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Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 2, 185–202

Persistent URL: http://dml.cz/dmlcz/102452

CZECHOSLOVAK MATHEMATICAL JOURNAL
Mathematical Institute of the Czechoslovak Academy of Sciences
V. *41* (*116*), PRAHA 14. 6. 1991, No 2

PERIODIC SOLUTIONS TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

JAROSLAV BARTÁK, OTTO VEJVODA, Praha

(Received March 4, 1988)

0. INTRODUCTION

In this paper we will study existence of periodic solutions to the equation

$$u_t(x, t) + a(x, t)\, u_x(x, t) + b(x, t)\, u(x, t) = g(x, t)\,, \quad (x, t) \in \Omega \subseteq R^2\,, \tag{0.1}$$

where a, b and g are ω-periodic in t and sufficiently smooth[1]). Solutions will be looked for as the classical ones, i.e., with continuous first derivatives on the region considered. As far as it is known to the authors, up to now the question of existence of periodic solutions to an equation of the first order has been studied only rarely. In her thesis [1] N. Klimperová deals with ω-periodic solutions to the equation

$$u_t + a(x, t)\, u_x = f(x, t)\,, \quad (x, t) \in R^2$$

with a and f ω-periodic in t and such that the solution $\chi(\tau; x, t)$ to the problem

$$\mathrm{d}\chi/\mathrm{d}\tau = a(\chi, \tau)\,, \quad \chi(t; x, t) = x$$

is strictly monotone in τ and

$$\left|\lim_{\tau \to +\infty} \chi(\tau; x, t)\right| = \left|\lim_{\tau \to -\infty} \chi(\tau; x, t)\right| = +\infty$$

for every fixed $(x, t) \in R^2$. Besides, she investigates the ω-periodic solutions to the equation

$$u_t + au_x = \varepsilon f(x, t, u, \varepsilon)$$

with $a = const.$

The results of papers [2], [3] by I. M. Vulpe and G. P. Choma, respectively, when applied to a single equation instead of systems, assert that the equation

$$u_t + au_x + bu = g(x, t) + \varepsilon f(x, t, u)\,,$$

[1]) A preliminary version of this paper was published as the 31st preprint of Mathematical Institute of the Czechoslovak Academy of Sciences.

with the constants $a \neq 0$, $b \neq 0$, with g and f ω-periodic in t and f Lipschitzian in u has an ω-periodic solution in t.

Somewhat more remote is the result by H. Brézis and L. Nirenberg [4] which asserts that there exists a solution 2π-periodic in x_j, $j = 1, \ldots, n$ to the equation

$$\sum_{j=1}^{n} a_j (\partial u / \partial x_j) + g(x, u) = 0 ,$$

with g 2π-periodic in each x_j and such that $g_u(x, u) > 0$ for all x, u.

We will make use of *the method of characteristics* and of *the Poincaré method.* Let us recall that *the characteristics* are the solutions to the equation

$$(0.2) \qquad (\mathrm{d}\xi/\mathrm{d}\tau) = a(\xi, \tau) .$$

Our assumptions on a will always ensure existence of a unique local solution to (0.2). Such a solution passing through a prescribed point (x, t) will be denoted by $\xi = \chi(\tau; x, t)$.

To be able to investigate existence of ω-periodic solutions we have moreover to suppose that for every point $(x, t) \in \Omega$ the function $\chi(.; x, t)$ is defined on the whole real axis R. Except for Sec. 4 the region Ω will be R^2; in that section Ω will be properly specified. As usual, the solution u is said to be ω-periodic in t if

$$(0.3) \qquad u(x, t) = u(x, t + \omega) , \quad t \in R .$$

In the sequel it turns out that an important role is played by characteristics called below *the periodic characteristics* for which

$$(0.4) \qquad \chi(\omega; x, 0) = x .$$

Points $x \in R$ having the property (0.4) are called *the singular points* and the set of all such points will be denoted by $\mathscr{S}$. Characteristics without this property are called *the regular characteristics* and points $x \in R$ without the property (0.4) are called *the regular points.* The set of regular points will be denoted by $\mathscr{R}$.

In Sec. 1 some auxiliary results are introduced. In Secs. 2 and 3 we will investigate the cases when all the plane R^2 is covered by regular or periodic characteristics, respectively. In Sec. 4 periodic solutions are looked for in a strip lying between two neighbouring periodic characteristics. Finally, Sec. 5 is devoted to the more general case when the plane R^2 is covered partly by regular and partly by periodic characteristics.

1. AUXILIARY RESULTS

We say that the region Ω has *the property P* if it has one of the following shapes:

either $\quad \Omega = R^2$

or $\quad \Omega = \{(x, t) \in R^2 \mid t \in R,\ x \in (-\infty, \chi(t; x_0, 0))\}$

or $\quad \Omega \in \{(x, t) \in R^2 \mid t \in R,\ x \in (\chi(t; x_0, 0), +\infty)\}$

or $\quad \Omega \in \{(x, t) \in R^2 \mid t \in R,\ x \in (\chi(t; x_1, 0), \chi(t; x_2, 0))\}$

where x_0, x_1, x_2 are singular points. In the sequel we will usually deal only with regions having this property. In particular, the property P implies that Ω is invariant under the ω-shift in the direction of t (see Lemma 1.3).

For regions Ω with the property P let us define the assumption $A(\Omega)$ as follows.

$A(\Omega)$: *the functions a, b, g are of the class $\mathscr{C}^{1,0}(\bar{\Omega})$ and are ω-periodic in t; for every point $(x, t) \in \bar{\Omega}$ the function $\chi(.; x, t)$ is defined on the whole R.*

(Let us recall that the function $f(x, t)$ is of the class $\mathscr{C}^{1,0}(Q)$ if it is continuous together with its first derivative with respect to x on Q.)

This assumption ensures that there is a unique solution $\chi(\tau; x, t)$ passing through any point (x, t).

Except for Lemma 1.8 let us suppose that the region Ω has the property P and that the assumption $A(\Omega)$ is satisfied.

1.1. Lemma. *The following assertions hold:*

(1.1) $\chi(s; \chi(t_1; x, t), t_1) = \chi(s; x, t)$,

(1.2) $\chi(s + k\omega; x, t + k\omega) = \chi(s; x, t)$,

(1.3) *$\chi(s; ., t)$ is a strictly growing function for every fixed s and t.*

Proof. The first proposition is evident. Denoting by $y_1(s)$ and $y_2(s)$ the left-hand and the right-hand sides of the relation (1.2), respectively, we see that both these functions are solutions to (0.2) and fulfil the same initial condition $y_i(t) = x$ ($i = 1, 2$). Consequently, $y_1(s) = y_2(s)$.

The property (1.3) follows immediately from the relation

$$\chi_x(s; x, t) = \exp\left(\int_t^s a_x(\chi(\sigma; x, t), \sigma)\, d\sigma\right) > 0 .$$

1.2. Definition. Let x_0 be an arbitrary regular point. Then the sequence $\{x_k\}_{k=-\infty}^{\infty}$, where

$$x_{k+1} = \chi(\omega; x_k, 0) \quad \text{for} \quad k = 0, 1, 2, \ldots ,$$
$$x_k = \chi(-\omega; x_{k+1}, 0) \quad \text{for} \quad k = -1, -2, \ldots$$

is called *a determining sequence corresponding to x_0.*

1.3. Lemma. *If x is a singular point then $\chi(\tau + k\omega; x, 0) = \chi(\tau; x, 0)$ for $k \in Z$.*

Proof. If suffices to prove the relation for $k = 1$. We have $\chi(\tau + \omega; x, 0) = \chi(\tau+\omega; \chi(\omega; x, 0), \omega) = \chi(\tau + \omega; x, \omega) = \chi(\tau; x, 0)$.

