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WEAK COHERENCE OF CONGRUENCES

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The concept of coherent congruences and algebras comes from D. Geiger [5]: An algebra A is *coherent* if for every its subalgebra B and each $\Theta \in \operatorname{Con} A$, if $[b]_\Theta \subseteq B$ for some $b \in B$, then $[x]_\Theta \subseteq B$ for each $x \in B$. In other words, A is coherent if every its subalgebra containing at least one congruence class of some $\Theta \in \operatorname{Con} A$ is the union of congruence classes of Θ. A variety $\mathscr{V}$ is *coherent* if each $A \in \mathscr{V}$ has this property.

D. Geiger [5] proved that $\mathscr{V}$ is coherent if and only if there exist an $(n + 1)$-ary term h and ternary terms t_i, $i = 1, \ldots, n$ such that

$$h(x, t_1(x, y, z), \ldots, t_n(x, y, z)) = y \quad \text{and}$$

$$t_i(x, x, z) = z \quad \text{for} \quad i = 1, \ldots, n\,.$$

Moreover, every coherent variety is *regular* (see [3]) and *permutable*. This led to the question whether, conversely, the permutability and regularity of $\mathscr{V}$ imply coherency. It was shown by W. Taylor [6] that this is not true; he introduced a variety which is regular (even uniform) and permutable but not coherent. The problem arose what conditions have to be added to permutability and regularity to obtain coherency. This problem was solved in [2]:

For a variety $\mathscr{V}$, the following conditions are equivalent:

(A) *$\mathscr{V}$ is coherent;*

(B) *$\mathscr{V}$ is regular, permutable and satisfies* CUT (see [2]),

where the condition CUT *is independent of regularity and/or permutability.*

Since coherency is a rather restrictive property, J. Duda [4] tried to modify it in a weaker form (the so called *coherency of ideals*) for varieties with a nullary operation 0. Unfortunately, this form of coherency implies neither regularity (or its weak form) nor permutability. The aim of this paper is to introduce another weak form of coherency which implies the above mentioned properties, and characterize it by a Mal'cev condition. Analogously as in [2], we give a weak form of CUT property which can be added to weak regularity and permutability to obtain weak coherence.

Let $\mathscr{V}$ be a variety containing a nullary operation 0 in its similarity type. We will say briefly that $\mathscr{V}$ is a *variety with* 0. If A is an algebra, denote by $\operatorname{Con} A$ its *con-*

gruence lattice. For $\Theta \in \mathrm{Con}\, A$ and $z \in A$ we denote by $[z]_\theta$ the *congruence class of Θ containing the element z*.

Definition 1. An algebra A with a nullary operation 0 is *weakly coherent* if for every subalgebra B of A and each $\Theta \in \mathrm{Con}\, A$,

$$[0]_\theta \subseteq B \quad \text{implies} \quad [x]_\theta \subseteq B$$

for each $x \in B$. A variety $\mathscr{V}$ with 0 is *weakly coherent* if each $A \in \mathscr{V}$ has this property.

Theorem 1. *Let $\mathscr{V}$ be a variety with 0. The following conditions are equivalent*:

(1) *$\mathscr{V}$ is weakly coherent*;

(2) *there exist an integer n, binary terms $b_1, \ldots, b_n$ and a $(1 + n)$-ary term w such that the identities*

$$y = w(x, b_1(x, y), \ldots, b_n(x, y)),$$
$$0 = b_i(x, x) \quad \textit{for} \quad i = 1, \ldots, n$$

hold in $\mathscr{V}$.

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{V}$ be weakly coherent and $A = F_V(x, y)$ (i.e. a free algebra of $\mathscr{V}$ generated by two free generators x, y). Let $\Theta = \Theta(x, y) \in \mathrm{Con}\, A$ and let B be a subalgebra of A generated by the set $\{x\} \cup [0]_\theta$. Then B contains $[0]_\theta$, also $x \in B$ and $y \in [x]_\theta$, thus, by (1), $y \in B$. Since B is generated by $\{x\} \cup [0]_\theta$ there exist a $(1 + n)$-ary term w and elements $d_1, \ldots, d_n \in [0]_\theta$ such that $y = w(x, d_1, \ldots, d_n)$. Since $d_i \in F_V(x, y)$, there exist binary terms b_i with $d_i = b_i(x, y)$. Moreover, $b_i(x, y) \in [0]_\theta$ for $\Theta = \Theta(x, y)$ implies $b_i(x, x) = 0$.

