. Further, the topology τ is stronger than the topology σ in the space X and hence the space $B_m(I)$ is continuously embedded into $B\,C_m(I)$.

A simple example shows that the space $B\,C_m(I)$ is not complete. Still, it is a subspace of the Fréchet space $C_m(I)$ which consists of all continuous real functions having continuous derivatives (not necessarily bounded) up to the m-th order on I, and which is provided with the sequence $\{p_n\}$ of seminorms given above. It can be shown that (a) $C_m(I)$ is the completion of the space $B\,C_m(I)$; (b) $B_m(I)$ is continuously embedded into $C_m(I)$, and (c) for each sequence $\{f_k\} \subset B_m(I)$ which is τ-bounded and $f_k \to_\sigma f \in C_m(I)$ we have $f \in B_m(I)$.

(iii) The topology ϱ is determined by the uncountable system of seminorms

$$p_{x,i}(f) = |f^{(i)}(x)| , \quad f \in X , \quad x \in I , \quad i = 0, 1, \dots, m .$$

The local base for the locally convex space (X, ϱ) which we denote by $q\,B_m(I)$ consists of the neighbourhoods

$$U(x_1, x_2, \dots, x_k;\ i_1, i_2, \dots, i_k;\ n_1, n_2, \dots, n_k) =$$
$$= \{f \in X: |f^{(i_1)}(x_1)| < n_1^{-1},\ |f^{(i_2)}(x_2)| < n_2^{-1}, \dots, |f^{(i_k)}(x_k)| < n_k^{-1}\}$$

determined for all natural numbers k, all k-tuples $(x_1, x_2, \dots, x_k)$ of real numbers from I, all k-tuples $(i_1, i_2, \dots, i_k)$ of integers i_j, $0 \leqq i_j \leqq m$, and all k-tuples $(n_1, \dots, n_k)$ of natural numbers. Clearly there exists no countable local base and hence $q\,B_m(I)$ is not metrizable. The convergence $f_k \to_\varrho f$ mans the pointwise convergence $f_k^{(j)}(x) \to f^{(j)}(x)$ at each point $x \in I$ for $j = 0, 1, \dots, m$. The set $K \subset X$ is ϱ-sequentially relatively compact iff every sequence $\{f_k\} \subset K$ contains a subsequence $\{f_{kl}\}$ such $f_{kl}^{(j)}(x) \to f^{(j)}(x)$ at each point $x \in I$ for $j = 0, 1, \dots, m$ and for some $f \in X$. If f belongs to K, then K is ϱ-sequentially compact.

(iv) The weak topology τ_w in the space X associated with the topology τ. We denote the space (X, τ_w) by $w\,B_m(I)$. When the interval I is compact, then the space $B_m(I)$ is separable and by [6], p. 374, the sequence $\{f_k\} \subset B_m(I)$ converges weakly to $f \in B_m(I)$ iff it is strongly bounded and $f_k \to_\varrho f$. In the general case when I is noncompact, Theorem 31, [6], p. 305 and Definition 10 in [6], p. 291 imply that $f_k \to_{\tau_w} f$ iff $|f_k|_{B_m}$ is bounded, $f_k \to_\varrho f$ and for each $\varepsilon > 0$, any increasing subsequence $\{n_k\}$

of natural numbers, each $j \in \{0, 1, \ldots, m\}$ and each $n_{k_0} \in \{n_k\}$ there exist finitely many indices $n_{k_0} < n_{k_1} < n_{k_2} < \ldots < n_{k_l}$ such that

$$\min_{1 \leqq i \leqq l} \left|f^{(j)}_{n_{k_i}}(x) - f^{(j)}(x)\right| < \varepsilon \quad \text{for each} \quad x \in I\,.$$

The topologies ϱ and τ_w are determined by the total vector subspaces Γ_1, Γ_2, respectively, of the vector space X of all linear functionals over X. Γ_1 consists of the functionals

$$L_{x,i}(f) = f^{(i)}(x)\,, \quad f \in X\,,$$

for each $x \in I$, $i = 0, 1, \ldots, m$, while $\Gamma_2 = X'$ is the dual space of the space (X, τ). Since $\Gamma_1 \subset \Gamma_2$, the topology ϱ is weaker or equal to τ_w. If $\langle a, b\rangle \subset I$, then $L(f) = \int_a^b f(x)\,\mathrm{d}x \in \Gamma_2$ but $L \notin \Gamma_1$ and hence ϱ is weaker than τ_w.

