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ON A CLASS OF FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Dedicated to Professor Tosihusa Kimura on his sixtieth birthday

1. INTRODUCTION

We consider the neutral functional differential equations

$$\text{(A)}\qquad \frac{d}{dt}[x(t) - h(t)\,x(\tau(t))] = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))),$$

$$\text{(B)}\qquad \frac{d}{dt}[x(t) - h(t)\,x(\tau(t))] + \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) = 0.$$

under the standing hypotheses that:

(a) $h \in C([t_0, \infty), (0, \infty))$;
(b) $\tau \in C([t_0, \infty), \mathbb{R})$; τ is strictly increasing and $\lim_{t\to\infty} \tau(t) = \infty$;
(c) $q_i \in C([t_0, \infty), \mathbb{R})$, $q_i(t) \geqq 0$, $\not\equiv 0$, $1 \leqq i \leqq N$;
(d) $f_i \in C(\mathbb{R}, \mathbb{R})$; f_i is nondecreasing and $u f_i(u) > 0$ for $u \neq 0$, $1 \leqq i \leqq N$;
(e) $g_i \in C([t_0, \infty), \mathbb{R})$ and $\lim_{t\to\infty} g_i(t) = \infty$, $1 \leqq i \leqq N$.

Our primary concern is with the oscillatory (and nonoscillatory) behavior of solutions of equations (A) and (B). By a solution of (A) (or (B)) we mean a continuous function $x: [T_x, \infty) \to \mathbb{R}$ such that $x(t) - h(t)\,x(\tau(t))$ is continuously differentiable and satisfies equation (A) (or (B)) for all sufficiently large $t > T_x$. The solutions which vanish for all large t will be excluded from our consideration. A solution of (A) (or (B)) is said to be *oscillatory* if it has an infinite sequence of zeros tending to infinity; otherwise a solution is said to be *nonoscillatory*.

After classifying the set of possible nonoscillatory solutions of (A) (or (B)) according to their asymptotic behavior as $t \to \infty$, we establish criteria for oscillation of all solutions of (A) (or (B)), i.e., nonexistence of nonoscillatory solutions, on the basis of known results regarding the non-neutral functional differential inequalities. Then, we derive sufficient conditions for the existence of certain classes of nonoscillatory solutions of (A) (or (B)) with the aid of the Schauder-Tychonoff fixed point theorem. As a result, we are able to prove the following theorems indicating classes of neutral equations for which the situation for oscillation of all solutions can be characterized.

Theorem 1.1. *Suppose that* f_i, $1 \leqq i \leqq N$, *satisfy*

$$\int_0^m \frac{du}{f_i(u)} < \infty \quad \text{and} \quad \int_0^{-m} \frac{du}{f_i(u)} < \infty \quad \text{for any} \quad m > 0 .$$

(i) *All solutions of equation* (A) *subject to the conditions*

$$1 < \lambda_* \leqq h(t) \leqq \lambda^* , \quad \tau(t) > t \quad \text{and} \quad g_i(t) < \tau(t) , \quad 1 \leqq i \leqq N ,$$

λ_* *and* λ^* *being constants, are oscillatory if and only if*

$$\sum_{i=1}^{N} \int_{t_0}^{\infty} q_i(t)\, dt = \infty . \tag{1.1}$$

(ii) *All solutions of equation* (B) *subject to the conditions*

$$0 < h(t) \leqq \lambda^* < 1 , \quad \tau(t) < t \quad \text{and} \quad g_i(t) < t , \quad 1 \leqq i \leqq N ,$$

λ^* *being a constant, are oscillatory if and only if* (1.1) *is satisfied.*

Theorem 1.2. *Suppose that* f_i, $1 \leqq i \leqq N$, *satisfy*

$$\int_M^{\infty} \frac{du}{f_i(u)} < \infty \quad \text{and} \quad \int_{-M}^{-\infty} \frac{du}{f_i(u)} < \infty \quad \text{for any} \quad M > 0 .$$

(i) *All solutions of equation* (A) *subject to the conditions*

$$0 < h(t) \leqq \lambda^* < 1 , \quad \tau(t) > t \quad \text{and} \quad g_i(t) > t , \quad 1 \leqq i \leqq N ,$$

λ^* *being a constant, are oscillatory if and only if* (1.1) *is satisfied.*

(ii) *All solutions of equation* (B) *subject to the conditions*

$$1 < \lambda_* \leqq h(t) \leqq \lambda^* , \quad \tau(t) < t \quad \text{and} \quad g_i(t) > \tau(t) , \quad 1 \leqq i \leqq N ,$$

λ_* *and* λ^* *being constants, are oscillatory if and only if* (1.1) *is satisfied.*

The problem of oscillation and nonoscillation for neutral functional differential equations has received considerable attention in recent years; see e.g. [1–5, 7–14] and the references cited therein. Most of the literature, however, centers around linear equations with constant coefficients and constant deviations to which the theory of characteristic equations applies, and very little is known about nonlinear equations with general coefficients and deviating arguments. To the best of the authors' knowledge, the papers [2–4, 7, 11] are the only references aiming at a systematic investigation in the latter direction. The principal feature of this paper is that unlike [2–4, 7, 11] the four cases

$$\{h(t) < 1, \tau(t) < t\} , \quad \{h(t) < 1, \tau(t) > t\} , \quad \{h(t) > 1, \tau(t) < t\} ,$$
$$\{h(t) > 1, \tau(t) > t\}$$

are examined for both equations (A) and (B).

2. CLASSIFICATION OF NONOSCILLATORY SOLUTIONS

We begin by analysing the asymptotic behavior of possible nonoscillatory solutions of equations (A) and (B). The following notation will be used extensively:

$$\tau^0(t) \equiv t\,, \quad \tau^i(t) = \tau(\tau^{i-1}(t))\,, \quad \tau^{-i}(t) = \tau^{-1}(\tau^{-(i-1)}(t))\,, \quad i = 1, 2, \ldots, \tag{2.1}$$

where $\tau^{-1}(t)$ denotes the inverse function of $\tau(t)$.

