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NUMERICAL INTEGRATION WITH WEIGHT FUNCTIONS $\cos kx$, $\sin kx$ ON $\left[0, \frac{2\pi}{t}\right]$, $t = 1, 2, \ldots$

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1. INTRODUCTION

Many theoretical and applied problems lead to the computation of integrals with the weight function

$$w(kx) = \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix}$$

(k – integer) appearing in the integrand, i.e.

$$\int_0^T f(x)\, w(kx)\, \mathrm{d}x \tag{1}$$

where $T = 2\pi/t$, $t = 1, 2, \ldots$

With increasing k in (1), the frequency of the subintegral function increases along with the difficulties of numerical computation by current methods. In this paper a specific interpolation method for computation of (1) is described, its convergence is investigated, some weight coefficients (further only coef.) of this method are tabulated, and a method of computation of (1) is suggested on the basis of their properties. The results are demonstrated on numerical experiments. The paper is a continuation of [2].

2. DESCRIPTION OF THE METHOD

Consider the quadrature formula

$$\int_0^1 f(x)\, w(2\pi kx)\, \mathrm{d}x = \sum_{i=0}^{n} A_i^{[k]} f(x_i^{(n)}) + R_n(f)\,, \tag{2}$$

coef. $A_i^{[k]}$ being chosen at given knots of integration $x_i^{(n)} \in [0,1]$ in a way that formula (2) be accurate for $f(x) = 1, x, x^2, \ldots, x^n$, i.e.

$$(3) \qquad \sum_{i=0}^{n} A_i^{[k]} (x_i^{(n)})^m = W_m \begin{Bmatrix} c \\ s \end{Bmatrix}, \quad m = 0, 1, 2, \ldots, n,$$

where

$$W_m \begin{Bmatrix} c \\ s \end{Bmatrix} = \int_0^1 x^m \begin{Bmatrix} \cos 2\pi kx \\ \sin 2\pi kx \end{Bmatrix} \mathrm{d}x .$$

From systems of equations (3) there follows [2]:

1. the existence and unicity of coef. $A_i^{[k]}$,

2. the coef. $A_i^{[k]}$ calculated for $[0,2\pi]$ are equal to the coef. computed for an arbitrary interval $[2l\pi, (2l + 2)\pi]$; they differ in signs from the coef. for $[(2l + 1)\pi, (2l + 3)\pi]$, l integer,

3. the coef. $A_i^{[p]}$ are equal in each interval where $w(kx)$ have p-period, $p = 1, 2, \ldots$, e.g. $A_i^{[p]}$ for $[0, 2\pi/p]$, $p = 1, 2, \ldots$ are equal. Thus, when $k = t \cdot p \cdot d$ then coef. $A_i^{[k]}$ for $[0, 2\pi]$ are equal to $A_i^{[p]}$ for $[2\pi(l - 1)/td, 2\pi l/td]$, $l = 1, 2, \ldots, d$.

It is not possible to carry out the accurate computation of $A_i^{[k]}$ for greater n by solving system (3). Coef. $A_i^{[k]}$ have been therefore calculated from well-known relations for interpolation quadrature, i.e. from

$$(4) \qquad A_i^{[k]} = \frac{1}{\omega_n'(x_i^{(n)})} \int_0^1 \frac{\omega_n(x)}{x - x_i^{(n)}} w(2\pi kx) \, \mathrm{d}x, \quad i = 0, 1, \ldots, n$$

where $\omega_n(x) = \prod_{j=0}^{n} (x - x_j^{(n)})$.

For computing of (1) by means of (2) there holds

Theorem 1. *Let $A_i^{[p]}$ be coef. (4) calculated for $w(2\pi px)$, let $x_i^{[l]} = (2\pi/td)(l - 1 + x_i^{(n)})$ where $x_i^{(n)}$ are the knots from (2), d is the number of equal subintervals $[0, T]$, $T = 2\pi/t$, $t = 1, 2, 3, \ldots$, $k = t \cdot p \cdot d$. Then*

$$(5) \qquad \int_0^T f(x) \, w(kx) \, \mathrm{d}x = \frac{2\pi}{td} \sum_{l=1}^{d} \sum_{i=0}^{n} A_i^{[p]} f(x_i^{[l]}) + R_k^{[d]}(f) .$$

If $f(x) \in C^{n+1}[0, T]$ and $|f^{(n+1)}(x)| \leqq M$, $x \in [0, T]$ then

$$(6) \qquad |R_k^{[d]}(f)| \leqq \frac{M T^{n+1} 4 \max_{[0,1]} |\omega_n(x)|}{t d^{n+1} (n + 1)!} .$$

Proof. Substituting $dtz = x$ we get

$$\int_0^T f(z)\, w(kz)\, \mathrm{d}z = \sum_{l=1}^{d} \int_{a_{l-1}}^{a_l} f(z)\, w(pdtz)\, \mathrm{d}z = \frac{1}{td} \sum_{l=1}^{d} \int_{2\pi(l-1)}^{2\pi l} f\left(\frac{x}{td}\right) w(px)\, \mathrm{d}x$$

where $[a_{l-1}, a_l] \equiv [(2\pi/td)(l-1), (2\pi/td)\, l]$.