Further, the space Γ_0 of all linear functionals $L_{x,0}(f) = f(x), f \in X$, is also total and thus Γ_0 determines a topology which is weaker than ϱ provided $m > 0$. Hence $q\ B_m(I)$ as well as $w\ B_m(I)$ are not minimal and they are both weak. This, on the basis of Theorem 22. 16 in [2], p. 60, implies that the following lemma is true.

Lemma 13. *Neither of the spaces $q\ B_m(I)$ $(m \geqq 1)$ and $w\ B_m(I)$ is complete.*

Since $B_m(I)$ is a Banach space, the set $K \subset B_m(I)$ is weakly compact iff it is sequentially weakly compact. The relation between the weak convergence and the ϱ-convergence gives the result. If I is a compact interval, then the subset $K \subset B_m(I)$ is relatively weakly compact iff it is strongly bounded and ϱ-sequentially relatively compact. If $I = \langle a, \infty)$, the situation is more complicated. A sufficient condition for the weak compactness in this case is given in the following lemma.

Lemma 14. *Suppose that $I = \langle a, \infty)$, where $a \in R$. Then the set $K \subset B_m(I)$ is weakly compact if the following conditions are satisfied:*

(i) *K is strongly bounded;*
(ii) *K is weakly closed;*
(iii) *K is ϱ-sequentially relatively compact;*
(iv) *for each $j \in \{0, 1, \ldots, m\}$ and each $f \in K$ there exists a finite $\lim_{x \to \infty} f^{(j)}(x)$;*
(v) *for each sequence $\{f_k\} \subset K$, each $j \in \{0, 1, \ldots, m\}$ and each $\varepsilon > 0$ there exists a neighbourhood $U(\infty)$ of ∞ and a k_0 such that*

$$\left|f_k^{(j)}(x_1) - f_k^{(j)}(x_2)\right| < \varepsilon$$

for each $k \geqq k_0$ and any two points $x_1, x_2 \in U(\infty)$.

Proof. Let $\{f_l\}$ be a sequence in K. By (iii), there is a subsequence $\{f_k\}$ of that sequence such that

$$f_k \to_\varrho f \quad \text{as} \quad k \to \infty\,. \tag{16}$$

(i) implies that $|f_k|_{B_m}$ is bounded. By what was said above, $\{f_k\}$ will weakly converge to f if for each $\varepsilon > 0$, any increasing subsequence $\{k_p\}$ of the sequence $\{k\}$, each $j \in \{0, 1, \ldots, m\}$ and each k_{p_0} there exist finitely many indices $k_{p_0} < k_{p_1} < \ldots < k_{p_r}$

such that

$$(17) \qquad \min_{1 \leqq i \leqq r} \left|f_{k_{p_i}}^{(j)}(x) - f^{(j)}(x)\right| < \varepsilon \quad \text{for each} \quad x \in I .$$

Let $\varepsilon > 0$, $j \in \{0, 1, \ldots, m\}$, a subsequence $\{k_p\}$ of the sequence $\{k\}$ and a k_{p_0} be given. We shall denote this subsequence by $\{k\}$. By (iv), (v), (16), there exist $U(\infty)$ and $k_0 \geqq k_{p_0}$ such that for each $k \geqq k_0$, each $x \in U(\infty)$ and a fixed $x_1 \in U(\infty)$ we have

$$(18) \qquad \left|f_k^{(j)}(x) - f^{(j)}(x)\right| \leqq \left|f_k^{(j)}(x) - f_k^{(j)}(x_1)\right| + \left|f_k^{(j)}(x_1) - f^{(j)}(x_1)\right| + \\ + \left|f^{(j)}(x_1) - f^{(j)}(x)\right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

Since $I - U(\infty) = K$ is a compact set and all functions involved in the consideration are continuous, the Arzela theorem ([6], p. 291) implies that the convergence of the sequence $f_k^{(j)}$ to $f^{(j)}$ in K is quasiuniform. Hence, there are finitely many indices $k_0 < k_1 < \ldots < k_r$ such that

$$(19) \qquad \min_{1 \leqq i \leqq r} \left|f_{k_i}^{(j)}(x) - f^{(j)}(x)\right| < \varepsilon \quad \text{for each} \quad x \in K .$$

The inequalities (18) and (19) imply that (17) is true (with f_{k_i} instead of $f_{k_{p_i}}$, $i = 1, 2, \ldots, r$). (ii) gives that $f \in K$ and the proof of the lemma is complete.

Remarks. 1. If K is convex, then (ii) can be replaced by the assumption

(ii') K is strongly closed.