1.4. Lemma. *Let x_0 be an arbitrary regular point and $\{x_k\}_{k=-\infty}^{\infty}$ the corresponding determining sequence. Then*

(1.4) $x_k = \chi(k\omega; x_0, 0)$ *for* $k \in Z$.

The proof follows readily by induction.

Evidently, the set of regular points $\mathscr{R}$ is an open set in R and its components are open intervals.

1.5. Lemma. *Let (α, β) be a component of the set of regular points. Then either*

$$\chi(-\omega; x, 0) < x < \chi(\omega; x, 0)$$

or

$$\chi(-\omega; x, 0) > x > \chi(\omega; x, 0)$$

for all $x \in (\alpha, \beta)$. (In particular, every determining sequence is strictly monotone.)

Proof. Let there be two points $x_1, x_2 \in (\alpha, \beta)$ such that $x_1 < \chi(\omega; x_1, 0)$ and $x_2 > \chi(\omega; x_2, 0)$. Then the continuous function $F(\eta) = \eta - \chi(\omega; \eta, 0)$ is negative for $\eta = x_1$ and positive for $\eta = x_2$. Necessarily, there exists a point $\eta_0 \in (x_1, x_2)$ for which $F(\eta_0) = 0$, i.e. $\eta_0 \in \mathscr{S}$, which contradicts the fact that $(\alpha, \beta) \subseteq \mathscr{R}$.

By Lemma 1.1 the inequalities $x < \chi(\omega; x, 0)$ and $x > \chi(\omega; x, 0)$ imply that $\chi(-\omega; x, 0) < x$ and $\chi(-\omega; x, 0) > x$, respectively, which completes the proof.

1.6. Lemma. *Let (α, β) be a component of the set of regular points. Let $\{x_k\}_{k=-\infty}^{\infty}$ be a determining sequence corresponding to x_0, $x_0 \in (\alpha, \beta)$. Then either*

$$\lim_{k \to +\infty} x_k = \alpha\,, \quad \lim_{k \to -\infty} x_k = \beta$$

or

$$\lim_{k \to -\infty} x_k = \alpha\,, \quad \lim_{k \to +\infty} x_k = \beta\,.$$

Proof. Let for instance $x_0 < x_1$. As the characteristic starting at x_0 cannot intersect the periodic characteristic $\chi(s; \beta, 0)$, by virtue of (1.4) we have $x_k < \beta$. (For $\beta = +\infty$ this inequality is trivial.) If we denote $x^* = \lim_{k \to +\infty} x_k$, the relation $x_{k+1} = \chi(\omega; x_k, 0)$ yields $x^* = \chi(\omega; x^*, 0)$ because of the continuity of $\chi(\omega; ., 0)$. However, this implies $x^* = \beta$. For $k < 0$ the proof proceeds similarly.

If $x_1 < x_0$ the proof is quite analogous.

1.7. Lemma. *Let $\{x_k\}_{k=-\infty}^{\infty}$ be a determining sequence corresponding to x_0.*

(i) *If $x \in [x_0, x_1]$ then $\chi(k\omega; x, 0) \in [x_k, x_{k+1}]$;*

(ii) *if $x \in [x_k, x_{k+1}]$ then $\chi(0; x, k\omega) \in [x_0, x_1]$ for $k \in Z$.*

Proof. The relations (1.1)–(1.4) imply

(i) $x_k = \chi(k\omega; x_0, 0) \leqq \chi(k\omega; x, 0) \leqq \chi(k\omega; x_1, 0) = \chi(k\omega; \chi(\omega; x_0, 0), 0) =$
$= \chi((k+1)\,\omega; \chi(\omega; x_0, 0), \omega) = \chi((k+1)\,\omega; x_0, 0) = x_{k+1}$,

(ii) $x_0 = \chi(0; x_k, k\omega) \leqq \chi(0; x, k\omega) \leqq \chi(0; x_{k+1}, k\omega) = x_1$.

Further, let us recall the formula for the solution to the initial problem given by the equation (0.1) and the initial condition

$$(1.5) \qquad u(x, 0) = \varphi(x)\,, \quad x \in I \quad (I \subseteq R \text{ an interval})\,.$$

1.8. Lemma. *Let the functions a, b, g be of the class $\mathscr{C}^{1,0}(\Omega)$, where Ω is the set covered by characteristics starting for $t = 0$ from the points of a given interval $I \subseteq R$. Let $\varphi \in \mathscr{C}^1(I)$. Then there exists in Ω a unique solution of (0.1), (1.5) and it* *is given by the formula*

$$(1.6)\qquad u(x, t) = \varphi(\chi(0; x, t)) \exp\left(- \int_0^t b(\chi(\sigma; x, t), \sigma)\, d\sigma\right) + \\ + \int_0^t g(\chi(s; x, t), s) \exp\left(- \int_s^t b(\chi(\sigma; x, t), \sigma)\, d\sigma\right) ds\,.$$

This formula may be verified by direct inspection. Uniqueness may be proved by standard methods.

1.9. Lemma. *The solution $u \in \mathscr{C}^1(\bar{\Omega})$ to (0.1) is ω-periodic if and only if*

$$(1.7)\qquad u(x, 0) = u(x, \omega) \quad \textit{for} \quad x \in \{x \in R \mid (x, 0) \in \operatorname{Int} \Omega\}\,.$$

Proof for $(x, t) \in \operatorname{Int} \Omega$ follows readily from the ω-periodicity of the functions a, b, g in t and from the fact that the solution of the equation (0.1) is uniquely determined by its initial function. The ω-periodicity on the boundary $\partial\Omega$ is a consequence of the continuity of a solution, since then in the equality $u(x,t + \omega) = u(x, t)$ we can pass with (x, t) to the boundary.

2. THE CASE OF REGULAR CHARACTERISTICS

In this section we deal with the case when the set of regular points $\mathscr{R}$ coincides with R and the condition $A(R^2)$ is fulfilled. The procedure how to construct a periodic solution to the equation

$$(2.1)\qquad u_t + a(x, t)\, u_x + b(x, t)\, u = g(x, t)\,, \quad (x, t) \in \Omega = R^2$$

in this case is the following. For definiteness we will suppose that $x < \chi(\omega; x, 0)$ for all x. Let us choose an arbitrary point $x_0 \in R$ and an arbitrary initial function φ_0 of the class $\mathscr{C}^1([x_0, x_1])$, where $x_1 = \chi(\omega; x_0, 0)$. The formula (1.6) defines the values of the solution $u_1(x, t)$ for $x \in [x_1, x_2]$, $t = \omega$, where $x_2 = \chi(\omega; x_1, 0)$ .Since the solution has to be ω-periodic the initial function φ_0 has to be extended onto the interval $[x_1, x_2]$ by the relation

$$\varphi_1(x) = u_0(x, \omega) = \varphi_0(\chi(0; x, \omega)) \exp\left(- \int_0^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) + \\ + \int_0^\omega g(\chi(s; x, \omega), s) \exp\left(- \int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds\,.$$

Making use of the induction, we find easily that the restrictions φ_k on $[x_k, x_{k+1}]$ (where x_k form the determining sequence corresponding to x_0) of the initial condition φ are given by

$$(2.2)\qquad \varphi_k(x) = \varphi_0(\chi(0; x, k\omega)) \exp\left(- \int_0^{k\omega} b(\chi(\sigma; x, k\omega), \sigma)\, d\sigma\right) + \\ + \int_0^{k\omega} g(\chi(s; x, k\omega), s) \exp\left(- \int_s^{k\omega} b(\chi(\sigma; x, k\omega), \sigma)\, d\sigma\right) ds\,, \quad k \in Z\,.$$