(2) $\Rightarrow$ (1): Let $A \in \mathscr{V}$, let B be a subalgebra of A and for any $\Theta \in \mathrm{Con}\, A$ let $[0]_\theta \subseteq B$. Let $x \in B$ and $y \in [x]_\theta$. We proceed to show that also $y \in B$:

Since $\langle y, x\rangle \in \Theta$, we have

$$\langle b_i(x, y), 0\rangle = \langle b_i(x, y), b_i(x, x)\rangle \in \Theta(y, x) \subseteq \Theta,$$

thus $b_i(x, y) \in [0]_\theta \subseteq B$. But $x \in B$, thus also

$$y = w(x, b_1(x, y), \ldots, b_n(x, y)) \in B,$$

i.e. $[x]_\theta \subseteq B$ for each $x \in B$ which proves weak coherence.

Example 1. Every variety of loops is weakly coherent. More generally, every variety of CHQ-algebras (see [8]) with a nullary operation is weakly coherent.

A variety $\mathscr{V}$ with 0 is *weakly regular* (see [3]) if for each $A \in \mathscr{V}$ and every $\Theta, \Psi \in \mathrm{Con}\, A$,

$$[0]_\theta = [0]_\Psi \quad \text{implies} \quad \Theta = \Psi,$$

i.e. if every congruence on A is determined by its congruence class containing 0.

B. Csákány [3] proved that $\mathscr{V}$ with 0 is weakly regular if and only if there exist binary terms $b_1, \ldots, b_n$ (for an integer $n \geqq 1$) such that

$$[b_1(x, y) = 0 \text{ and } \ldots \text{ and } b_n(x, y) = 0] \quad \text{if and only if} \quad x = y.$$

Corollary 1. *If $\mathscr{V}$ with 0 is weakly coherent then $\mathscr{V}$ is weakly regular.*

Proof. Let $\mathscr{V}$ be weakly coherent. By (2) of Theorem 1, there exist binary terms $b_1, \ldots, b_n$ such that $x = y \Rightarrow b_i(x, y) = 0$ for $i = 1, \ldots, n$. Conversely, let $b_i(x, y) = 0$ for $i = 1, \ldots, n$. Then, by (2), we have

$$y = w(x, b_1(x, y), \ldots, b_n(x, y)) = w(x, 0, \ldots, 0) = $$
$$= w(x, b_1(x, x), \ldots, b_n(x, x)) = x\,,$$

proving the converse implication.

Corollary 2. *If $\mathscr{V}$ with 0 is weakly coherent then $\mathscr{V}$ is permutable.*

Proof. Put $t(x, y, z) = w(z, b_1(y, x), \ldots, b_n(y, x))$. Then

$$t(x, z, z) = w(z, b_1(z, x), \ldots, b_n(z, x)) = x$$

and

$$t(x, x, z) = w(z, b_1(x, x), \ldots, b_n(x, x)) = w(z, 0, \ldots, 0) =$$
$$= w(z, b_1(z, z), \ldots, b_n(z, z)) = z\,,$$

thus $t(x, y, z)$ is a Mal'cev term and $\mathscr{V}$ is permutable.

A natural question arises whether weak regularity and permutability imply weak coherence. For this reason, let us introduce the following concept:

Definition 2. An algebra A with a nullary operation 0 has *subalgebras closed under translations of congruence* 0*-classes*, briefly A satisfies 0-*CUT*, if for each subalgebra B of A, every n-ary algebraic function φ over A and each $x \in A$, $y \in B$, if $[0]_\Theta \subseteq B$ and $\varphi(0, \ldots, 0) = y$ then $\varphi([0]_\Theta) \subseteq B$, where $\Theta = \Theta(x, y)$ and $\varphi(C) = \{\varphi(c_1, \ldots, c_n);\ c_i \in C\}$. A variety $\mathscr{V}$ with 0 satisfies 0-*CUT* if each $A \in \mathscr{V}$ has this property.

Remark. We will show that 0-CUT, weak regularity and permutability are independent properties (the independency of weak regularity and permutability is well-known).

Example 2. Let $\mathscr{S}$ be a variety of $\vee$-semilattices with 0. Then $\mathscr{S}$ satisfies 0-CUT but $\mathscr{S}$ is neither permutable nor weakly regular.

It is well-known that $\mathscr{S}$ is neither permutable nor weakly regular. Let us prove that $\mathscr{S}$ satisfies 0-CUT: Let B be an at least two-element subsemilattice of A (containing 0), let $x \in A$, $0 \neq y \in B$, $\Theta = \Theta(x, y)$ and suppose $[0]_\Theta \subseteq B$. Then, evidently,

$$\text{either} \quad [0]_\Theta = \{0\} \quad \text{or} \quad 0 \in [y]_\Theta$$

since $x \geqq 0$, $y \geqq 0$. Let φ be an n-ary algebraic function over A and $\varphi(0, \ldots, 0) = y$.