If we use (2) for the last integrals we get (5) in which

$$R_k^{[d]}(f) = \sum_{l=1}^{d} R_l$$

and

$$(7) \qquad R_l = \frac{1}{(n+1)!} \int_{a_{l-1}}^{a_l} (z - z_0^{[l]})(z - z_1^{[l]}) \dots (z - z_n^{[l]})\, w(kz)\, f^{(n+1)}(\xi_l)\, \mathrm{d}z$$

where $z_i^{[l]}, \xi_l \in [a_{l-1}, a_l]$.

Knots $z_i^{[l]}$ were obtained by transforming the knots of $\omega_n(x)$, i.e. $x_i^{(n)}$ into $[a_{l-1}, a_l]$, $z_i^{[l]} = (2\pi/td)(l - 1 + x_i^{(n)})$. Assuming that for $z \in [a_{l-1}, a_l]$ there is $|f^{(n+1)}(z)| \leqq M_l$, $l = 1, 2, \dots, d$ then after substitution $z = (2\pi/td)(l - 1 + x)$ we get from (7)

$$|R_l| \leqq \left(\frac{2\pi}{td}\right)^{n+2} \frac{M_l}{(n+1)!} \int_0^1 |w(2\pi px)\, \omega_n(x)|\, \mathrm{d}x\,.$$

Since $\int_0^1 |w(2\pi px)|\, \mathrm{d}x = 2/\pi$ there is

$$|R_l| \leqq \left(\frac{2\pi}{t}\right)^{n+1} \frac{M_l 4 \max_{[0,1]} |\omega_n(x)|}{td^{n+2}(n+1)!} \qquad l = 1, 2, \dots, d\,.$$

If $\max_l M_l = M$ then estimate (6) holds. (It is obvious that $\max_{[0,1]} |\omega_n(x)| \leqq 1$ always holds.)

The minimizing of estimate (6) is possible by the choice of $x_i^{(n)}$, $i = 0, 1, \dots, n$ in $\omega_n(x)$, $(x_i^{(n)} \neq x_j^{(n)}, i \neq j)$. If we choose for knots $x_i^{(n)}$ the roots of Chebyshev polynomials after transformation into $[0, 1]$ then $\max_{[0,1]} |\omega_n(x)| = 1/2^{2n+1}$, while any other choice of $x_i^{(n)}$ does not diminish $\max_{[0,1]} |\omega_n(x)|$ below this limit.

The estimate in this case will be

$$|R_k^{[d]}(f)| \leqq \frac{\pi^{n+1} M}{2^{n-2} d^{n+1} t^{n+2} (n+1)!}$$

which justifies the great accuracy of method (5) especially for greater k by using Theorem 1. For practical reasons, equidistant knots are often advantageous.

3. CONVERGENCE OF FORMULA (2)

Consider the Newton-Cotes quadrature formula with the general weight function $W(x)$

$$(8) \qquad \int_{-1}^{1} f(x)\, W(x)\, \mathrm{d}x = \sum_{i=0}^{n} A_i f(x_i^{(n)}) + \bar{R}_n(f)$$

where

$$(9) \qquad A_i = \frac{1}{\omega_n'(x_i^{(n)})} \int_{-1}^{1} \frac{\omega_n(x)}{x - x_i^{(n)}}\, W(x)\, \mathrm{d}x\,, \quad i = 0, 1, \ldots, n$$

and $\omega_n(x) = \prod_{j=0}^{n} (x - x_j^{(n)}),\ x_j^{(n)} \in [-1, 1]$.

It is known that in order to make the interpolation quadrature process (8) at $n = 0, 1, 2, \ldots$ converge for each $f(x) \in C[-1, 1]$ i.e.

$$\lim_{n\to\infty} \sum_{i=0}^{n} A_i f(x_i^{(n)}) = \int_{-1}^{1} f(x)\, W(x)\, \mathrm{d}x$$

it is necessary and sufficient (Polya, Steklov) that there exists such K that the inequality

$$(10) \qquad \sum_{i=0}^{n} |A_i| \leqq K$$

is satisfied, $n = 0, 1, 2, \ldots$ It has been proved (Kuzmin) that the coef. of quadrature (8) with equidistant knots do not satisfy (10).