Thus we have got the solution $u(x, t)$ consisting of its restrictions u_k, $k \in Z$, where u_k is the solution in the strip between the characteristics $\chi(.\,; x_k, 0)$, $\chi(.\,; x_{k+1}, 0)$ with the initial condition $u_k(x, 0) = \varphi_k(x)$ for $k \in Z$ whose values for $t = 0$ and $t = \omega$

coincide. By Lemma 1.9 this guarantees that this solution is ω-periodic in t. This solution belongs to the class $\mathscr{C}^1(R^2)$ if $\varphi \in \mathscr{C}^1(R)$. It is easy to find that the function φ defined by its restriction φ_k, $k \in Z$ is continuous together with its first derivative at the point x_1 if and only if the following compatibility conditions are satisfied:

$$\varphi_0(x_1) = \varphi_0(x_0) \exp\left(-\int_0^\omega b(\chi(\sigma; x_1, \omega), \sigma)\, d\sigma\right) + \int_0^\omega g(\chi(s; x_1, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x_1, \omega), \sigma)\, d\sigma\right) ds\,, \tag{2.3}$$

$$\varphi_0'(x_1) = \left[\varphi_0'(x_0)\, \chi_x(0; x_1, \omega) - \varphi_0(x_0) \int_0^\omega b_x(\chi(\sigma; x_1, \omega), \sigma) \cdot \chi_x(\sigma; x_1, \omega)\, d\sigma\right] \exp\left(-\int_0^\omega b(\chi(\sigma; x_1, \omega), \sigma)\, d\sigma\right) + \int_0^\omega \left[g_x(\chi(s; x_1, \omega), s)\, \chi_x(s; x_1, \omega) - g(\chi(s; x_1, \omega), s) \cdot \int_s^\omega b_x(\chi(\sigma; x_1, \omega), \sigma)\, \chi_x(\sigma; x_1, \omega)\, d\sigma\right] \exp\left(-\int_s^\omega b(\chi(\sigma; x_1, \omega), \sigma)\, d\sigma\right) ds. \tag{2.4}$$

Thus we have proved the following theorem.

2.1. Theorem. *Let the assumption $A(R^2)$ be fulfilled and let $\mathscr{R} = R$. Then for every point $x_0 \in R$ and every function $\varphi_0 \in \mathscr{C}^1([x_0, \chi(\omega; x_0, 0)])$ satisfying the compatibility conditions (2.3), (2.4) there exists a unique ω-periodic solution $u \in \mathscr{C}^1(R^2)$ to (2.1) such that $u(x, 0) = \varphi_0(x)$ for $x \in [x_0, \chi(\omega; x_0, 0)]$. This ω-periodic solution is given by the formula (1.6) with the function φ such that $\varphi_{|[x_k, x_{k+1}]} = \varphi_k$, where $x_k = \chi(k\omega; x_0, 0)$ and φ_k are given by the formula (2.2).*

2.2. Example. Let us look for a 2π-periodic solution to the equation

$$u_t + au_x = x \sin t\,, \quad \text{where} \quad a \neq 0 \quad \text{is a constant}\,.$$

Evidently $\chi(\tau; x, t) = x + a(\tau - t)$. Then (2.2) implies

$$\varphi_k(x) = \varphi_0(x - 2\pi ak) - 2\pi ak\,, \quad x \in [x_0 + 2\pi ak\,, \;\; x_0 + 2\pi a(k + 1)]\,,$$

where $\varphi_0 \in \mathscr{C}^1([x_0, x_0 + 2\pi a])$ with

$$\varphi_0(x_0 + 2\pi a) = \varphi_0(x_0) - 2\pi a\,, \quad \varphi_0'(x_0) = \varphi_0'(x_0 + 2\pi a)\,.$$

Thanks to the simplicity of this problem, we can readily get the solution in a closed form. Writing down the condition of the 2π-periodicity

$$\varphi(x) = \varphi(x - 2\pi a) - 2\pi a$$

or equivalently

$$\varphi(x) + x = \varphi(x - 2\pi a) + x - 2\pi a\,,$$

we see that

$$\varphi(x) = \psi(x) - x\,,$$

where $\psi(x) \in \mathscr{C}^1(R)$ is an arbitrary $2\pi a$-periodic function. This agrees with the solution found above. Then the 2π-periodic solution is given by (1.6).

3. THE CASE OF PERIODIC CHARACTERISTICS

In this section we deal with the case when the set of singular points $\mathscr{S}$ coincides with R and the condition $A(R^2)$ is fulfilled. The condition of ω-periodicity

$$\varphi(x) = \varphi(\chi(0; x, \omega)) \exp\left(-\int_0^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) + \\ + \int_0^\omega g(\chi(s; x, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds$$

reduces in our case to

$$\varphi(x)\left[1 - \exp\left(-\int_0^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right]\right. = \\ = \int_0^\omega g(\chi(s; x, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds\,.$$

Denote

$$\mathscr{N} = \{x \in \mathscr{S} \mid \int_0^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma = 0\}\,, \quad \mathscr{M} = \mathscr{S} - \mathscr{N}\,.$$

Clearly, if $x \in \mathscr{N}$ then an ω-periodic solution exists only if

$$\int_0^\omega g(\chi(s; x, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds = 0$$

and $\varphi(x)$ remains undetermined.

On the other hand, if $x \in \mathscr{M}$ then the value of $\varphi(x)$ is given by $\varphi(x) = \varphi_{\mathscr{M}}(x)$, where

(3.1) $$\varphi_{\mathscr{M}}(x) = \left[1 - \exp\left(-\int_0^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right)\right]^{-1} . \\ . \int_0^\omega g(\chi(s; x, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds\,.$$

The following theorem is almost evident.

3.1. Theorem. *Let the assumption $A(R^2)$ be fulfilled and let every point $x \in R$ be a singular point. Further, let*

(i) $\int_0^\omega g(\chi(s; x, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds = 0$ *for* $x \in \mathscr{N}$,

(ii) $\varphi_{\mathscr{M}} \in \mathscr{C}^1(\mathrm{cl}\,\mathscr{M})$ (*this extension will be denoted by* $\varphi_{\mathrm{cl}\mathscr{M}}$).

If $\mathrm{cl}\,\mathscr{M} = R$ *then there exists a unique solution, and it is given by the formula* (1.6) *with* $\varphi = \varphi_{\mathrm{cl}\mathscr{M}}$.

If $\mathrm{cl}\,\mathscr{M} \neq R$ *then for an arbitrary* $\varphi \in \mathscr{C}^1(R)$ *such that* $\varphi = \varphi_{\mathrm{cl}\mathscr{M}}$ *on* $\mathrm{cl}\,\mathscr{M}$ *there exists a unique* ω*-periodic solution* $u \in \mathscr{C}^1(R^2)$ *such that* $u(x, 0) = \varphi(x)$.

3.2. Corollary. *Let the assumptions of Theorem* 3.1 *be satisfied, let* $\mathscr{N} = R$ *and let*

$$\int_0^\omega g(\chi(s; x, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma\right) ds \equiv 0\,.$$

Then the initial function φ *is an arbitrary function from the class* $\mathscr{C}^1(R)$.

3.3. Remark. In the preceding theorem it need not be always clear whether the function φ with the required properties exists. The situation is more lucid under more restrictive assumptions on $\mathscr{M}$. Namely, we can suppose that

(3.2) *the boundary of* $\mathscr{M}$ *has not finite points of accumulation*.

If $\mathscr{M} \neq 0$, let us write $\mathscr{M} = \bigcup_k J_k$, $J_k = (\gamma_k, \delta_k)$, where the union is the nonvoid union of at most countably many disjoint open intervals. Then the problem of the possibility of the $\mathscr{C}^1$-extension of $\varphi_{\mathscr{M}}$ onto cl $\mathscr{M}$ reduces to the existence of the finite limits $\lim_{x\to\gamma_k+} \varphi_{\mathscr{M}}(x)$, $\lim_{x\to\delta_k-} \varphi_{\mathscr{M}}(x)$, $\lim_{x\to\gamma_k+} \varphi'_{\mathscr{M}}(x)$ and $\lim_{x\to\delta_k-} \varphi'_{\mathscr{M}}(x)$ and, if $\delta_k = \gamma_{k+1}$, to the equality of the limits from the left and from the right at this point.