(a) If $[0]_\Theta = \{0\}$, then $\varphi([0]_\Theta) = \varphi(\{0\}) = \{y\} \subseteq B$.

(b) If $0 \in [y]_\Theta$ then $[0]_\Theta = [y]_\Theta$. Suppose $a_1, \ldots, a_n \in [0]_\Theta$.

Then $a_i \in [y]_\Theta$, i.e. $\langle a_i, y\rangle \in \Theta$ for $i = 1, \ldots, n$. Moreover, the associativity, commutativity and idempotency of $\vee$ imply $\varphi(0, \ldots, 0) = 0 \vee a$ for some $a \in A$. Since

$y = \varphi(0, \ldots, 0)$, we have $y = 0 \vee a$ whence $a = y$, thus

$$\varphi(y, \ldots, y) = y \vee a = y .$$

Therefore,

$$\langle \varphi(a_1, \ldots, a_n), y \rangle = \langle \varphi(a_1, \ldots, a_n), \varphi(y, \ldots, y) \rangle \in \Theta ,$$

i.e. $\varphi(a_1, \ldots, a_n) \in [y]_\theta = [0]_\theta \subseteq B$ which proves $\varphi([0]_\theta) \subseteq B$ also in this case.

Example 3. A variety $\mathscr{V}$ of implication algebras is weakly regular but $\mathscr{V}$ is neither permutable nor 0-CUT.

By [1], the variety $\mathscr{V}$ of implication algebras is of the type (2, 0) where the following identities hold:

$$(xy)\,x = x ,$$
$$(xy)\,y = (yx)\,x ,$$
$$x(yz) = y(xz) ,$$
$$xx = 1$$

(the last identity is a consequence of the three foregoing ones, see [1]). Put $b_1(x, y) = xy$, $b_2(x, y) = yx$. As was proved in Corollary of Theorem 2 in [1],

$$[b_1(x, y) = 1 \text{ and } b_2(x, y) = 1] \quad \text{if and only if} \quad x = y ,$$

thus $\mathscr{V}$ is weakly regular (the nullary operation 1 is considered as a zero in this case). It is well-known that $\mathscr{V}$ is 3-permutable but not permutable. It remains to show that $\mathscr{V}$ does not satisfy 1-CUT:

Let A be a free algebra of $\mathscr{V}$ generated by x, y. By [1], A has exactly 6 elements, namely

$$1, x, y, xy, yx, (xy)\,y ,$$

and the binary operation is given as follows:

	1	x	y	xy	yx	$(xy)\,y$
1	1	x	y	xy	yx	$(xy)\,y$
x	1	1	xy	xy	1	1
y	1	yx	1	1	yx	1
xy	1	x	$(xy)\,y$	1	yx	$(xy)\,y$
yx	1	$(xy)\,y$	y	xy	1	$(xy)\,y$
$(xy)\,y$	1	yx	xy	xy	yx	1

Let $\Theta = \Theta(x, y)$. Then we can easily verify that Θ has two classes only, namely $B_0 = \{x, y, (xy)\,y\}$ and $B_1 = \{xy, yx, 1\}$. Moreover, B_1 is a subalgebra of A. Put $B = B_1$. Then B is a subalgebra of A containing the class $[1]_\theta$. Let φ be a unary algebraic function over A given by

$$\varphi(t) = (xt)\,y .$$

Then $\varphi(1) = (x1)\, y = 1y = y$. However, $xy \in [1]_\theta$ but

$$\varphi(xy) = (x(xy))\, y = (xy)\, y \notin B\,,$$

thus $\mathscr{V}$ does not satisfy 1-CUT.