Let $x(t)$ be a nonoscillatory solution of (A) (or (B)). Put

$$y(t) = x(t) - h(t)\,x(\tau(t))\,. \tag{2.2}$$

Then, from (A) (or (B)), $y(t)$ is eventually monotone, so that $y(t)$ has to be eventually of constant sign. Therefore, either

$$x(t)\,y(t) > 0 \tag{2.3}$$

or

$$x(t)\,y(t) < 0 \tag{2.4}$$

for all sufficiently large t. Denote by $\mathcal{N}^+$ [or $\mathcal{N}^-$] the set of all nonoscillatory solutions $x(t)$ of (A) (or (B)) such that (2.3) [or (2.4)] is satisfied. Let us introduce the following subclasses of $\mathcal{N}^+$ and $\mathcal{N}^-$:

$$\mathcal{N}_0^+ = \{x \in \mathcal{N}^+ : \lim_{t\to\infty} y(t) = 0\}\,,$$

$$\mathcal{N}_c^+ = \{x \in \mathcal{N}^+ : \lim_{t\to\infty} |y(t)| = \text{const} > 0\}\,,$$

$$\mathcal{N}_\infty^+ = \{x \in \mathcal{N}^+ : \lim_{t\to\infty} |y(t)| = \infty\}\,.$$

$$\mathcal{N}_0^- = \{x \in \mathcal{N}^- : \lim_{t\to\infty} y(t) = 0\}\,,$$

$$\mathcal{N}_c^- = \{x \in \mathcal{N}^- : \lim_{t\to\infty} |y(t)| = \text{const} > 0\}\,,$$

$$\mathcal{N}_\infty^- = \{x \in \mathcal{N}^- : \lim_{t\to\infty} |y(t)| = \infty\}\,.$$

It is easy to see that the totality $\mathcal{N}$ of all possible nonoscillatory solutions of (A) (or (B)) has, in general, the following decomposition:

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_\infty^+ \cup \mathcal{N}_c^+ \cup \mathcal{N}_0^- \cup \mathcal{N}_c^- \quad \text{for} \quad (A)\,, \\ \mathcal{N} &= \mathcal{N}_c^+ \cup \mathcal{N}_0^+ \cup \mathcal{N}_c^- \cup \mathcal{N}_\infty^- \quad \text{for} \quad (B)\,. \end{aligned} \tag{2.5}$$

The simple lemmas below indicate that additional restrictions upon $h(t)$ and $\tau(t)$ may force some of the nonoscillatory solution classes appearing in (2.5) to be empty.

Lemma 2.1. *In addition to the conditions* (a) *and* (b) *suppose that*

$$0 < h(t) \leqq 1 \quad \text{for} \quad t \geqq t_0\,. \tag{2.6}$$

Let $x(t)$ *be a continuous nonoscillatory solution of the functional inequality*

$$x(t)\,[x(t) - h(t)\,x(\tau(t))] < 0 \tag{2.7}$$

defined in a neighborhood of infinity.

(i) *Suppose that* $\tau(t) < t$ *for* $t \geqq t_0$. *Then* $x(t)$ *is bounded. If, moreover,*

$$0 < h(t) \leqq \lambda^* < 1\,, \quad t \geqq t_0\,, \tag{2.8}$$

for some positive constant λ^*, *then* $\lim_{t\to\infty} x(t) = 0$.

(ii) *Suppose that* $\tau(t) > t$ *for* $t \geqq t_0$. *Then* $x(t)$ *is bounded away from zero, that is, there exists a positive constant* c *such that* $|x(t)| \geqq c$ *for all large* t. *If, moreover,* (2.8) *holds, then* $\lim_{t\to\infty} |x(t)| = \infty$.

Lemma 2.2. *In addition to* (a) *and* (b) *suppose that*

$$1 \leqq h(t) \quad for \quad t \geqq t_0\,. \tag{2.9}$$

Let $x(t)$ *be a continuous nonoscillatory solution of the functional inequality*

$$x(t)\,[x(t) - h(t)\,x(\tau(t))] > 0 \tag{2.10}$$

defined in a neighborhood of infinity.

(i) *Suppose that* $\tau(t) > t$ *for* $t \geqq t_0$. *Then* $x(t)$ *is bounded. If, moreover,*

$$1 < \lambda_* \leqq h(t)\,, \quad t \geqq t_0\,, \tag{2.11}$$

for some positive constant λ_*, *then* $\lim_{t\to\infty} x(t) = 0$.

(ii) *Suppose that* $\tau(t) < t$ *for* $t \geqq t_0$. *Then* $x(t)$ *is bounded away from zero. If, moreover,* (2.11) *holds, then* $\lim_{t\to\infty} |x(t)| = \infty$.

Proof of Lemma 2.1. Let $x(t)$ be a nonoscillatory solution of (2.7). We may assume that $x(t) > 0$ on $[T_x, \infty)$.

(i) Let $\tau(t) < t$ for $t \geqq t_0$. Then, in view of (2.6),

$$x(t) < h(t)\,x(\tau(t)) \leqq x(\tau(t)) \quad \text{for all large } t\,,$$

which implies that $x(t)$ is bounded. If (2.8) holds, we have by iteration

$$x(\tau^{-n}(t)) < (\lambda^*)^n\,x(t) \to 0 \quad \text{as} \quad n \to \infty\,,$$

which implies that $\lim_{t\to\infty} x(t) = 0$.

(ii) Let $\tau(t) > t$ for $t \geqq t_0$. Then, using (2.6) we have

$$x(\tau(t)) \geqq h(t)\,x(\tau(t)) > x(t) \quad \text{for} \quad t \geqq T_x\,,$$

which implies that $x(t)$ has to be bounded away from zero. If (2.8) holds, we obtain

$$x(\tau^n(t)) > (1/\lambda^*)^n\,x(t) \to \infty \quad \text{as} \quad n \to \infty\,,$$

whence it follows that $\lim_{t\to\infty} x(t) = \infty$. This completes the proof.

The proof of Lemma 2.2 is similar, and so we omit it.