3.4. Example. Let the equation

$$u_t + x \cos t\, u_x + x\, u = x^2$$

be given.

Evidently $\chi(\tau; x, t) = x\, e^{\sin\tau - \sin t}$, $\int_0^{2\pi} b(\chi(\sigma; x, 2\pi), \sigma)\, d\sigma = \int_0^{2\pi} x\, e^{\sin\sigma}\, d\sigma = Ax$, where $A = \int_0^{2\pi} e^{\sin\sigma}\, d\sigma > 0$ so that $\mathscr{S} = R$, $\mathscr{N} = \{0\}$, $\mathscr{M} = R - \{0\}$. Further,

$$\varphi_{\mathscr{M}}(x) =$$
$$= [1 - e^{-Ax}]^{-1} \int_0^{2\pi} x^2 \exp(2 \sin s) \exp\left(-\int_s^{2\pi} x \exp(\sin\sigma)\, d\sigma\right) ds\,.$$

We find easily that $\lim_{x\to 0} \varphi_{\mathscr{M}}(x) = 0$ and $\lim_{x\to 0} \varphi'_{\mathscr{M}}(x) = A^{-1} \int_0^{2\pi} \exp(2 \sin s)\, ds$. Hence cl $\mathscr{M} = R$ and the function $\varphi_{\mathrm{cl}\mathscr{M}} \in \mathscr{C}^1(R)$ defines the unique 2π-periodic solution to the given equation.

4. PERIODIC SOLUTIONS ON REGULAR COMPONENTS

In this section we study the existence of ω-periodic solutions to the equation

$$(4.1) \qquad u_t + a(x, t)\, u_x + b(x, t)\, u = g(x, t)$$

on a set covered by characteristics starting for $t = 0$ from a component of regular points. We suppose that the set of regular points is not the whole set R (this case was investigated in Sec. 2). Evidently, components of the set of regular points are open intervals. At most two of them may be infinite, say $(\alpha, +\infty)$ and/or $(-\infty, \beta)$, whereas the other must be finite, say (α, β). Let I be one of these intervals. Let us denote

$$\Omega(I) = \{(x, t) \in R^2 \mid t \in R,\ x \in (\chi(t; \alpha, 0), +\infty)\} \quad \text{if} \quad I = (\alpha, +\infty)\,,$$
$$\Omega(I) = \{(x, t) \in R^2 \mid t \in R,\ x \in (-\infty, \chi(t; \beta, 0))\} \quad \text{if} \quad I = (-\infty, \beta)\,,$$
$$\Omega(I) = \{(x, t) \in R^2 \mid t \in R,\ x \in (\chi(t; \alpha, 0), \chi(t; \beta, 0))\} \quad \text{if} \quad I = (\alpha, \beta)\,.$$

We will investigate the existence of ω-periodic solutions on the open set $\Omega(I)$ and, under rather strong conditions, also on its closure cl $\Omega(I)$. Under the assumption $A(\Omega(I))$, let us perform on I the same construction as in Sec. 2. Choosing arbitrarily $x_0 \in I$ and the function $\varphi_0 \in \mathscr{C}^1([x_0, x_1])$, $x_1 = \chi(\omega; x_0, 0)$ satisfying the compatibility conditions (2.3), (2.4), we obtain for $k \to \pm\infty$ monotone sequences of points x_k and of functions φ_k as in Sec. 2. The following statement can be proved analogously as Theorem 2.1.

4.1. Theorem. *Let the assumption $A(\Omega(I))$ be fulfilled. Then for every point*

$x_0 \in I$ *and every function* $\varphi_0 \in \mathscr{C}^1([x_0, \chi(\omega; x_0, 0)])$ *satisfying the compatibility conditions* (2.3), (2.4) *there exists a unique* ω*-periodic solution* $u \in \mathscr{C}^1(\Omega(I))$ *to* (4.1) *such that* $u(x, 0) = \varphi_0(x)$ *for* $x \in [x_0, \chi(\omega; x_0, 0)]$. *This solution is defined by the formula* (1.6) *with the function* φ *such that* $\varphi_{|[x_k, x_{k+1}]} = \varphi_k$, φ_k *being defined by the formula* (2.2).

Disregarding the intermediate cases when u is looked for on the union of $\Omega(I)$ only with its left or right boundary characteristic, let us look for ω-periodic solutions on the whole cl $\Omega(I)$.

By Lemma 1.6,

$$\text{either} \quad \lim_{k \to -\infty} x_k = \alpha \quad \text{and} \quad \lim_{k \to +\infty} x_k = \beta\,,$$
$$\text{or} \quad \lim_{k \to -\infty} x_k = \beta \quad \text{and} \quad \lim_{k \to +\infty} x_k = \alpha\,.$$

For the sake of simplicity we will use the symbol $\lim_k$, which will mean one of the limits $\lim_{k \to -\infty}$ or $\lim_{k \to +\infty}$. By (2.2) we have

$$\begin{aligned}
\varphi_k(x_k) &= \varphi_0(\chi(0; \chi(k\omega; x_0, 0), k\omega)) \,. \\
&\quad . \exp\left(- \textstyle\int_0^{k\omega} b(\chi(\sigma; \chi(k\omega; x_0, 0), k\omega), \sigma)\, \mathrm{d}\sigma\right) + \\
&\quad + \textstyle\int_0^{k\omega} g(\chi(s; \chi(k\omega; x_0, 0), k\omega), s)\,. \\
&\quad . \exp\left(- \textstyle\int_s^{k\omega} b(\chi(\sigma; \chi(k\omega; x_0, 0), k\omega), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s = \\
&= \varphi_0(x_0) \exp\left(- \textstyle\int_0^{k\omega} b(\chi(\sigma; x_0, 0), \sigma)\, \mathrm{d}\sigma\right) + \\
&\quad + \textstyle\int_0^{k\omega} g(\chi(s; x_0, 0), s) \exp\left(- \textstyle\int_s^{k\omega} b(\chi(\sigma; x_0, 0), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\varphi_0(x_0) &= \varphi_k(x_k) \exp\left(\textstyle\int_0^{k\omega} b(\chi(\sigma; x_0, 0), \sigma)\, \mathrm{d}\sigma\right) - \\
&\quad - \textstyle\int_0^{k\omega} g(\chi(s; x_0, 0), s) \exp\left(\textstyle\int_0^{s} b(\chi(\sigma; x_0, 0), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s\,.
\end{aligned} \tag{4.2}$$

Here the values $\varphi_k(x_k)$ are almost arbitrary, say $\psi(x_k)$ with the function ψ such that the limit of the right-hand side exists (due to the independence of the left-hand side of k) and such that the resulting function φ is smooth enough. This leads us to looking for a fuction $\psi(x)$ such that the limit

$$\begin{aligned}
\varphi(x) &= \lim_k \{\psi(\chi(k\omega; x, 0)) \exp\left(\textstyle\int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, \mathrm{d}\sigma\right) - \\
&\quad - \textstyle\int_0^{k\omega} g(\chi(s; x, 0), s) \exp\left(\textstyle\int_0^{s} b(\chi(\sigma; x, 0), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s\}
\end{aligned} \tag{4.3}$$

exists. Then this limit gives an initial condition defining a periodic solution as the following theorem shows.

4.2. Theorem. *Let the assumption* $A(\Omega(I))$ *be satisfied and let there exist a function* ψ *defined on* I *such that the function* φ *defined by* (4.3) *for* $x \in I$ (*for at least one choice of orientation* $k \to +\infty$ *or* $k \to -\infty$) *can be extended to* $\bar{I}$ *as a continuously differentiable function* (*this extension is denoted by the same symbol*

$\varphi(x)$). *Then there exists an* ω*-periodic solution* $u \in \mathscr{C}^1(\mathrm{cl}\,\Omega(I))$. *This solution is given by the formula* (1.6) *with the initial function* $\varphi(x)$.