Prove now that for (8) there holds

Theorem 2. *Let $W(x)$ satisfy on $[-1, 1]$ these assumptions: $W''(x)$ exists, is integrable and $W'(x)$ has bounded variation. If $x_i^{(n)}$ in (8) are roots of the Chebyshev polynomial $T_{n+1}(x)$ then (10) holds, i.e. (8) converges for each $f(x) \in C[-1, 1]$.*

Proof. Let in (9)

$$(11) \qquad L_i(x) = \frac{\omega_n(x)}{\omega_n'(x_i^{(n)})\,(x - x_i^{(n)})}$$

where $x_i^{(n)} = \cos \varphi_i$ are roots of $T_{n+1}(x)$, $\varphi_i = ((2i + 1)/2(n + 1))\,\pi$, $i = 0, 1, 2, \ldots$ $\ldots, n$. For (11) it holds that [1]

$$(12) \qquad L_i(\cos \varphi) = \frac{1}{n + 1} + \frac{2}{n + 1} \sum_{m=1}^{n} \cos m\varphi_i \cos m\varphi\,, \quad i = 0, 1, \ldots, n$$

since $L_i(\cos \varphi_i) = 1$, $L_i(\cos \varphi_j) = 0$, $j = 0,1,., i - 1, i + 1, \ldots, n - 1, n$. Substituting $x = \cos \varphi$ in (9) we get

$$A_i = \int_0^\pi L_i(\cos \varphi) \sin \varphi \, W(\cos \varphi) \, d\varphi \,. \tag{13}$$

Substituting (12) into (13) we have

$$A_i = \frac{\pi}{n + 1} S_n(\varphi_i) \tag{14}$$

where

$$S_n(\varphi_i) = \frac{1}{\pi} \int_0^\pi \sin \varphi \, W(\cos \varphi) \, d\varphi + \sum_{m=1}^{n} \frac{2}{\pi} \int_0^\pi \sin \varphi \, W(\cos \varphi) \cos m\varphi \, d\varphi \cos m\varphi_i \tag{15}$$

is the Fourier series on $[0, \pi]$ of the function $F(\varphi) = \sin \varphi \, W(\cos \varphi)$ at the point φ_i. Coef. (14) can be estimated by

$$|A_i| \leqq \frac{\pi}{n + 1} \sum_{m=0}^{n} |c_m| \tag{16}$$

where c_m are Fourier coefficients of $F(\varphi)$ in (15). After double integration of c_m by parts we get for $m > 0$

$$c_m = \frac{2}{\pi} \int_0^\pi F(\varphi) \cos m\varphi \, d\varphi = \frac{2}{\pi m^2} \left[(-1)^{m+1} W(-1) - W(1) - \int_0^\pi F''(\varphi) \cos m\varphi \, d\varphi \right]$$

and thus by using the properties of $W(x)$ we obtain $|c_m| \leqq 2C/\pi m^2$ where

$$C = |W(-1)| + |W(1)| + \overset{\pi}{\underset{0}{V}}(F') \,. \tag{17}$$

Since $W(x)$ satisfies on $[-1, 1]$ Lipschitz's condition, it has bounded variation there, i.e. $F'(\varphi)$ is the function of bounded variation, too. As $|W(x)| \leqq M$ we have for (16)

$$|A_i| \leqq \frac{2}{n + 1} \left(M + C \sum_{m=1}^{n} \frac{1}{m^2} \right)$$

and thus (10) is satisfied for

$$K = 2M + \frac{\pi^2 C}{3} .$$

Remark: Since it holds

$$\sin \varphi_i \, W(\cos \varphi_i) = S_n(\varphi_i) + R(\varphi_i) \tag{18}$$

where

$$(19) \qquad R(\varphi_i) = \frac{2}{\pi} \sum_{m=n+2}^{\infty} \int_0^{\pi} \sin \varphi \, W(\cos \varphi) \cos m\varphi \, d\varphi \cos m\varphi_i$$

we can write (14) as

$$(20) \qquad A_i = \frac{\pi}{n+1} \sin \varphi_i \, W(\cos \varphi_i) - \frac{\pi}{n+1} R(\varphi_i) .$$

Since the Fourier series of $F(\varphi)$ converges uniformly, we get from (20) an asymptotic expression for A_i

$$(21) \qquad A_i \approx \frac{\pi}{n+1} \sin \varphi_i \, W(\cos \varphi_i)$$

i.e. it holds from sufficiently high n

$$\operatorname{sign} A_i = \operatorname{sign} W(\cos \varphi_i) , \quad i = 0, 1, \ldots, n .$$

Quadrature formula (8) in this case reads

$$(22) \qquad \int_{-1}^{1} f(x) \, W(x) \, dx = \frac{\pi}{n+1} \sum_{i=0}^{n} \sin \varphi_i \, W(\cos \varphi_i) f(\cos \varphi_i) + \tilde{R}_n(f)$$

where

$$(23) \qquad \tilde{R}_n(f) = \bar{R}_n(f) - \frac{\pi}{n+1} \sum_{i=0}^{n} \sum_{m=n+2}^{\infty} f(\cos \varphi_i) \, c_m \cos m\varphi_i .$$