Motivated by the above lemmas we will distinguish the eight cases for $\{h(t), \tau(t)\}$ listed below:

$$(2.12)\qquad \begin{aligned} &\text{(I) } 0 < h(t) \leqq 1\,, \quad \tau(t) < t \text{ for } t \geqq t_0\,;\\ &\text{(I') } 0 < h(t) \leqq \lambda^* < 1\,, \quad \tau(t) < t \text{ for } t \geqq t_0\,;\\ &\text{(II) } 1 \leqq h(t) \leqq \lambda^*\,, \quad \tau(t) > t \text{ for } t \geqq t_0\,;\\ &\text{(II') } 1 < \lambda_* \leqq h(t) \leqq \lambda^*\,, \quad \tau(t) > t \text{ for } t \geqq t_0\,;\\ &\text{(III) } 1 \leqq h(t) \leqq \lambda^*\,, \quad \tau(t) < t \text{ for } t \geqq t_0\,;\\ &\text{(III') } 1 < \lambda_* \leqq h(t) \leqq \lambda^*\,, \quad \tau(t) < t \text{ for } t \geqq t_0\,;\\ &\text{(IV) } 0 < h(t) \leqq 1\,, \quad \tau(t) > t \text{ for } t \geqq t_0\,;\\ &\text{(IV') } 0 < h(t) \leqq \lambda^* < 1\,, \quad \tau(t) > t \text{ for } t \geqq t_0\,. \end{aligned}$$

In what follows, for simplicity, the equation (A) subject to the case (I) etc. will be referred to as equation (A-I) etc. From Lemmas 2.1 and 2.2 we easily conclude that

$$\begin{aligned} &\mathcal{N}_\infty^- = \emptyset \quad \text{for} \quad \text{(B-I)}\,; \quad \mathcal{N}_\infty^+ = \emptyset \quad \text{for} \quad \text{(A-II)}\,;\\ &\mathcal{N}_c^- = \emptyset \quad \text{for} \quad \text{(A-I')} \quad \text{and} \quad \text{(B-I')}\,; \quad \text{and}\\ &\mathcal{N}_c^+ = \emptyset \quad \text{for} \quad \text{(A-II')} \quad \text{and} \quad \text{(B-II')}\,. \end{aligned}$$

It follows in particular that

$$\mathcal{N}^- = \emptyset \quad \text{for} \quad \text{(B-I')} \quad \text{and} \quad \mathcal{N}^+ = \emptyset \quad \text{for} \quad \text{(A-II')}\,,$$

that is, (B-I′) has no nonoscillatory solution $x(t)$ satisfying (2.4), and (A-II′) has no nonoscillatory solution satisfying (2.3).

3. OSCILLATION THEOREMS

We intend to establish criteria for oscillation of all solutions of equations (A) and (B) subject to (I)–(IV) by imposing suitable restrictions on $q_i(t)$, $g_i(t)$, $f_i(u)$, $1 \leqq i \leqq N$, which preclude all the solution classes appearing in (2.5). The following result is needed for this purpose.

Lemma 3.1. *Let the conditions* (c), (d) *and* (e) *be satisfied and suppose that*

$$(3.1)\qquad \sum_{i=1}^{N} \int_{t_0}^{\infty} q_i(t)\,\mathrm{d}t = \infty\,.$$

Then the inequality

$$(3.2)\qquad \{u'(t) + \sum_{i=1}^{N} q_i(t) f_i(u(g_i(t)))\} \operatorname{sgn} u(t) \leqq 0$$

has no nonoscillatory solution $u(t)$ *such that* $\liminf_{t\to\infty} |u(t)| > 0$, *and the inequality*

$$(3.3)\qquad \{u'(t) - \sum_{i=1}^{N} q_i(t) f_i(u(g_i(t)))\} \operatorname{sgn} u(t) \geqq 0$$

has no nonoscillatory solution $u(t)$ *such that* $\limsup_{t\to\infty} |u(t)| < \infty$.

Proof. Assume for contradiction that (3.2) has a nonoscillatory solution $u(t)$ such that $\liminf_{t\to\infty} |u(t)| > 0$. We may suppose with no loss of generality that $u(t)$ is eventually positive. Then there are constants $T \geqq t_0$ and $c > 0$ such that $u(g_i(t)) \geqq c$ for $t \geqq T$ and $1 \leqq i \leqq N$. By condition (d), $f_i(u(g_i(t))) \geqq f_i(c)$ for $t \geqq T$, $1 \leqq i \leqq N$, and so (3.2) implies

$$(3.4) \qquad u'(t) \leqq -\sum_{i=1}^{N} q_i(t) f_i(c), \quad t \geqq T.$$

Integrating (3.4) on $[T, t]$, $t > T$, and letting $t \to \infty$, we have $\lim_{t\to\infty} u(t) = -\infty$ because of (3.1), which contradicts the positivity of $u(t)$.

Now assume that (3.3) has a nonoscillatory solution $u(t)$ which is eventually positive and such that $\limsup_{t\to\infty} u(t) < \infty$. Since, by (3.3), $u'(t) > 0$ for all large t, there exist $T \geqq t_0$ and $c > 0$ such that $u(g_i(t)) \geqq c$ for $t \geqq T$, and so (3.3) implies

$$(3.5) \qquad u'(t) \geqq \sum_{i=1}^{N} q_i(t) f_i(c), \quad t \geqq T.$$

Integration of (3.5) yields a contradiction: $\lim_{t\to\infty} u(t) = \infty$. This completes the proof.

Let us now turn to the original neutral equations (A) and (B). Lemma 3.1 enables us to conclude that condition (3.1) preclude the classes $\mathscr{N}_c^+$ and $\mathscr{N}_c^-$ in (2.5), since it can be shown that:

(i) if $x \in \mathscr{N}^+$ for (A), then the function $y(t)$ defined by (2.2) satisfies

$$(3.6) \qquad \{y'(t) - \sum_{i=1}^{N} q_i(t) f_i(y(g_i(t)))\} \operatorname{sgn} y(t) \geqq 0 \quad \text{for all large } t;$$

(ii) if $x \in \mathscr{N}^-$ for (A-I) or (A-IV), then the function $z(t) = -y(t)$ with $y(t)$ given by (2.2) satisfies

$$(3.7) \qquad \{z'(t) + \sum_{i=1}^{N} q_i(t) f_i(z(\tau^{-1} \circ g_i(t)))\} \operatorname{sgn} z(t) \leqq 0 \quad \text{for all large } t,$$

(iii) if $x \in \mathscr{N}^-$ for (A-II) or (A-III), then the function $w(t) = -y(t)/\lambda^*$ with $y(t)$ given by (2.2) satisfies

$$(3.8) \qquad \left\{w'(t) + \frac{1}{\lambda^*} \sum_{i=1}^{N} q_i(t) f_i(w(\tau^{-1} \circ g_i(t)))\right\} \operatorname{sgn} w(t) \leqq 0 \quad \text{for all large } t;$$