Proof. By (1.6) we have for $x \in I$

$$u(x, \omega) = \lim_k \{\psi(\chi(k\omega; \chi(0; x, \omega), 0)) \,.$$

$$. \exp \left(\int_0^{k\omega} b(\chi(\sigma; \chi(0; x, \omega), 0), \sigma) \, \mathrm{d}\sigma\right) -$$

$$- \int_0^{k\omega} g(\chi(s; \chi(0; x, \omega), 0), s) \exp \left(\int_0^s b(\chi(\sigma; \chi(0; x, \omega), 0), \sigma) \, \mathrm{d}\sigma\right) \mathrm{d}s\} \,.$$

$$. \exp \left(- \int_0^{\omega} b(\chi(\sigma; x, \omega), \sigma) \, \mathrm{d}\sigma\right) +$$

$$+ \int_0^{\omega} g(\chi(s; x, \omega), s) \exp \left(- \int_s^{\omega} b(\chi(\sigma; x, \omega), \sigma) \, \mathrm{d}\sigma\right) \mathrm{d}s \,.$$

Using the properties of the characteristics from Sec. 1 and the ω-periodicity of the functions b and g we obtain

$$u(x, \omega) = \lim_k \{\psi(\chi((k-1)\,\omega; x, 0)) \exp \left(\int_0^{(k-1)\omega} b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) -$$

$$- \int_0^{(k-1)\omega} g(\chi(s; x, 0), s) \exp \left(\int_0^s b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) \mathrm{d}s\} = \varphi(x) \,.$$

By Lemma 1.9 this proves the ω-periodicity on the closure $\bar{\Omega}$.

4.3. Remark. If ψ from Theorem 4.2 happens to determine an initial condition of an ω-periodic solution then the function φ defined by the formula (4.3) equals ψ.

Indeed, if u is the ω-periodic solution with the initial condition ψ, then

$$(1) \qquad \psi(x) = u(x, -k\omega) \,, \quad k \in Z \,.$$

Using (1.6) and (1.2) we get

$$u(x, -k\omega) = \psi(\chi(k\omega; x, 0)) \exp \left(\int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) -$$

$$- \int_0^{k\omega} g(\chi(s; x, 0), s) \exp \left(\int_0^s b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) \mathrm{d}s \,.$$

This, together with (1) implies

$$\psi(x) = \psi(\chi(k\omega; x, 0)) \exp \left(\int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) -$$

$$- \int_0^{k\omega} g(\chi(s; x, 0), s) \exp \left(\int_0^s b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) \mathrm{d}s$$

which after passing to the limit yields $\psi = \varphi$.

4.4. Remark. If the function ψ and the corresponding limit in (4.3) exist for one of the orientation $k \to +\infty$ or $k \to -\infty$ in Theorem 4.2, then there exists a function ψ_1 such that the corresponding limit exists for the other orientation of k and is also of the class $\mathscr{C}^1(\bar{I})$.

Proof. Let φ be the initial condition defined by the limit (4.3), $\varphi \in \mathscr{C}^1(\bar{I})$. Choosing $\psi = \varphi$ we get similarly as in Remark 4.3 that

$$\varphi(x) = \varphi(\chi(k\omega; x, 0)) \exp \left(\int_0^{k\omega} b(\chi(\sigma; x, 0)\, \sigma), \mathrm{d}\sigma\right) -$$

$$- \int_0^{k\omega} g(\chi(s; x, 0), s) \exp \left(\int_0^s b(\chi(\sigma; x, 0), \sigma) \, \mathrm{d}\sigma\right) \mathrm{d}s$$

for an arbitrary $k \in Z$. The independence of k of the left-hand side of this relation implies that the limits $\lim_{k\to+\infty}$ and $\lim_{k\to-\infty}$ exist in (4.3) if we put $\psi = \varphi$. Obviously it suffices to take $\psi_1 = \varphi$ and the formula (4.3) defines the same function φ again.

4.5. Remark. If $\gamma \in \mathcal{N} \cap \{\alpha, \beta\}$ then the necessary condition for the existence of a function ψ from Theorem 4.2 is that

$$\int_0^\omega g(\chi(s; \gamma, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; \gamma, \omega), \sigma)\, d\sigma\right) ds = 0\,.$$

On the other hand, if $\gamma \in \mathcal{M} \cap \{\alpha, \beta\}$ then $\varphi(\gamma) = \varphi_{\mathrm{cl}\mathcal{M}}(\gamma)$.

4.6. Example. Let an equation

$$u_t + xu_x + bu = x\, p(t)$$

with a constant b and an ω-periodic function $p \in \mathscr{C}^1(R)$ be given. Here $\chi(\tau; x, t) = x \exp(\tau - t)$ and the points $x_k = \chi(2\pi k; x_0, 0) = x_0 \exp(2\pi k)$ converge to the singular point $x = 0$ for $k \to -\infty$ on both components of regular points $(-\infty, 0)$ and $(0, +\infty)$. Hence let $k \to -\infty$.

If $b \neq 0$ then $0 \in \mathcal{M}$ and the value of $\varphi(0)$ is given by $\varphi(0) = \varphi_{\mathcal{M}}(0) = 0$. Further,

(1) $$\varphi(x) = \lim_{k\to-\infty} \{\psi(x \exp(k\omega)) \exp(bk\omega) - \int_0^{k\omega} x \exp(s)\, p(s) \exp(bs)\, ds\}\,.$$

(i) If $b < -1$, then in general the integral in (1) is divergent for $k \to -\infty$ and due to the relation

$$\begin{aligned} &-\int_0^{k\omega} x \exp(s)\, p(s) \exp(bs)\, ds = \\ &= x \sum_{j=0}^{-k-1} \int_{-(j+1)\omega}^{-j\omega} p(s) \exp((b+1)s)\, ds = \\ &= x \int_0^\omega p(s) \exp((b+1)(s-\omega)) \sum_{j=0}^{-k-1} \exp(-j(b+1)\omega)\, ds = \\ &= x \int_0^\omega p(s) \exp((b+1)(s-\omega))\, ds\, \frac{1 - \exp((b+1)k\omega)}{1 - \exp(-(b+1)\omega)} \end{aligned}$$

we can put

$$\psi(x) = x \frac{\int_0^\omega p(s) \exp((b+1)(s-\omega))\, ds}{1 - \exp(-(b+1)\omega)} + C|x|^{-b}\,.$$

Then

$$\varphi(x) = x \frac{\int_0^\omega p(s) \exp((b+1)s)\, ds}{\exp((b+1)\omega) - 1} + C|x|^{-b}, \quad C \in R$$

on both components of regular points with $\lim_{x\to0} \varphi(x) = 0$, which agrees with the value of $\varphi_{\mathcal{M}}(0)$.

(ii) If $b = -1$ and $\int_0^\omega p(s)\, ds = 0$, then putting $\psi(x) = Cx$ we get $\varphi(x) = Cx$, which makes the desired extension on each of the both regular components possible.

(iii) If $b = -1$ and $\int_0^\omega p(s)\,ds \neq 0$ we should have to put

$$\psi(x) = \frac{1}{\omega}\int_0^\omega p(s)\,ds\, x \ln |x|$$

in order to cancel the terms tending to infinity, which leads to

$$\varphi(x) = \frac{1}{\omega}\int_0^\omega p(s)\,ds\, x \ln |x| ,$$

but this is not a function from $\mathscr{C}^1$ on the closure of any of the two investigated components.