4. COMPUTATION OF COEF. $A_i^{[k]}$

Let $\omega_n(x) = x^{n+1} + a_1 x^n + \ldots + a_n x + a_{n+1}$ have real roots $x_i \in [0, 1]$ (instead of $x_i^{(n)}$ we write x_i). Computation of coef. $A_i^{[k]}$ was carried out from

$$(24) \qquad A_i^{[k]} = \frac{1}{g(x_i)} \sum_{j=0}^{n} b_j W_{n-j} \begin{Bmatrix} c \\ s \end{Bmatrix} , \quad i = 0, 1, \ldots, n ,$$

whereby in $\omega_n(x)$

$$a_r = -\frac{1}{r} \sum_{i=0}^{r-1} a_i s_{r-i} , \quad s_r = \sum_{i=0}^{n} x_i^r , \quad r = 1, 2, \ldots, n+1$$

where x_i are

a) roots of Chebyshev polynomials $T_{n+1}(x) = (1/2^n) \cos [(n+1) \arccos x]$ for $[0, 1]$ i.e.

$$(25) \qquad x_{n-i} = \frac{1}{2}\left(\cos \frac{i + 0.5}{n+1}\pi + 1\right), \quad i = n, n-1, \dots 1, 0$$

b) (26) $$x_i = \frac{i}{n}, \quad i = 0, 1, \dots, n \quad \text{respectively.}$$

For the function $g(x)$ there is

$$g(x) = \frac{\omega_n(x)}{x - x_i} = \sum_{j=0}^{n} b_j x^{n-j} .$$

For the moments $W_m \begin{Bmatrix} c \\ s \end{Bmatrix}$ we get from their reccurent relations

$$(27) \quad W_0(c) = W_1(c) = 0\,, \quad W_m(c) = \frac{m}{q^2}\left[1 - (m-1)\, W_{m-2}(c)\right], \quad m = 2, 3, \dots$$

and

$$W_m(s) = \frac{-q}{m+1}\, W_{m+1}(c)$$

where $q = 2\pi k$.

We have the following estimate of moments

$$(28) \qquad \left| W_m \begin{Bmatrix} c \\ s \end{Bmatrix} \right| \leqq \frac{1}{m+1} .$$

Algorithms for the computation of $W_m \begin{Bmatrix} c \\ s \end{Bmatrix}$ are very unstable. E.g. in single precision arithmetics in (27) for $k = 1$ the inequality (28) is satisfied only up to $m = 29$. At $m = 49$ is $|W_{49}(c)| > 0{\cdot}32 \,.\, 10^{16}$. In double precision arithmetics (28) is satisfied up to $m = 43$ while $|W_{49}(c)| > 0{\cdot}17 \,.\, 10^{5}$.

The following assertion holds on coef. (9):

Theorem 3. *Let A_i be Cotes coef. with the weight function $W(x)$ i.e. (9). If $W(x)$ is even (odd) on $[-1, 1]$ and $x_i = -x_{n-i}$, $i = 0, 1, \dots, n$ then there holds $A_i = A_{n-i}(A_i = -A_{n-i})$, $i = 0, 1, 2, \dots, n$.*

Proof. Consider the expressions $A_i \mp A_{n-i}$. Since $\omega_n'(x_{n-i}) = (-1)^n\, \omega_n'(x_i)$,

$$A_i \mp A_{n-i} = \frac{1}{\omega_n'(x_i)} \int_{-1}^{1} W(x)\, P(x)\, [x - x_{n-i} \mp (-1)^n (x - x_i)]\, dx$$

where $P(x) = \omega_n(x)/(x - x_i)(x - x_{n-i})$.

Consider further a) n even, b) n odd number. Since $x_i + x_{n-i} = 0$ there is

a)

$$A_i - A_{n-i} = \frac{x_i - x_{n-i}}{\omega_n'(x_i)} \int_{-1}^{1} W(x)\, P(x)\, dx\,, \quad A_i + A_{n-i} = \frac{2}{\omega_n'(x_i)} \int_{-1}^{1} W(x)\, x\, P(x)\, dx$$

b)

$$A_i - A_{n-i} = \frac{2}{\omega_n'(x_i)} \int_{-1}^{1} W(x)\, x\, P(x)\, dx\,, \quad A_i + A_{n-i} = \frac{x_i - x_{n-i}}{\omega_n'(x_i)} \int_{-1}^{1} W(x)\, P(x)\, dx$$

respectively.

Function $P(x)$ is on $[-1, 1]$ odd in case a), even in case b). Function $x\,P(x)$ has the reverse property. If $W(x)$ is even then $A_i - A_{n-i} = 0$ i.e. $A_i = A_{n-i}$ in both cases. If $W(x)$ is odd then similarly holds $A_i = -A_{n-i}$, $i = 0, 1, \ldots, n$.

A constant step of integration enables us to make certain simplifications in practical computation. The symmetry of coef. $A_i^{[k]}$ allows in (5) to use the values $f(x)$ already calculated in the boundary points of subintervals. In case $W(x) = \cos 2\pi kx$ the total number of knots is $N = dn + 1$, if $W(x) = \sin 2\pi kx$ then $N = d(n - 1) + 2$ (for n even there is $N = d(n - 2) + 2$ since the subintegral function in (4) is odd on $[0, 1]$ and consequently $A_{n/2}^{[k]} = 0$).