(iv) if $x \in \mathscr{N}^+$ for (B), then the function $y(t)$ satisfies

$$(3.9) \qquad \{y'(t) + \sum_{i=1}^{N} q_i(t) f_i(y(g_i(t)))\} \operatorname{sgn} y(t) \leqq 0 \quad \text{for all large } t;$$

(v) if $x \in \mathscr{N}^-$ for (B-I) or (B-IV), then $z(t) = -y(t)$ satisfies

$$(3.10) \qquad \{z'(t) - \sum_{i=1}^{N} q_i(t) f_i(z(\tau^{-1} \circ g_i(t)))\} \operatorname{sgn} z(t) \geqq 0 \quad \text{for all large } t;$$

(vi) if $x \in \mathcal{N}^-$ for (B-II) or (B-III), then $w(t) = -y(t)/\lambda^*$ satisfies

$$(3.11)\qquad \left\{w'(t) - \frac{1}{\lambda^*}\sum_{i=1}^{N} q_i(t) f(w(\tau^{-1}\circ g_i(t)))\right\} \operatorname{sgn} w(t) \geqq 0 \quad \text{for all large } t\,.$$

In fact, if $x \in \mathcal{N}^+$ for (A) and if $x(t)$ is eventually positive, then $0 < y(t) \leqq x(t)$ by (2.2) and from (A) we have

$$y'(t) = \sum_{i=1}^{N} q_i(t) f_i(x(g_i(t))) \geqq \sum_{i=1}^{N} q_i(t) f_i(y(g_i(t))) \quad \text{for all large } t\,,$$

which implies (3.6). A similar argument shows that (3.6) also holds for an eventually negative solution $x \in \mathcal{N}^+$ of (A). This proves the statement (i). To verify (ii) and (iii) let $x \in \mathcal{N}^+$ for (A) and let it be eventually positive (that is, $y(t) < 0$ for large t). (A similar argument holds if $x(t)$ is eventually negative.) Then, in the case (I) or (IV), we see from (2.2) that

$$(3.12)\qquad x(t) = \frac{-y(\tau^{-1}(t)) + x(\tau^{-1}(t))}{h(\tau^{-1}(t))} \geqq -\frac{y(\tau^{-1}(t))}{h(\tau^{-1}(t))} \geqq -y(\tau^{-1}(t))$$

for large t, while in the case (II) and (III), using (2.2) again, we have

$$(3.13)\qquad x(t) = \frac{-y(\tau^{-1}(t)) + x(\tau^{-1}(t))}{h(\tau^{-1}(t))} \geqq -\frac{y(\tau^{-1}(t))}{\lambda^*}$$

for large t. Combining (A) with (3.12), we obtain

$$y'(t) \geqq \sum_{i=1}^{N} q_i(t) f_i(-y(\tau^{-1}\circ g_i(t))) \quad \text{for all large } t\,,$$

which, if rewritten for $z(t) = -y(t)$, reduces to (3.7). From (A) and (3.13) it follows that

$$y'(t) \geqq \sum_{i=1}^{N} q_i(t) f_i(-y(\tau^{-1}\circ g_i(t))/\lambda^*) \quad \text{for all large } t\,,$$

so that the function $w(t) = -y(t)/\lambda^*$ satisfies (3.8).

The statements (iv)–(vi) can be proved analogously.

Furthermore, we claim that the condition (3.1) also ensures that $\mathcal{N}_0^+ = \emptyset$ for (B-III) and $\mathcal{N}_0^- = \emptyset$ for (A-IV). In fact, let $x \in \mathcal{N}_0^+$ for (B-III). Then, according to Lemma 2.2 (ii) $x(t)$ is bounded away from zero, and an argument similar to that used in the proof of Lemma 3.1 shows that the function $y(t) = x(t) - h(t)\,x(\tau(t))$ satisfies $\lim_{t\to\infty} y(t) = -\infty$ if $x(t) > 0$ and $\lim_{t\to\infty} y(t) = \infty$ if $x(t) < 0$, which is clearly impossible. On the other hand, if $x \in \mathcal{N}_0^-$ for (A-IV), then since $x(t)$ must be bounded away from zero, we see from (A) and (3.1) that $\lim_{t\to\infty} |y(t)| = \infty$, which contradicts the boundedness of $y(t)$.

Summarizing the above observations, we conclude that under the condition (3.1) the classification (2.5) of the set $\mathcal{N}$ of all nonoscillatory solutions of equations (A) and (B) now specializes to:

$$(3.14)\qquad \begin{aligned} \mathcal{N} &= \mathcal{N}_\infty^+ \cup \mathcal{N}_0^- &&\text{for} \quad (\text{A-I}),\\ \mathcal{N} &= \mathcal{N}_0^+ &&\text{for} \quad (\text{B-I}),\\ \mathcal{N} &= \mathcal{N}_0^- &&\text{for} \quad (\text{A-II}),\\ \mathcal{N} &= \mathcal{N}_0^+ \cup \mathcal{N}_\infty^- &&\text{for} \quad (\text{B-II}),\\ \mathcal{N} &= \mathcal{N}_\infty^+ \cup \mathcal{N}_0^- &&\text{for} \quad (\text{A-III}),\\ \mathcal{N} &= \mathcal{N}_\infty^- &&\text{for} \quad (\text{B-III}),\\ \mathcal{N} &= \mathcal{N}_\infty^+ &&\text{for} \quad (\text{A-IV}),\\ \mathcal{N} &= \mathcal{N}_0^+ \cup \mathcal{N}_\infty^- &&\text{for} \quad (\text{B-IV}). \end{aligned}$$