(iv) If $b \in (-1, 0)$, the integral has a finite limit for $k \to -\infty$ and we may put $\psi(x) = C|x|^{-b}$. Then

$$\varphi(x) = C|x|^{-b} + x \int_{-\infty}^0 p(s) \exp((b+1)\,s)\,ds .$$

This function is of class $\mathscr{C}$ on the closure of each of the regular components with $\lim_{x\to 0} \varphi(x) = 0$, but it is of class $\mathscr{C}^1$ on them only for $C = 0$. Thus $\varphi(x) = x \int_{-\infty}^0 p(s) \exp((b+1)\,s)\,ds$.

(v) If $b = 0$ then the singular point 0 belongs to $\mathscr{N}$ and the value $\varphi(0)$ is not defined. Further, for $x \neq 0$ we have

$$\varphi(x) = \lim_{k\to-\infty} \{\psi(x \exp(k\omega)) - \int_0^{k\omega} x\, p(s) \exp(s)\,ds\} =$$
$$= \psi(0) + x \int_{-\infty}^0 p(s) \exp(s)\,ds = C + x \int_{-\infty}^0 p(s) \exp(s)\,ds$$

with an arbitrary constant C.

(vi) If $b > 0$ then putting $\psi(x) = 1$ we get

$$\varphi(x) = \lim_{k\to-\infty} - \int_0^{k\omega} x \exp(s)\, p(s) \exp(bs)\,ds =$$
$$= x \int_{-\infty}^0 p(s) \exp((b+1)\,s)\,ds .$$

(The choice $\psi(x) = C|x|^{-b}$ which would cancel the asymptotic behaviour of $\exp(bk\omega)$ in (1) is not possible since it leads to the function which cannot be smoothly extended at $x = 0$.)

Now let us state *an existence and "uniqueness" theorem* under some more restrictive conditions than in Theorem 4.2.

4.7. Theorem. *Let the assumption $A(\Omega(I))$ be fulfilled. Further, let there exist limits*

$$B(x) = \lim_k \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\,d\sigma , \quad x \in I$$

and

$$G(x) = \lim_k \int_0^{k\omega} g(\chi(s; x, 0), s) \exp\left(\int_0^s b(\chi(\sigma; x, 0), \sigma)\,d\sigma\right) ds , \quad x \in I ,$$

which can be extended onto $\bar{I}$ as functions from $\mathscr{C}^1(\bar{I})$ (the notation B and G will be preserved also for the extended functions).

Then there exists a one-parameter family of ω-periodic solutions of the class $\mathscr{C}^1(\text{cl}\,\Omega(I))$. These solutions are given by the formula (1.6) *with initial conditions of the form*

$$\varphi(x) = C \exp(B(x)) - G(x)\,, \quad x \in \bar{I}\,, \tag{4.4}$$

where $C \in R$ is an arbitrary parameter.

Moreover, this one-parameter family includes all ω-periodic solutions of the class $\mathscr{C}^1(\text{cl}\,\Omega(I))$.

Proof. Putting $\psi(x) = C$ we see that the function $\varphi(x) = C \exp(B(x)) - G(x)$ satisfies all assumptions of Theorem 4.2. Thus every function (4.4) determines an ω-periodic solution. It remains to prove that there are no other ω-periodic solutions than those determined by the initial conditions (4.4).

Let u_1 be an arbitrary ω-periodic solution and let u_2 be a solution belonging to the family (4.4), then the function $v = u_1 - u_2$ satisfies the equalities

$$v_t + av_x + bv = 0\,, \quad v(x, t) - v(x, t + \omega) = 0\,. \tag{1}$$

Let us continue in two steps.

First let us show that the equation

$$w_t + aw_x = 0 \tag{2}$$

can have only a constant as an ω-periodic solution. Indeed, constructing an ω-periodic solution as indicated at the beginning of this section, we see that if the function φ_0 on $[x_0, x_1]$ has a range $[r_1, r_2]$ then the functions φ_k have the same range for all $k \in Z$. Evidently φ may have limits for $x \to \alpha$ and $x \to \beta$ only if the interval $[r_1, r_2]$ reduces to one point, which means that φ is a constant. Now, let us show that the problem (1) has only solutions determined by the initial functions $C \exp(B(x))$. In fact, by the assertion of the theorem one solution of this problem is

$$u_0(x, t) = \exp(B(\chi(0; x, t))) \exp\left(-\int_0^t b(\chi(\sigma; x, t), \sigma)\, d\sigma\right) > 0\,.$$

Looking for other solutions to (1) in the form $z(x, t)\, u_0(x, t)$ we readily find that z solves the equation (2), which completes the proof.

4.8. Remark. In Theorem 4.7 let there exist

$$\lim_{k \to +\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma = B_+(x)$$

or

$$\lim_{k \to -\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma = B_-(x)\,.$$

Then there exists also

$$\lim_{k \to -\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma = B_-(x)$$

or

$$\lim_{k \to +\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma = B_+(x)\,,$$

respectively, and

$$B_-(x) = B_+(x) - B_+(\lim_{k\to-\infty} x_k) \quad \text{if} \quad \lim_{k\to-\infty} x_k \neq \pm\infty$$

and

$$B_-(x) = B_+(x) + B_-(\lim_{k\to+\infty} x_k) \quad \text{if} \quad \lim_{k\to+\infty} x_k \neq \pm\infty\,.$$

An analogous assertion may be formulated for functions G_+ and G_-.

Proof. For definiteness let there exist

$$B_+(x) = \lim_{k\to+\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma$$

(the proof for the other orientation is quite similar).

Using (1.1) and (1.2) we get

$$(1) \qquad -\int_0^{-k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma =$$
$$= \int_0^{h\omega} b(\chi(\sigma; x_{-k}, 0), \sigma)\, d\sigma - \int_0^{(h-k)\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma$$

for a determining sequence $\{x_k\}_{k=-\infty}^{k=+\infty}$ and arbitrary $k, h \in Z$. Passing to the limit for $h \to +\infty$ in (1) we obtain

$$-\int_0^{-k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma = B_+(x_{-k}) - B_+(x)\,.$$

Now letting $k \to +\infty$ we have

$$-B_-(x) = B_+(\lim_{k\to+\infty} x_{-k}) - B_+(x)\,.$$

4.9. Remark. Let the assumptions of Theorem 4.7 be satisfied. Let $\gamma \in \{\alpha, \beta\}$, $\gamma \neq \pm\infty$. Then $\gamma \in \mathcal{N}$ and $\int_0^\omega g(\chi(s; \gamma, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; \gamma, \omega), \sigma)\, d\sigma\right) ds = 0$.

Proof. Let for example $\gamma = \alpha$. Let for definiteness $\lim_{k\to-\infty} x_k = \alpha$ for every determining sequence. By Remark 4.8 we can suppose without loss of generality that

$$B(x) = \lim_{k\to-\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma\,.$$

(The other cases of orientation of limits can be handled similarly.) If on the contrary α were from $\mathcal{M}$, i.e.

$$\int_0^\omega b(\chi(\sigma; \alpha, \omega), \sigma)\, d\sigma \neq 0\,, \quad \text{say} \quad \int_0^\omega b(\chi(\sigma; \alpha, \omega), \sigma)\, d\sigma > 0\,,$$

then there should exist $\varepsilon > 0$ and $c > 0$ such that

$$\int_0^\omega b(\chi(\sigma; x, \omega), \sigma)\, d\sigma \geqq c > 0 \quad \text{for} \quad x \in (\alpha, \alpha + \varepsilon)\,.$$

Taking $x \in (\alpha, \alpha + \varepsilon)$ we would have (for $k < 0$)

$$\int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, d\sigma = -\sum_{n=k+1}^{0} \int_{(n-1)\omega}^{n\omega} (\chi(\sigma; x, 0), \sigma)\, d\sigma =$$
$$= -\sum_{n=k+1}^{0} \int_0^\omega b(\chi(\sigma + (n-1)\,\omega; x, 0), \sigma)\, d\sigma =$$

$$= - \sum_{n=k+1}^{0} \int_0^\omega b(\chi(\sigma - \omega; x_n, 0), \sigma)\, \mathrm{d}\sigma =$$

$$= - \sum_{n=k+1}^{0} \int_0^\omega b(\chi(\sigma; x_n, \omega), \sigma)\, \mathrm{d}\sigma \leqq kc\,;$$

note that $x_n \to \alpha$ for $n \to -\infty$ monotonicaly and thus $x_n \in (\alpha, \alpha + \varepsilon)$ for every $n < 0$.