Coef. $A_i^{[k]}$ for knots (25) and (26) for $W(x) = \cos 2\pi kx (A_i^{[k]} = A_{n-i}^{[k]})$ and $W(x) = \sin 2\pi kx (A_i^{[k]} = -A_{n-i}^{[k]})$, $k = 1(1)\,20, 50, 100$, $n = 2(1)\,20$ have been calculated, for $k = 1, 2, 3, 5$ and for given n they are tabulated in Table 5. The given number of decimals was checked by the computation of all equations (3) for $m = 0, 1, 2, \ldots$ $\ldots, n + 1$. Formula (2) — similarly as all quadratures of Newton-Cotes type — yields precise results at $n + 1$ odd for the polynomial of the $n + 1$-st degree, at $n + 1$ even for the polynomial of the n-th degree.

5. NUMERICAL EXAMPLES

We show now some numerical experiments which point out the possibilities of using formula (5).

Let us introduce the conception of the so called characteristics (further only char.) of formula (5). It will be the symbol $(n.t.p.d)$ consisting of the parameters of formula (5) $(k = t.p.d)$.

Example 1. For coefficients (further cff.) a_k, b_k of the development of a given function into the Fourier series holds

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\, dx\,, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\, dx \tag{29}$$

Table 1

$f(x)$	$k = p\,.\,d$	1		10		50		100	
	knots of integration	(25)	(26)	(25)	(26)	(25)	(26)	(25)	(26)
	$(p\,.\,d)$	(1.1)		(10.1)		(10.5)		(20.5)	
e^x $e^x \cos x$	a_k b_k	$2{\cdot}2\,.\,10^{-8}$ $1{\cdot}7\,.\,10^{-7}$	$2{\cdot}0\,.\,10^{-6}$ $4{\cdot}5\,.\,10^{-5}$	$6{\cdot}2\,.\,10^{-8}$ $1{\cdot}7\,.\,10^{-5}$	$2{\cdot}9\,.\,10^{-7}$ $1{\cdot}5\,.\,10^{-4}$	$5{\cdot}7\,.\,10^{-10}$ $9{\cdot}0\,.\,10^{-10}$	$1{\cdot}2\,.\,10^{-8}$ $9{\cdot}5\,.\,10^{-11}$	$5{\cdot}3\,.\,10^{-11}$ $3{\cdot}6\,.\,10^{-10}$	$6{\cdot}4\,.\,10^{-9}$ $1{\cdot}4\,.\,10^{-9}$

$f(x)$	$k = p\,.\,d$	200		300		400		500	
	knots of integration	(25)	(26)	(25)	(26)	(25)	(26)	(25)	(26)
	$(p\,.\,d)$	(50.4)		(50.6)		(50.8)		(50.10)	
e^x $e^x \cos x$	a_k b_k	$2{\cdot}7\,.\,10^{-11}$ $1{\cdot}2\,.\,10^{-10}$	$1{\cdot}5\,.\,10^{-9}$ $1{\cdot}0\,.\,10^{-9}$	$1{\cdot}6\,.\,10^{-11}$ $1{\cdot}1\,.\,10^{-10}$	$1{\cdot}0\,.\,10^{-9}$ $6{\cdot}0\,.\,10^{-10}$	$1{\cdot}1\,.\,10^{-11}$ $7{\cdot}4\,.\,10^{-11}$	$7{\cdot}6\,.\,10^{-10}$ $4{\cdot}0\,.\,10^{-10}$	$8{\cdot}4\,.\,10^{-12}$ $7{\cdot}4\,.\,10^{-11}$	$6{\cdot}1\,.\,10^{-10}$ $3{\cdot}2\,.\,10^{-10}$

Table 2

$f(x)$		char.	$k = 2pd$						
			2	4	6	8	10	12	16
$\frac{1}{x+3}$		(12.2.1.d)	$9{\cdot}5\,.\,10^{-10}$	$3{\cdot}0\,.\,10^{-11}$	$8{\cdot}0\,.\,10^{-12}$	$3{\cdot}4\,.\,10^{-11}$	$5{\cdot}2\,.\,10^{-12}$	$4{\cdot}2\,.\,10^{-12}$	$2{\cdot}3\,.\,10^{-12}$
e^{rx}	$r = 10$ $r = 15$	(13.2.1.d) (12.2.1.d)	$2{\cdot}1\,.\,10^{-5}$ * $5{\cdot}5\,.\,10^{-5}$ *	$1{\cdot}5\,.\,10^{-7}$ * $2{\cdot}0\,.\,10^{-6}$ *	$5{\cdot}2\,.\,10^{-6}$ $4{\cdot}2\,.\,10^{-8}$ *	$2{\cdot}2\,.\,10^{-6}$ $1{\cdot}7\,.\,10^{-8}$ *	$8{\cdot}0\,.\,10^{-7}$ $6{\cdot}0\,.\,10^{-8}$ *	$4{\cdot}1\,.\,10^{-7}$ $3{\cdot}6\,.\,10^{-8}$ *	$2{\cdot}2\,.\,10^{-7}$ $3{\cdot}5\,.\,10^{-7}$ *