Therefore, in order to establish the desired criteria for oscillation of all solutions of (A) and (B), it suffices to find additional conditions under which the solution classes $\mathcal{N}_0^+, \mathcal{N}_0^-, \mathcal{N}_\infty^+$ and $\mathcal{N}_\infty^-$ are eliminated from the above list. To do this, we introduce the conditions

$$(\text{f})\qquad \int_0^m \frac{du}{f_i(u)} < \infty \quad \text{and} \quad \int_0^{-m} \frac{du}{f_i(u)} < \infty \quad \text{for any} \quad m > 0\,, \quad 1 \leqq i \leqq N\,;$$

$$(\text{g})\qquad \int_M^\infty \frac{du}{f_i(u)} < \infty \quad \text{and} \quad \int_{-M}^{-\infty} \frac{du}{f_i(u)} < \infty \quad \text{for any} \quad M > 0\,, \quad 1 \leqq i \leqq N\,;$$

and the notation:

$$\begin{aligned} \mathcal{A}(g_i) &= \{t \in [t_0, \infty) \colon g_i(t) > t\}\,,\\ \mathcal{A}(g_i, \tau) &= \{t \in [t_0, \infty) \colon g_i(t) > \tau(t) \geqq t_0\}\,,\\ \mathcal{R}(g_i) &= \{t \in [t_0, \infty) \colon t_0 \leqq g_i(t) < t\}\,,\\ \mathcal{R}(g_i, \tau) &= \{t \in [t_0, \infty) \colon t_0 \leqq g_i(t) < \tau(t)\}\,, \end{aligned}$$

The main results of this section are as follows.

Theorem 3.1. *Suppose that* (f) *and* (g) *are satisfied. If*

$$(3.15)\qquad \sum_{i=1}^N \int_{\mathcal{A}(g_i)} q_i(t)\, dt = \infty$$

and

$$(3.16)\qquad \sum_{i=1}^N \int_{\mathcal{R}(g_i, \tau)} q_i(t)\, dt = \infty\,,$$

then all solutions of equations (A-I) *and* (A-III) *are oscillatory.*

Theorem 3.2. *Suppose that* (f) *is satisfied. If* (3.16) *holds, then all solutions of equation* (A-II) *are oscillatory.*

Theorem 3.3. *Suppose that* (g) *is satisfied. If* (3.15) *holds, then all solutions of equation* (A-IV) *are oscillatory.*

Theorem 3.4. *Suppose that* (f) *and* (g) *are satisfied. If*

$$(3.17)\qquad \sum_{i=1}^{N} \int_{\mathscr{R}(g_i)} q_i(t)\,\mathrm{d}t = \infty$$

and

$$(3.18)\qquad \sum_{i=1}^{N} \int_{\mathscr{A}(g_i,\tau)} q_i(t)\,\mathrm{d}t = \infty\,,$$

then all solutions of equations (B-II) *and* (B-IV) *are oscillatory.*

Theorem 3.5. *Suppose that* (f) *is satisfied. If* (3.17) *holds, then all solutions of equation* (B-I) *are oscillatory.*

Theorem 3.6. *Suppose that* (g) *is satisfied. If* (3.18) *holds, then all solutions of equation* (B-III) *are oscillatory.*

Proof of Theorem 3.1. Note that either of (3.15) and (3.16) implies (3.1). In view of (3.14) it suffices to prove that $\mathscr{N}^+_\infty = \mathscr{N}^-_0 = \emptyset$ for (A-I) and (A-III).

If there exists a solution $x(t) \in \mathscr{N}^+_\infty$ of (A-I) or (A-III), then the function $y(t) = x(t) - h(t)\,x(\tau(t))$ (see (2.2)) is a nonoscillatory solution of the differential inequality (3.6) such that $\lim_{t\to\infty} |y(t)| = \infty$. This however, is impossible since a result of Kitamura and Kusano [6, Theorem 1] shows that all solutions of (3.6) must be oscillatory provided (g) and (3.15) are satisfied. Let $x \in \mathscr{N}^-_0$ be a solution of (A). Then, $z(t) = h(t)\,x(\tau(t)) - x(t)$ is a nonoscillatory solution of (3.7) such that $\lim_{t\to\infty} z(t) = 0$ if case (I) holds, and $w(t) = [h(t)\,x(\tau(t)) - x(t)]/\lambda^*$ is a nonoscillatory solution of (3.8) such that $\lim_{t\to\infty} w(t) = 0$ if case (III) holds. This is also a contradiction, since, according to another result of [6, Theorem 2], (f) and (3.16) guarantee the oscillation of all solutions of (3.7) and (3.8). It follows that both $\mathscr{N}^+_\infty = \emptyset$ and $\mathscr{N}^-_0 = \emptyset$ for (A-I) and (A-III), as desired. This completes the proof.

The proof of Theorem 3.4 is similar. Apply the above-mentioned results of [6] to the inequalities (3.9), (3.10) and (3.11) which are satisfied by possible nonoscillatory solutions $x \in \mathscr{N}^+_0 \cup \mathscr{N}^-_\infty$ of (B-II) and (B-III). The proofs of the remaining theorems are simpler.

4. NONOSCILLATION THEOREMS

This section is concerned with the existence of nonoscillatory solutions of equations (A) and (B) which can be unified as

$$(\mathrm{C})\qquad \frac{\mathrm{d}}{\mathrm{d}t}[x(t) - h(t)\,x(\tau(t))] + \sigma \sum_{i=1}^{N} q_i(t)\,f_i(x(g_i(t))) = 0\,,$$

where $\sigma = -1$ or $+1$. First, criteria are given for equation (C) subject to cases (I′)–(IV′) (see (2.12)) to have bounded nonoscillatory solutions which are bounded away from zero, and then conditions are derived under which equation (C) subject to (III') or (IV') possesses an unbounded nonoscillatory solution which is asymptotic as $t \to \infty$ to a positive solution of the functional equation $x(t) - h(t)\,x(\tau(t)) = 0$.

The following notation is employed:

$$(4.1) \qquad H_0(t) \equiv 1\,, \quad H_i(t) = \prod_{j=0}^{i-1} h(\tau^j(t))\,, \quad i = 1, 2, \ldots .$$

Theorem 4.1. *Suppose that one of the cases* (I')–(IV') *holds. Equations* (A) *and* (B) *possess bounded nonoscillatory solutions which are bounded away from zero if and only if*

$$(4.2) \qquad \sum_{i=1}^{N} \int_{t_0}^{\infty} q_i(t)\,\mathrm{d}t < \infty\,.$$

Proof. (The "only if" part) Let $x(t)$ be a bounded nonoscillatory solution of (C) such that $\liminf_{t\to\infty} |x(t)| > 0$. Integrating (C) from T to t, $T > t_0$, being sufficiently large, and then letting $t \to \infty$, we easily see that

$$(4.3) \qquad \sum_{i=1}^{N} \int_{T}^{\infty} q_i(t)\,|f_i(x(g_i(t)))|\,\mathrm{d}t < \infty\,,$$

irrespective of the cases (I')–(IV'). The desired inequality (4.2) follows from (4.3) and the fact that $\liminf_{t\to\infty} |x(t)| > 0$.