This would imply $B(x) = \lim_{k \to -\infty} \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, \mathrm{d}\sigma = -\infty$, which is a contradiction. Necessarily $\gamma \in \mathcal{N}$. The proof that

$$\int_0^\omega g(\chi(s; \gamma, \omega), s) \exp\left(-\int_s^\omega b(\chi(\sigma; \gamma, \omega), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s = 0$$

proceeds along the same lines.

4.10. Example. Let the equation

$$u_t + \cos^2 x\, u_x + \cos t\, u = \exp(-\mathrm{tg}^2\, x - \sin t)\,,$$

$$x \in \left(-\tfrac{1}{2}\pi + h\pi, \tfrac{1}{2}\pi + h\pi\right),$$

$$u_t + \cos t\, u = 0\,, \quad x = \pm\tfrac{1}{2}\pi + h\pi\,,$$

$h \in Z$ be given.

Here $\chi(\tau; x, t) = \mathrm{arctg}(\mathrm{tg}\, x + \tau - t) + h\pi$ for $x \in \left(-\frac{1}{2}\pi + h\pi, \frac{1}{2}\pi + h\pi\right)$, $\chi(\tau; \pm\frac{1}{2}\pi + h\pi, t) = \pm\frac{1}{2}\pi + h\pi$. Evidently $B(x) = 0$ for $x \in \left[-\frac{1}{2}\pi + h\pi, \frac{1}{2}\pi + h\pi\right]$,

$$G(x) = \lim_{k \to +\infty} \int_0^{2k\pi} \exp\left(-\mathrm{tg}^2(\mathrm{arctg}(\mathrm{tg}\, x + s) + h\pi) - \sin s\right).$$

$$.\exp\left(\int_0^s \cos\sigma\, \mathrm{d}\sigma\right) \mathrm{d}s = \lim_{k \to +\infty} \int_0^{2k\pi} \exp\left(-(\mathrm{tg}\, x + s)^2\right) \mathrm{d}s \quad \text{for}$$

$$x \in \left(-\tfrac{1}{2}\pi + h\pi, \tfrac{1}{2}\pi + h\pi\right)$$

and thus

$$G(x) = \tfrac{1}{2}\sqrt{\pi}\,(1 - \mathrm{erf}(\mathrm{tg}\, x)) \quad \text{for} \quad x \in \left(-\tfrac{1}{2}\pi + h\pi, \tfrac{1}{2}\pi + h\pi\right),$$

$$G\left(-\tfrac{1}{2}\pi + h\pi\right) = \sqrt{\pi}\,,$$

$$G\left(\tfrac{1}{2}\pi + h\pi\right) = 0\,.$$

Obviously $B, G \in \mathscr{C}^1\left(\left[-\frac{1}{2}\pi + h\pi, \frac{1}{2}\pi + h\pi\right]\right)$.

According to Theorem 4.7 the system of functions

$$\varphi(x) = C - \tfrac{1}{2}\sqrt{\pi} + \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(\mathrm{tg}\, x)\,, \quad x \in \left(-\tfrac{1}{2}\pi + h\pi, \tfrac{1}{2}\pi + h\pi\right),$$

$$\varphi\left(-\tfrac{1}{2}\pi + h\pi\right) = C - \sqrt{\pi}\,,$$

$$\varphi\left(\tfrac{1}{2}\pi + h\pi\right) = C$$

with a parameter C is the system of all initial conditions generating periodic solutions to the given equation. The corresponding periodic solutions are

$$u(x, t) = \left[C - \tfrac{1}{2}\sqrt{\pi} + \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(\mathrm{tg}\, x)\right] \exp(-\sin t)\,,$$

$$x \in \left(-\tfrac{1}{2}\pi + h\pi, \tfrac{1}{2}\pi + h\pi\right),$$

$$u\left(-\tfrac{1}{2}\pi + h\pi, t\right) = (C - \sqrt{\pi}) \exp(-\sin t)\,,$$

$$u\left(\tfrac{1}{2}\pi + h\pi, t\right) = C \exp(-\sin t)\,.$$

In this section we investigate the existence of ω-periodic solutions to the equation

$$(5.1) \qquad u_t + a(x, t)\, u_x + b(x, t)\, u = g(x, t)$$

on the whole plane R^2. The cases $\mathscr{R} = R$ and $\mathscr{S} = R$ were investigated in Secs. 2 and 3 and thus we will assume that $\mathscr{R} \neq \emptyset$, $\mathscr{S} \neq \emptyset$. The set $\mathscr{R}$ of regular points is an open set and hence we can write it as a union of disjoint open intervals:

$$(5.2) \qquad \mathscr{R} = \bigcup_{q \in Q} I_q\,, \quad I_q = (\alpha_q, \beta_q)\,,$$

where the set $Q \subseteq Z$ of indices is either finite or countable. If Q is not finite, we will suppose in addition that

(5.3) *the points α_q, β_q have no finite point of accumulation .*

Then the intervals I_q can be ordered so that

$$(5.4) \qquad \alpha_q < \beta_q \leqq \alpha_{q+1} \quad \textit{for} \quad q, q + 1 \in Q\,.$$

We look for the initial function φ which gives rise to an ω-periodic solution. The restrictions $\varphi_{|[\alpha_q, \beta_q]}$ will be simply denoted by φ_q.

First we suppose that on every I_q, $q \in Q$ the assumptions of Theorem 4.7 are fulfilled. Then the ω-periodic solutions on $\mathrm{cl}\, \Omega(I_q)$ form a one-parameter family determined by initial functions

$$(5.5) \qquad \varphi_q(x) = C_q \exp\left(B_q(x)\right) - G_q(x)\,, \quad x \in \bar{I}_q\,, \quad C_q \in R\,.$$

We will use the symbols $\mathscr{M}$, $\mathscr{N}$, $\varphi_{\mathscr{M}}$ and $\varphi_{\mathrm{cl}\mathscr{M}}$ introduced in Sec. 3. The value of $\varphi(x)$ for $x \in \mathscr{N}$ is not determined from the ω-periodicity condition, whereas for $x \in \mathscr{M}$ necessarily $\varphi(x) = \varphi_{\mathscr{M}}(x)$, the function φ being uniquely defined on the set $\mathscr{M}$.