Table 3

r	$k = 4pd$				
	4	8	12	16	20
10	$3{\cdot}6 \,.\, 10^{-7}$ *	$7{\cdot}9 \,.\, 10^{-7}$	$2{\cdot}2 \,.\, 10^{-7}$	$1{\cdot}7 \,.\, 10^{-7}$	$1{\cdot}2 \,.\, 10^{-7}$
15	$1{\cdot}9 \,.\, 10^{-6}$ *	$1{\cdot}2 \,.\, 10^{-8}$ *	$8{\cdot}0 \,.\, 10^{-10}$ *	$2{\cdot}6 \,.\, 10^{-5}$	$1{\cdot}9 \,.\, 10^{-5}$
20	$1{\cdot}4 \,.\, 10^{-6}$ *	$2{\cdot}8 \,.\, 10^{-8}$ *	$4{\cdot}4 \,.\, 10^{-9}$ *	$4{\cdot}0 \,.\, 10^{-8}$ *	$9{\cdot}0 \,.\, 10^{-7}$ *

For $k = 1, 10, 50, 100(100)\,500$ integrals (29) were computed for functions a) $f(x) = e^x$, b) $f(x) = e^x \cos x$ with char. $(12.1.p.d)$. Table 1 gives the absolute errors of computation if the knots (25) and (26), respectively, are used. Instead of the char. only $(p.d)$ is given.

We see that method (5) achieves good results especially for high k, knots (25) giving less error than (26). In char. we can choose n, p, d in various ways. Thus e.g. at $f(x) = e^x$ for a_{100} by char. $(12.1.p.d)$ let $(p.d) \equiv (100.1), (50.2), (20.5), (10.10)$ respectively. Absolute errors in these cases at knots (25) are (in the parentheses stays the actual number of knots of integration N): $2{\cdot}39 \,.\, 10^{-9}(13)$; $5{\cdot}84 \,.\, 10^{-11}(25)$; $5{\cdot}28 \,.\, 10^{-11}(61)$; $3{\cdot}28 \,.\, 10^{-10}(121)$.

Example 2. For coef. c_k of the development of a given function $f(x)$ into the series by means of Chebyshev polynomials holds

$$c_k = \frac{2}{\pi} \int_0^{\pi} f(\cos x) \cos kx \, \mathrm{d}x \,. \tag{30}$$

Table 4

p	n	d				
		1	2	3	4	5
1	6	$1{\cdot}3 \,.\, 10^{-3}$	$2{\cdot}0 \,.\, 10^{-6}$	$8{\cdot}7 \,.\, 10^{-8}$	$1{\cdot}0 \,.\, 10^{-8}$	$1{\cdot}9 \,.\, 10^{-9}$
	10	$7{\cdot}8 \,.\, 10^{-8}$	$3{\cdot}4 \,.\, 10^{-10}$	$2{\cdot}5 \,.\, 10^{-10}$	$2{\cdot}2 \,.\, 10^{-10}$	$1{\cdot}5 \,.\, 10^{-10}$
2	6	$2{\cdot}7 \,.\, 10^{-2}$	$6{\cdot}1 \,.\, 10^{-5}$	$2{\cdot}8 \,.\, 10^{-6}$	$3{\cdot}5 \,.\, 10^{-7}$	$7{\cdot}1 \,.\, 10^{-8}$
	10	$1{\cdot}4 \,.\, 10^{-6}$	$3{\cdot}6 \,.\, 10^{-10}$	$1{\cdot}0 \,.\, 10^{-10}$	$4{\cdot}0 \,.\, 10^{-10}$	$1{\cdot}5 \,.\, 10^{-10}$
3	6	$3{\cdot}0 \,.\, 10^{-2}$	$7{\cdot}3 \,.\, 10^{-5}$	$3{\cdot}3 \,.\, 10^{-6}$	$4{\cdot}2 \,.\, 10^{-7}$	$8{\cdot}5 \,.\, 10^{-8}$
	10	$2{\cdot}0 \,.\, 10^{-5}$	$1{\cdot}2 \,.\, 10^{-9}$			
5	6	$7{\cdot}7 \,.\, 10^{-3}$	$2{\cdot}0 \,.\, 10^{-5}$	$9{\cdot}8 \,.\, 10^{-7}$	$1{\cdot}2 \,.\, 10^{-7}$	$2{\cdot}2 \,.\, 10^{-8}$
	10	$2{\cdot}6 \,.\, 10^{-6}$	$2{\cdot}3 \,.\, 10^{-10}$			

Integrals (30) were computed with knots (25) for functions a) $f(x) = (x + 3)^{-1}$, b) $f(x) = e^{rx}$, $r = 10, 15$ with char. and errors given in Table 2 ($d = 1(1)\,8$). For $|c_k| > 10^3$ we give relative errors (marked with $*$), otherwise the absolute errors.