(The "if" part) (i) Case (I'): $\{0 < h(t) \leqq \lambda^* < 1,\ \tau(t) < t\}$. Let $c > 0$ be a fixed constant and choose $T > t_0$ large enough that

$$(4.4) \qquad T_0 = \min\{\tau(T),\ \inf_{t\geqq T} g_1(t), \ldots, \inf_{t\geqq T} g_N(t)\} \geqq t_0$$

and

$$(4.5) \qquad \sum_{i=1}^{N} \int_{T}^{\infty} q_i(t)\,f_i\left(\frac{c}{1-\lambda^*}\right)\mathrm{d}t \leqq (1-\lambda^*)\,c\,.$$

Let X denote the set

$$(4.6) \qquad X = \{x \in C[T_0, \infty):\ \lambda^* c \leqq x(t) \leqq c \text{ for } t \geqq T,\ x(t) = x(T) \text{ for } T_0 \leqq t \leqq T\}\,,$$

and with every $x \in X$ we associate the function $\hat{x}$: $[T_0, \infty) \to \mathbb{R}$ defined by

$$(4.7) \qquad \begin{cases} \hat{x}(t) = \displaystyle\sum_{i=0}^{n(t)-1} H_i(t)\,x(\tau^i(t)) + \frac{x(T)}{1-h(T)}\,H_{n(t)}(t)\,, & t > T\,, \\[2ex] \hat{x}(t) = \dfrac{x(T)}{1-h(T)}\,, & T_0 \leqq t \leqq T\,, \end{cases}$$

where $n(t)$ is the least positive integer such that $T_0 < \tau^{n(t)}(t) \leqq T$. It is easy to verify that $\hat{x}(t)$ is continuous on $[T_0, \infty)$ and satisfies

$$(4.8) \qquad \lambda^* c \leqq \hat{x}(t) \leqq \frac{c}{1-\lambda^*} \quad \text{for} \quad t \geqq T_0$$

and

$$(4.9)\qquad \hat{x}(t) - h(t)\,\hat{x}(\tau(t)) = x(t) \quad \text{for} \quad t \geqq T.$$

Define the mapping $\mathscr{F}\colon X \to C[T_0, \infty)$ by

$$(4.10)\qquad \begin{cases} \mathscr{F}x(t) = \delta c + \sigma \int_t^\infty \sum_{i=1}^N q_i(s) f_i(\hat{x}(g_i(s)))\,\mathrm{d}s\,, & t \geqq T, \\ \mathscr{F}x(t) = \delta c + \sigma \int_T^\infty \sum_{i=1}^N q_i(s) f_i(\hat{x}(g_i(s)))\,\mathrm{d}s\,, & T_0 \leqq t \leqq T, \end{cases}$$

where $\delta = \lambda^*$ if $\sigma = 1$ and $\delta = 1$ if $\sigma = -1$. It is not difficult to verify that $\mathscr{F}$ is continuous and maps X into a compact subset of X. Therefore, by the Schauder-Tychonoff fixed point theorem, $\mathscr{F}$ has a fixed element $x \in X$, which satisfies

$$(4.11)\qquad x(t) = \delta c + \sigma \int_t^\infty \sum_{i=1}^N q_i(s) f_i(\hat{x}(g_i(s)))\,\mathrm{d}s\,, \quad t \geqq T.$$

In view of (4.9), (4.11) is rewritten as

$$(4.12)\qquad \hat{x}(t) - h(t)\,\hat{x}(\tau(t)) = \delta c + \sigma \int_t^\infty \sum_{i=1}^N q_i(s) f_i(\hat{x}(g_i(s)))\,\mathrm{d}s\,, \quad t \geqq T.$$

Differentiation of (4.12) shows that $\hat{x}(t)$ is a solution of equation (C) on $[T, \infty)$. From (4.8) it follows that $\hat{x}(t)$ is bounded and bounded away from zero.

(ii) Case (II'): $\{1 < \lambda_* \leqq h(t) \leqq \lambda^*,\ \tau(t) > t\}$. Take an arbitrary constant $c > 0$ and let $T > t_0$ be so large that (4.4) with $\tau(T)$ replaced by $\tau^{-1}(T)$ holds and

$$(4.13)\qquad \sum_{i=1}^N \int_T^\infty q_i(t) f_i\left(\frac{c\lambda_*}{\lambda_* - 1}\right) \mathrm{d}t \leqq (\lambda_* - 1)\,c\,.$$

Define X by

$$(4.14)\qquad \begin{aligned} X = \{x \in C[T_0\colon & -c\lambda_* \leqq x(t) \leqq -c \text{ for } t \geqq T, \\ & x(t) = x(T) \text{ for } T_0 \leqq t \leqq T\} \end{aligned}$$

and with every $x \in X$ associate the function $\hat{x}\colon [T_0, \infty) \to \mathbb{R}$ given by

$$(4.15)\qquad \begin{cases} \hat{x}(t) = -\sum_{i=1}^{n(t)} \dfrac{x(\tau^{-i}(t))}{H_i(\tau^{-i}(t))} - \dfrac{x(T)}{(h(\tau^{-1}(T)) - 1))\,H_{n(t)}(\tau^{-n(t)}(t))}\,, & t > T, \\ \hat{x}(t) = -\dfrac{x(T)}{h(\tau^{-1}(T)) - 1}\,, & T_0 \leqq t \leqq T, \end{cases}$$

where $n(t)$ is the least positive integer such that $T_0 < \tau^{-n(t)}(t) \leqq T$. Then, $\hat{x}(t)$ is continuous on $[T_0, \infty)$ and satisfies (4.9) and

$$(4.16)\qquad \frac{c}{\lambda^*} \leqq \hat{x}(t) \leqq \frac{c\lambda_*}{\lambda_* - 1}\,, \quad t \geqq T_0\,.$$

We consider the mapping $\mathscr{F}\colon X \to C[T_0, \infty)$ defined by (4.10) with δ replaced by $\delta = -c\lambda_*$ for $\sigma = 1$ and $\delta = -c$ for $\sigma = -1$. As in the case (I'), $\mathscr{F}$ is shown to have a fixed element x in X which gives rise to a solution $\hat{x}(t)$ of (4.12). This function $\hat{x}(t)$ is a nonoscillatory solution of (C) on $[T, \infty)$ with the required boundedness property.