5.1. Theorem. *In addition to the assumptions $A(R^2)$, $\mathscr{R} \neq \emptyset$, $\mathscr{S} \neq \emptyset$ let the following assumptions be satisfied:*

(i) $\mathscr{R} = \bigcup_{q \in Q} I_q$, $I_q = (\alpha_q, \beta_q)$, $Q \subseteq Z$, $\alpha_q < \beta_q \leqq \alpha_{q+1}$ *for $q, q + 1 \in Q$, and the set of points α_q, β_q has no finite point of accumulation (the first and last intervals, provided they exist, may be infinite);*

(ii) *for all $q \in Q$ and all $x \in I_q$ there exist limits*

$$B_q(x) = \lim_k \int_0^{k\omega} b(\chi(\sigma; x, 0), \sigma)\, \mathrm{d}\sigma,$$

$$G_q(x) = \lim_k \int_0^{k\omega} g(\chi(s; x, 0), s) \exp\left(\int_0^s b(\chi(\sigma; x, 0), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s$$

and they can be extended onto $\bar{I}_q$ as functions from $\mathscr{C}^1(\bar{I}_q)$;

(iii) $\int_0^\omega g(\chi(s; x, \omega), s) \exp\left(- \int_s^\omega b(\chi(\sigma; x, \omega), \sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s = 0$ *for $x \in \mathscr{N}$;*

(iv) *if there exists a subset $Q' \subseteq Q$ such that $\beta_j = \alpha_{j+1}$ for $j, j + 1 \in Q'$ then there exist constants C_j, $j \in Q'$ such that the function $\varphi = \varphi_j$ for $x \in (\alpha_j, \beta_j)$*

with $\varphi_j(x) = C_j \exp(B_j(x)) - G_j(x)$, $j \in Q'$ *is of the class* $\mathscr{C}^1(\bar{I}_{Q'})$ *where* $\bar{I}_{Q'} = [\alpha_{Q'}, \beta_{Q'}]$, $\alpha_{Q'} = \inf_{j \in Q'} \alpha_j$ $\beta_{Q'} = \sup_{j \in Q'} \beta_j$;

(v) *the boundary of* $\mathscr{M}$ *has no finite point of accumulation and* $\varphi_{\mathrm{cl}\mathscr{M}} \in \mathscr{C}^1(\mathrm{cl}\,\mathscr{M})$;

(vi) $\partial\mathscr{M} \cap \partial\mathscr{R} = \emptyset$.

Then there exists an ω*-periodic solution* $u \in \mathscr{C}^1(R^2)$ *to* (5.1). *The initial function* φ *corresponding to the* ω*-periodic solution is given by* (5.5) *on each* $\bar{I}_q$. *In the case of „adjacent" intervals* I_j, $j \in Q'$ *the constants* C_j *are determined by the assumption* (iv), *and in the opposite case* C_q *is arbitrary; further,* φ *is equal to* $\varphi_{\mathrm{cl}\mathscr{M}}$ *on* cl $\mathscr{M}$ *and finally,* $\varphi_{|\mathscr{N}}$ *is an arbitrary function such that the resulting function* φ *is of the class* $\mathscr{C}^1(R)$.

Proof. By the assumption (ii), on each interval $\bar{I}_q$ the initial function $\varphi_{|\bar{I}_q}$ of the sought periodic solution is determined according to Theorem 4.7 by the formula (5.5). If there is a sequence of „adjacent" intervals I_j then the $\mathscr{C}^1$ continuity on the union of these intervals is guaranteed by the condition (iv).

On a bounded component $[\beta_j, \alpha_{j+1}]$, $\beta_j < \alpha_{j+1}$ of $\mathscr{S}$ there exists at most a finite number of intervals of points from $\mathscr{M}$ on the closure of which the function $\varphi_{|\mathrm{cl}\mathscr{M}} = \varphi_{\mathrm{cl}\mathscr{M}}$ is defined uniquely by (v). Since by the assumption (vi) the set $\mathscr{M}$ is separated from the points β_j, α_{j+1} by intervals of points from $\mathscr{N}$,it is always possible to extend the initial function to the whole interval $[\alpha_j, \beta_{j+1}]$ as a function from $\mathscr{C}^1$ due to the arbitrariness of φ on $\mathscr{N}$.

The construction of φ on the unbounded components of $\mathscr{S}$ (if there are any) is quite similar.

5.2. Remark. It is clear from the proof which assumptions of the theorem can be altered. For example, if the intervals from $\mathscr{R}$ are separated by intervals of points from $\mathscr{N}$ then the existence Theorem 4.2 may be used instead of Theorem 4.7.

5.3. Remark. In the case of at most two "adjacent" intervals, say $I_j = (\alpha_j, \beta_j)$ and $I_{j+1} = (\alpha_{j+1}, \beta_{j+1})$, of regular points the condition (iv) can be replaced by the assumption of solvability of the system of equations

$$C_j \exp(B_j(\beta_j)) - C_{j+1} \exp(B_{j+1}(\beta_j)) = G_j(\beta_j) - G_{j+1}(\beta_j)\,,$$
$$C_j B_j'(\beta_j) \exp(B_j(\beta_j)) - C_{j+1} B_{j+1}'(\beta_j) \exp(B_{j+1}(\beta_j)) =$$
$$= G_j'(\beta_j) - G_{j+1}'(\beta_j)\,.$$

5.4. Example. Let the equation

$$u_t + \cos^2 x\, u_x + \cos t\, u = \exp(-\mathrm{tg}^2 x - \sin x)\,, \quad x \neq \tfrac{1}{2}\pi + h\pi$$
$$u_t + \cos t\; u = 0\,, \quad x = \tfrac{1}{2}\pi + h\pi\,, \quad h \in Z$$

be given.

Here $I_q = (-\frac{1}{2}\pi + q\pi, \frac{1}{2}\pi + q\pi)$, $Q = Z$, $\mathscr{S} = \mathscr{N} = \{\frac{1}{2}\pi + h\pi \mid h \in Z\}$, $\mathscr{M} = \emptyset$, and according to Example 4.10 the limits $B_q(x)$, $G_q(x)$ exist, are of the class $\mathscr{C}^1(\bar{I}_q)$ for every $q \in Z$ and the resulting function is given by

$$\varphi_q(x) = C_q - \tfrac{1}{2}\sqrt{\pi} + \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(\mathrm{tg}\,x)\,, \quad x \in (-\tfrac{1}{2}\pi + q\pi, \tfrac{1}{2}\pi + q\pi)\,,$$
$$\varphi_q(-\tfrac{1}{2}\pi + q\pi) = C_q - \sqrt{\pi}\,,$$
$$\varphi_q(\tfrac{1}{2}\pi + q\pi) = C_q\,, \quad C_q \in R\,.$$

Thus the conditions (i)–(iii), (v) and (vi) of Theorem 5.1 are satisfied. It remains to find constants C_j from the condition (iv). In our case $Q' = Q$. Let us fix the constant C_0 on the interval $I_0 = (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, say $C_0 = C$. Then the constants C_q are given by the relation $C_q = C + q\sqrt{\pi}$ and the sought initial conditions defining periodic solutions are functions

$$\varphi(x) = C + (q - \tfrac{1}{2})\sqrt{\pi} + \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(\mathrm{tg}\,x)\,,$$
$$x \in (-\tfrac{1}{2}\pi + q\pi, \tfrac{1}{2}\pi + q\pi)\,,$$
$$\varphi(-\tfrac{1}{2}\pi + q\pi) = C + (q - 1)\sqrt{\pi}\,,$$
$$\varphi(\tfrac{1}{2}\pi + q\pi) = C + q\sqrt{\pi}\,,$$

$q \in Z$ and $C \in R$ arbitrary, and the periodic solutions are

$$u(x, t) = [C + (q - \tfrac{1}{2})\sqrt{\pi} + \tfrac{1}{2}\sqrt{\pi}\,\mathrm{erf}(\mathrm{tg}\,x)]\exp(-\sin t)\,,$$
$$x \in (-\tfrac{1}{2}\pi + q\pi, \tfrac{1}{2}\pi + q\pi)\,, \quad t \in R\,,$$
$$u(-\tfrac{1}{2}\pi + q\pi, t) = (C + (q - 1)\sqrt{\pi})\exp(-\sin t)\,, \quad t \in R\,,$$
$$u(\tfrac{1}{2}\pi + q\pi, t) = (C + q\sqrt{\pi})\exp(-\sin t)\,, \quad t \in R\,,$$

$C \in R$ arbitrary.

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Authors' addresses: Jaroslav Barták, M. D. Rettigové 4, 116 39 Praha 1, (Katedra matematiky pedagogické fakulty UK); Otto Vejvoda Žitná 25, 115 67 Praha 1, Czechoslovakia (Matematický ústav ČSAV).