Example 3. For modified Bessel functions of the first kind $I_k(r)$ (k-even integer) the relation

$$I_k(r) = (-1)^{k/2} \frac{2}{\pi} \int_0^{\pi/2} \mathrm{ch}\,(r \sin x) \cos kx \,\mathrm{d}x \tag{31}$$

holds. For $k = 4(4)\,20$, $r = 10, 15, 20$ integrals (31) were calculated ($d = 1(1)\,5$) with knots (25) and char. (12.4.1.d). The results are in Table 3. Relative errors ($*$) are again in Table for $|I_k(r)| > 10^3$.

Example 4. Let us compute cff. b_k of function $f(x) = x \cos x$ for $p = 1, 2, 3, 5$; $d = 1(1)\,20$, i.e. $k = 1(1)\,20$, $2(2)\,40$, $3(3)\,60$, $5(5)\,100$. Absolute errors of results obtained with char. (6.1.$p.d$) and (10.1.$p.d$) are given in Table 4.

First 30 Fourier cff. b_k of this function were obtained with char. (12.1.1.d), $d = 1(1)\,30$ with the absolute error $6{\cdot}09 \,.\, 10^{-9}$ at b_1 and with the maximal error $5 \,.\, 10^{-10}$ at b_k, $k = 2(1)\,30$. In the computations carried out no symptom of any unstability of the computation process was observed when increasing p and d in (5).

Method (5) is also applicable on the Fourier method of solution of boundary value problems for partial differential equations, for computation of Fourier cff. of more variables [2] and for Fourier transformation.

All calculations were carried out by the Danish computer GIER in GIER-ALGOL III in double precision arithmetics.

Acknowledgement. The autor wishes to thank Petr Přikryl for his careful attention to the manuscript and valuable suggestions.

d					
6	7	8	9	10	11 … 20
$3{\cdot}0 \,.\, 10^{-10}$	$\leqq 1 \cdot 10^{-10}$				
$3{\cdot}0 \,.\, 10^{-10}$	$1{\cdot}3 \,.\, 10^{-10}$	$\leqq 1 \,.\, 10^{-10}$			
$1{\cdot}8 \,.\, 10^{-8}$	$4{\cdot}8 \,.\, 10^{-9}$	$2{\cdot}5 \,.\, 10^{-9}$	$1{\cdot}2 \,.\, 10^{-10}$	$4{\cdot}0 \,.\, 10^{-10}$	$\leqq 1 \,.\, 10^{-10}$
$\leqq 1 \cdot 10^{-10}$					
$2{\cdot}2 \,.\, 10^{-8}$	$7{\cdot}7 \,.\, 10^{-9}$	$3{\cdot}0 \,.\, 10^{-9}$	$1{\cdot}1 \,.\, 10^{-9}$	$5{\cdot}0 \,.\, 10^{-10}$	$\leqq 1 \,.\, 10^{-10}$
$\leqq 1 \cdot 10^{-10}$					
$6{\cdot}7 \,.\, 10^{-9}$	$2{\cdot}3 \,.\, 10^{-9}$	$8{\cdot}0 \,.\, 10^{-10}$	$4{\cdot}0 \,.\, 10^{-10}$	$\leqq 1 \,.\, 10^{-10}$	
$\leqq 1 \cdot 10^{-10}$					