(iii) Case (III'): $\{1 < \lambda_* \leqq h(t) \leqq \lambda^*, \ \tau(t) < t\}$. For a fixed $c > 0$ take $T > t_0$ large enough that (4.4) holds and

$$(4.17) \qquad \sum_{i=1}^{N} \int_T^\infty q_i(t) f_i\left(\frac{c}{\lambda_* - 1}\right) \mathrm{d}t \leqq (\lambda_* - 1)\, c\,.$$

Let X and $\mathscr{F}$ be as in the case (II') except that (4.15) is replaced by the following formula:

$$(4.18) \qquad \begin{cases} \hat{x}(t) = -\sum_{i=1}^{\infty} \dfrac{x(\tau^{-i}(t))}{H_i(\tau^{-i}(t))}\,, & t \geqq T\,, \\ \hat{x}(t) = \hat{x}(T)\,, \quad T_0 \leqq t \leqq T\,. \end{cases}$$

Noting that $\hat{x}(t)$ given by (4.18) for $x \in X$ is continuous on $[T_0, \infty)$ and satisfies (4.9) and (4.16), we can show, via the Schauder-Tychonoff fixed point theorem, the existence of a fixed $x \in X$ of $\mathscr{F}$ giving rise to the required solution of (C) on $[T, \infty)$.

(iv) Case (IV'): $\{0 < h(t) \leqq \lambda^* < 1, \ \tau(t) > t\}$. For a fixed $c > 0$ choose $T > t_0$ so that (4.4) (with $\tau(T)$ replaced by $\tau^{-1}(T)$) and (4.5) hold. Consider the set $X \subset$ $\subset C[T_0, \infty)$ and the mapping $\mathscr{F}: X \to C[T_0, \infty)$ defined by (4.6) and (4.10), respectively, except that (4.7) is replaced by

$$(4.19) \qquad \begin{cases} \hat{x}(t) = \sum_{i=0}^{\infty} H_i(t)\, x(\tau^i(t))\,, & t \geqq T\,, \\ \hat{x}(t) = x(T)\,, \quad T_0 \leqq t \leqq T\,. \end{cases}$$

Since, for every $x \in X$, $\hat{x}(t)$ is continuous and satisfies (4.8) and (4.9), it can be shown as above that there exists a fixed element $x \in X$ of $\mathscr{F}$ which generates a required nonoscillatory solution of (C). This completes the proof.

It is natural to ask if equations (A) and (B) possess unbounded nonoscillatory solutions. A partial asnwer will be given below.

Let Ω denote the set of all continuous positive solutions of the functional equation

$$(4.20) \qquad x(t) - h(t)\, x(\tau(t)) = 0$$

defined in a neighborhood of infinity. Suppose that the case (III') or (IV') holds. Then, Ω is non-empty and $\lim_{t \to \infty} \omega(t) = \infty$ for every $\omega \in \Omega$. All members of Ω have the same order of growth near infinity in the sense that for any $\omega, \tilde{\omega} \in \Omega$ there are positive constants c_1, c_2 such that $c_1\, \omega(t) \leqq \tilde{\omega}(t) \leqq c_2\, \omega(t)$ for all sufficiently large t.

Theorem 4.2. *Suppose that either case* (III') *or case* (IV') *holds. Let* $\omega \in \Omega$ *and suppose that*

$$(4.22) \qquad \sum_{i=1}^{N} \int_{t_0}^\infty q_i(t) f_i(\omega(g_i(t)) + a)\, \mathrm{d}t < \infty \quad \textit{for some constant} \quad a > 0\,.$$

Then, equations (A) *and* (B) *posses unbounded nonoscillatory solutions* $x(t)$ *such that* $\lim_{t\to\infty} x(t)/\omega(t) = \text{const} > 0$.

Proof. (i) Case (III'). Let $T > t_0$ be large enough so that (4.4) holds and

$$\sum_{i=1}^{N} \int_T^\infty q_i(t) f_i\left(\omega(g_i(t)) + \frac{c\lambda_*}{\lambda_* - 1}\right) dt \leqq (\lambda_* - 1)\, c\,, \tag{4.23}$$

where $c > 0$ is a fixed constant such that $c\lambda_*/(\lambda_* - 1) \leqq a$. Let X denote the set (4.14). By associating with every $x \in X$ the function

$$\begin{cases} \hat{x}(t) = \omega(t) - \sum_{i=1}^{\infty} \dfrac{x(\tau^{-i}(t))}{H_i(\tau^{-i}(t))}\,, & t \geqq T\,, \\ \hat{x}(t) = \hat{x}(T)\,, \quad T_0 \leqq t \leqq T\,, \end{cases} \tag{4.24}$$

we consider the mapping $\mathscr{F}$ defined by (4.10) where $\delta = -c\lambda_*$ for $\sigma = 1$ and $\delta = -c$ for $\sigma = -1$. It is a matter of easy computation to verify that $\mathscr{F}$ maps X continuously into a compact subset of X. The Schauder-Tychonoff theorem then implies the existence of a function $x \in X$ such that $x = \mathscr{F}x$. In view of the fact that $\hat{x}(t) - h(t)\,\hat{x}(\tau(t)) = x(t)$, $t \geqq T$, it follows that $\hat{x}(t)$ satisfies (4.12), that is, $\hat{x}(t)$ is a solution of (C) for $t \geqq T$. Since

$$\frac{c}{\lambda^*} + \omega(t) \leqq \hat{x}(t) \leqq \omega(t) + \frac{c\lambda_*}{\lambda_* - 1}\,, \quad t \geqq T\,,$$

$\hat{x}(t)$ is nonoscillatory and satisfies $\lim_{t\to\infty} \hat{x}(t)/\omega(t) = 1$.