Example 4. Let $\mathscr{V}$ be a variety with 0 and with a single non-nullary operation p which is ternary and satisfies

$$p(x, x, z) = z \quad \text{and} \quad p(x, z, z) = x\,.$$

Then $\mathscr{V}$ is permutable. It is well-known that $\mathscr{V}$ is not weakly regular. It remains to prove that $\mathscr{V}$ does not satisfy 0-CUT. Choose $A \in \mathscr{V}$ such that $A = \{0, a, b, c\}$, p is symmetrical in all variables and for triples of different elements of A we have

$$p(a, b, c) = 0\,,$$
$$p(a, b, 0) = a\,,$$
$$p(a, c, 0) = 0\,,$$
$$p(b, c, 0) = a\,.$$

Then clearly $B = \{0, a, b\}$ is a subalgebra of A. Put $x = c$, $y = a$. Then $y \in B$ and $\Theta(x, y)$ has exactly two classes, namely $\{0, b\}$ and $\{a, c\}$. Thus $[0]_\theta \subseteq B$. Put $\varphi(t) = p(t, c, b)$. Then

$$\varphi(0) = p(0, c, b) = a \in B \quad \text{but}$$
$$\varphi(b) = p(b, c, b) = c \notin B\,,$$

thus A (and also $\mathscr{V}$) does not satisfy 0-CUT.

A binary relation R on an algebra A is *compatible* if R is a subalgebra of the direct product $A \times A$. The set of all reflexive and compatible relations on A forms a complete lattice with respect to set inclusion ($A \times A$ is the greatest and the identity relation the least element of this lattice). Hence, for any subset $N \subseteq A \times A$ there exists the least reflexive and compatible relation on A containing N; we denote it by $R(N)$.

Lemma. *Let A be an algebra and $N \subseteq A \times A$. Then $\langle a, b\rangle \in R(N)$ if and only if there exist an n-ary algebraic function φ over A and elements $c_i, d_i \in A$, $i = 1, \ldots, n$ such that $\langle c_i, d_i\rangle \in N$ and $a = \varphi(c_1, \ldots, c_n)$, $b = \varphi(d_1, \ldots, d_n)$.*

The proof is evident.

Theorem 2. *Let $\mathscr{V}$ be a variety with 0. Then the following conditions are equivalent:*

(1) *$\mathscr{V}$ is weakly coherent;*

(2) *$\mathscr{V}$ is weakly regular, permutable and satisfies* 0-CUT.

Proof. (1) $\Rightarrow$ (2): By Corollaries 1 and 2, it remains to prove only that $\mathscr{V}$ satisfies 0-CUT. Let $A \in \mathscr{V}$, let B be a subalgebra of A, $x \in A$, $y \in B$ and $\Theta = \Theta(x, y)$. Let φ

be an n-ary algebraic function over A such that $\varphi(0, \ldots, 0) = y$, and suppose $[0]_\theta \subseteq B$.

If $a, b \in \varphi([0]_\theta)$ then clearly $\langle a, b\rangle \in \Theta$, thus $\varphi([0]_\theta)$ is contained in some congruence class C of Θ. Since $\varphi(0, \ldots, 0) = y$, we have $y \in C$. Since $y \in B$, the weak coherence of A implies $C \subseteq B$, i.e. $\varphi([0]_\theta) \subseteq B$ which proves 0-CUT.

(2) $\Rightarrow$ (1): Let $A = F_V(x, y)$. Put $\Theta = \Theta(x, y)$, $C = [0]_\theta$ and $N = \{0\} \times [0]_\theta$. Then $[0]_\theta$ is a class of $\Theta(N)$ and thus, by the weak regularity,

$$\Theta(N) = \Theta(x, y) \quad \text{which implies} \quad \langle x, y\rangle \in \Theta(N).$$

Since $\mathscr{V}$ is permutable, the theorem of Werner [7] implies

$$\Theta(N) = R(N), \quad \text{thus} \quad \langle x, y\rangle \in R(N).$$

By Lemma there exist an m-ary algebraic function φ over A and elements $d_1, \ldots, d_m \in \in [0]_\theta$ such that

$$x = \varphi(0, \ldots, 0) \text{ and } y = \varphi(d_1, \ldots, d_m).$$

Let B be a subalgebra of A generated by the set $\{x\} \cup [0]_\theta$. Then $x \in B$, $[0]_\theta \subseteq B$, $x = \varphi(0, \ldots, 0)$, thus, by 0-CUT, also $\varphi([0]_\theta) \subseteq B$, i.e.

$$y = \varphi(d_1, \ldots, d_m) \in B.$$

Since B is generated by $\{x\} \cup [0]_\theta$, there exist an $(n + 1)$-ary term w and elements $a_1, \ldots, a_n \in [0]_\theta$ such that

$$y = w(x, a_1, \ldots, a_n).$$

Since $a_i \in F_V(x, y)$, there exist binary terms $b_i(x, y)$, $i = 1, \ldots, n$ such that $a_i = = b_i(x, y)$ and, moreover, $b_i(x, y) \in [0]_{\theta(x,y)}$ implies $b_i(x, x) = 0$. By Theorem 1, $\mathscr{V}$ is weakly coherent.

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