2. When $I = (a, b)$, then Lemma 14 remains to be true if the following changes are made. In (iv) the existence of a finite $\lim f^{(j)}(t)$ should be assumed at both endpoints of (a, b). In (v), a neighbourhood of both endpoints a, b in the form $I - K_1$ where K_1 is a compact subinterval of I should be considered instead of $U(\infty)$.

References

[1] *Browder, F., E.:* The Fixed Point Theory of Multi-valued Mappings in Topological Vector Spaces. Math. Ann. *177* (1968), 283—301.

[2] *Choquet, G.:* Lectures on Analysis, Volume II. Representation Theory. W. A. Benjamin, Inc. New York, Amsterdam, 1969.

[3] *Conti, G. e Nistri, P.:* On surjectivity for *e*-quasibounded multivalued maps. Atti Acad. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. 61 (1976), 195—201.

[4] *Day, M., M.:* Normed Linear Spaces. Springer-Verlag, Berlin—Göttingen—Heidelberg 1958 (Russian translation: Izdat. inostr. lit. Moscow, 1961).

[5] *Deimling, K.:* Nonlinear Functional Analysis. Springer-Verlag, Berlin, Heidelberg, 1985.

[6] *Dunford, N., Schwartz, J., T., Bade, W., G., Bartle, R., G.:* Linear Operators. Part I: General Theory. Interscience Publishers, New York, London 1958 (Russian translation: Izdat. inostr. lit. Moscow 1962).

[7] *Edwards, R., E.:* Functional Analysis-Theory and Applications. Holt, Rinehart and Winston, New York 1965 (Russian translation: Izdat. Mir, Moscow, 1969).

[8] *Fan, K.:* Applications of a Theorem Concerning Sets with Convex Sections. Math. Ann. *163* (1966), 189—203.

[9] *Haščák, A.:* Fixed Point Theorems for Multivalued Mappings. Czech. Math. J. *35 (110)* (1985), 533—542.

[10] *Himmelberg, J.:* Fixed Points of Compact Multifunctions. J. Math. Anal. Appl. *38* (1972), 205—207.
[11] *Kapoor, O., P.:* On an Intersection Lemma. J. Math. Anal. and Appl. *45* (1974), 354—356.
[12] *Kelley, J., L.:* General Topology. Van Nostrand Co. Inc., New York, 1957 (Russian translation: Izdat. Nauka, Moscow, 1968).
[13] *Köthe, K.: Topological Vector Spaces, I. Springer-Verlag,* Berlin—Heidelberg—New York, 1969.
[14] *Reich, S.:* A Remark on Set-Valued Mappings that Satisfy the Leray-Schauder Condition. Atti Accad. Naz. Lincei, Rend. Cl. Sci. fis. mat. nat. *61* (1976), 193—194.
[15] *Robertson, A., P.* and *Robertson, W.:* Topological Vector Spaces. Cambridge Univ. Press 1964 (Russian translation: Izdat. Mir, Moscow, 1967).
[16] *Schaefer, H., H.:* Topological Vector Spaces. Mac Millan Co., New York, 1966 (Russian translation: Izdat. Mir, Moscow, 1971).
[17] *Schaefer, H.:* Über die Methode der a priori-Schranken. Math. Ann. *129* (1955), 415—416.
[18] *Seah, S., W.:* Asymptotic Equivalence of Multivalued Differential Systems. Boll. Un. Mat. Ital. (5), *17* (1980), 1124—1145.
[19] *Sehgal, V., M.* and *Singh, S., P.:* A Variant of a Fixed Point Theorem of Ky Fan. Indian J. Math. *25* (1983), 171—174.
[20] *Sobieszek, W., Kowalski, P.:* On the Different Definitions of the Lower Semicontinuity, Upper Semicontinuity. Upper Semicompacity, Closity and Continuity of the Point-to-Set Maps. Demonstratio Mathematica, *11* (1978), 1053—1063.
[21] *Švec, M.:* Fixpunktsatz und monotone Lösungen der Differentialgleichung $y^{(n)} + B(x, y, y', \ldots, y^{(n-1)})\, y = 0$, Archivum Mathematicum (Brno), *2* (1966), 43—55.
[22] *Taylor, A., E.:* Introduction to Functional Analysis. J. Wiley and Sons, Inc., New York, 1967 (Czech translation: Academia, Praha, 1973).
[23] *Yosida, K.:* Functional Analysis. Springer-Verlag, Berlin—Heidelberg—New York, 1980.
[24] *Zeidler, E.:* Vorlesungen über nichtlineare Funktionalanalysis *I*-Fixpunktsätze. BSB B. G. Teubner Verlagsgeselschaft, Leipzig, 1976.

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