(ii) Case (IV'). For a constant $c > 0$ such that $c/(1 - \lambda^*) \leqq a$ choose $T > t_0$ so that (4.4) (with $\tau(T)$ replaced by $\tau^{-1}(T)$) and the inequality

$$\sum_{i=1}^{N} \int_T^\infty q_i(t) f_i\left(\omega(g_i(t)) + \frac{c}{1 - \lambda^*}\right) dt \leqq (1 - \lambda^*)\, c \tag{4.25}$$

holds. Consider the set X given by (4.6) and with every $x \in X$ assign the function

$$\begin{cases} \hat{x}(t) = \omega(t) + \sum_{i=0}^{\infty} H_i(t)\, x(\tau^i(t))\,, & t \geqq T\,, \\ \hat{x}(t) = \hat{x}(T)\,, \quad T_0 \leqq t \leqq T\,. \end{cases} \tag{4.26}$$

With this function $\hat{x}(t)$ we define the mapping $\mathscr{F}: X \to C[T_0, \infty)$ by (4.10), where $\delta = \lambda^*$ for $\sigma = 1$ and $\delta = 1$ for $\sigma = -1$. Then, it can be shown that $\mathscr{F}$ has a fixed point $x \in X$, to which there corresponds a solution $\hat{x}(t)$ of (C). That $\hat{x}(t)$ has the desired asymptotic property follows from the inequalities

$$\omega(t) + c\lambda^* \leqq \hat{x}(t) \leqq \omega(t) + \frac{c}{1 - \lambda^*}\,, \quad t \geqq T\,.$$

The proof is thus complete.

Remark 4.1. Theorem 1.1 [or Theorem 1.2] stated in the Introduction now follows from Theorems 3.2, 3.5 and 4.1 [or Theorems 3.3, 3.6 and 4.1, respectively].

Remark 4.2. As is easily seen, the nonoscillatory solutions of (A) and (B) constructed in Theorems 4.1 and 4.2 belong either to class $\mathcal{N}_c^+$ or $\mathcal{N}_c^-$. Naturally, the remaining solution classes $\mathcal{N}_0^\pm$ and $\mathcal{N}_\infty^\pm$ may have members as the following examples show:

(i) The equation

$$\frac{d}{dt}[x(t) - \lambda x(t-\sigma)] = e^{\varrho}(1 - \lambda e^{-\sigma})\, x(t-\varrho),$$

λ, σ and ϱ being positive constants, has a solution $x(t) = e^t$, which is of class $\mathcal{N}_\infty^+$ if $1 - \lambda e^{-\sigma} > 0$ and of class $\mathcal{N}_\infty^-$ if $1 - \lambda e^{-\sigma} < 0$.

(ii) The equation

$$\frac{d}{dt}[x(t) - \lambda x(t-\sigma)] = e^{-\varrho}(\lambda e^{\sigma} - 1)\, x(t-\varrho),$$

λ, σ and ϱ being positive constants, has a solution $x(t) = e^{-t}$, which is of class $\mathcal{N}_0^+$ if $\lambda e^\sigma - 1 > 0$ and of class $\mathcal{N}_0^-$ if $\lambda e^\sigma - 1 < 0$.

It is an interesting but difficult problem to find criteria for the existence of solutions of classes $\mathcal{N}_0^\pm$ and $\mathcal{N}_\infty^\pm$ for equations (A) and (B).

We conclude with examples which illustrate some of our main results presented above.

Examples. Define functions $F_\alpha(u)$, $F_{\alpha,\beta}(u)$ by

$$F_\alpha(u) = |u|^\alpha \operatorname{sgn} u,$$

$$F_{\alpha,\beta}(u) = \begin{cases} F_\alpha(u) & \text{for} \quad |u| \leqq 1, \\ F_\beta(u) & \text{for} \quad |u| \geqq 1, \end{cases}$$

and consider the equations

$$\frac{d}{dt}[x(t) - \lambda x(t-\sigma)] = q(t)\, F_{\alpha,\beta}(x(t - 2\sigma \sin t)), \tag{4.27}$$

$$\frac{d}{dt}[x(t) - \lambda x(t+\sigma)] = q(t)\, F_{\alpha,\beta}(x(t + 2\sigma \sin t)), \tag{4.28}$$

for $t \geqq 0$, where α, β, λ and σ are positive constants, and $q(t)$ is a positive continuous function on $[0, \infty)$.

(i) Theorem 3.1 implies that all solutions of (4.27) are oscillatory if $\alpha < 1$, $\beta > 1$,

$$\sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{2k\pi} q(t)\, dt = \infty$$

and

$$\sum_{k=0}^{\infty} \int_{(2k+1/6)\pi}^{(2k+5/6)\pi} q(t)\, dt = \infty.$$

(ii) From Theorems 3.2 and 3.3 it follows that all solutions of (4.28) are oscillatory if

$$\alpha < 1\,, \quad \beta \text{ is arbitrary and } \quad \sum_{k=0}^{\infty} \int_{(2k+5/6)\pi}^{(2k+13/6)\pi} q(t)\,\mathrm{d}t = \infty$$

or if

$$\beta > 1\,, \quad \alpha \text{ is arbitrary and } \quad \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} q(t)\,\mathrm{d}t = \infty\,.$$

(iii) Let $\lambda \neq 1$. According to Theorem 4.1, equations (4.27) and (4.28) have bounded nonoscillatory solutions which are bounded away from zero if and only if

$$\int_0^\infty q(t)\,\mathrm{d}t < \infty\,.$$

(iv) Noting that $x(t) = \lambda^{t/\sigma}$ [or $x(t) = \lambda^{-t/\sigma}$] is a solution of the equation $x(t) - \lambda x(t - \sigma) = 0$ [or $x(t) - \lambda x(t + \sigma) = 0$ respectively] and applying Theorem 4.2 to (4.27) and (4.28), we conclude that equation (4.27) with $\lambda > 1$ has an unbounded nonoscillatory solution $x(t)$ such that $\lim_{t\to\infty} x(t)/\lambda^{t/\sigma} = \text{const} > 0$ if

$$\int_0^\infty q(t)\,\lambda^{(\beta/\alpha)t}\,\mathrm{d}t < \infty$$

and that equation (4.28) with $\lambda < 1$ has an unbounded nonoscillatory solution $x(t)$ such that $\lim_{t\to\infty} x(t)/\lambda^{-t/\sigma} = \text{const} > 0$ if

$$\int_0^\infty q(t)\,\lambda^{-(\beta/\alpha)t}\,\mathrm{d}t < \infty\,.$$

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