Table 5

k	n	„i" in $A_i^{[k]}$	weight functions [knots of integration]			
			cos $2\pi kx$ [(25)]	sin $2\pi kx$ [(25)]	cos $2\pi kx$ [(26)]	sin $2\pi kx$ [(26)]
1	6	0	0·04313 12924 13651	0·00448 18262 97686	0·04723 72763 22943	0·01194 35821 62640
		1	0·11265 29913 26385	0·08801 45654 36011	0·17687 68708 72712	0·15344 97288 68827
		2	−0·04533 61345 31402	0·19814 73828 40313	−0·22075 23176 16303	0·13473 46250 50110
		3	−0·22089 62984 17269	0·00000 00000 00000	−0·00672 36591 58704	0·00000 00000 00000
	8	0	0·02635 65835 98517	0·00162 90108 36047	0·03424 34928 14781	0·00619 31784 96551
		1	0·08193 78583 15369	0·03549 57238 70503	0·17226 10041 75172	0·09674 69099 81299
		2	0·05634 82226 98804	0·12056 82763 15560	−0·15858 50813 29226	0·11058 09041 65851
		3	−0·07673 05878 74287	0·14417 48061 29425	0·15723 43409 81252	0·10044 45201 05772
		4	−0·17582 41534 76808	0·00000 00000 00000	−0·41030 75132 83959	0·00000 00000 00000
	10	0	0·01769 40302 6166	0·00073 34670 9428	0·02651 51829 7640	0·00371 52437 8349
		1	0·05846 47460 7149	0·01653 91089 0649	0·15896 13977 4860	0·06619 56459 6767
		2	0·06624 35276 7379	0·06497 09772 3458	−0·15019 63030 8664	0·07803 36354 8096
		3	0·01615 31691 2634	0·11912 13262 6792	0·32315 59690 3850	0·11898 20824 9243
		4	−0·08752 21873 8853	0·10606 08675 6093	−0·61408 94957 0357	0·04035 08412 4354
		5	−0·14206 65714 8952	0·00000 00000 0000	0·51130 64980 5343	0·00000 00000 0000
	12	0	0·01268 96862 80	0·00037 67837 78	0·02146 02070 97	0·00244 43222 33
		1	0·04303 42955 89	0·00864 62450 31	0·14611 51138 36	0·04823 10425 19
		2	0·05755 28716 87	0·03626 25580 23	−0·16167 75997 82	0·05353 93751 66
		3	0·04493 09636 14	0·07880 53114 04	0·48143 85341 82	0·11856 87853 38
		4	−0·01233 43814 47	0·10634 55690 42	−0·93745 75327 92	0·02741 62678 47
		5	−0·08523 43039 91	0·08012 09108 33	1·22584 22297 36	0·07441 21201 75
		6	−0·12127 82634 64	0·00000 00000 00	−1·55144 19045 55	0·00000 00000 00
		0	0·00954 11424 3	0·00021 28388 1	0·01792 47369	0·00171 43854
		1	0·03278 23169 5	0·00493 29381 2	0·13512 43667	0·03683 02523

	14	2	0·04724 03741 4	0·02140 97880 4	−0·18043 01255	0·03627 81472
		3	0·04889 49180 7	0·05059 14589 9	0·64189 37468	0·11573 17836
		4	0·02271 19512 5	0·08152 67390 4	−1·40519 08088	−0·00874 09441
		5	−0·02731 13493 7	0·09158 41622 7	2·39647 80038	0·14855 72388
		6	−0·08164 59344 2	0·06225 10274 8	−3·47679 28608	−0·03410 71264
		7	−0·10442 68381 2	0·00000 00000 0	3·74198 58819	0·00000 00000
2	6	0	0·04846 58055 81301	0·00235 43571 52061	0·04197 41205 11685	0·02395 41811 83135
		1	0·01191 06552 98096	0·15346 17842 98659	0·01914 50146 25500	0·16806 86970 95223
		2	−0·14882 04608 06572	−0·09841 14116 93128	−0·22637 44799 11910	−0·16926 75231 02010
		3	0·17688 79998 54349	0·00000 00000 00000	0·33051 06895 49448	0·00000 00000 00000
	8	0	0·02558 67697 04821	0·00380 57339 25138	0·03222 00176 31199	0·01231 76559 20014
		1	0·06260 41132 82519	0·06325 67774 51772	0·08715 49518 21864	0·14407 82791 90190
		2	−0·08897 10990 86319	0·10662 00782 28196	−0·28460 02153 25964	−0·03272 10982 34229
		3	−0·08272 77491 58309	−0·13884 11972 79665	0·24377 07473 77238	−0·09775 33785 98379
		4	0·16701 59305 14574	0·00000 00000 00000	−0·15709 10030 08676	0·00000 00000 00000
	10	0	0·01773 43291 6454	0·00142 98429 6130	0·02553 32976 2056	0·00737 94957 830
		1	0·05133 83403 8409	0·03184 62216 8423	0·11038 00031 2517	0·11111 92764 826
		2	0·00262 17859 0063	0·09342 23199 0495	−0·26132 97719 3933	0·02213 72454 506
		3	−0·11579 75940 7090	0·03041 09459 8541	0·26849 97971 5282	−0·00774 73441 305
		4	−0·02739 50986 5231	−0·13435 23711 8378	−0·49318 63339 0960	−0·13440 42752 069
		5	0·14299 64745 4788	0·00000 00000 0000	0·70020 60159 0077	0·00000 00000 000
	12	0	0·01268 11347 81	0·00075 41089 97	0·02091 44910 96	0·00485 67943 3
		1	0·04040 20295 05	0·01692 88825 18	0·11625 69346 70	0·08595 03123 0
		2	0·02963 41147 31	0·06161 00066 26	−0·24358 94169 67	0·03326 58360 3
		3	−0·04635 65437 24	0·07725 54149 19	0·39156 12337 29	0·07331 42713 9
		4	−0·10489 30120 83	−0·02349 18408 47	−0·92263 88954 14	−0·16509 37045 1
		5	0·00835 29761 65	−0·11706 95369 91	1·31648 07560 92	−0·00424 62475 6
		6	0·12035 86012 48	0·00000 00000 00	−1·35797 02064 13	0·00000 00000 0

Table 5

k	n	„i“ in $A_i^{[k]}$	weight functions [knots of integration]			
			cos $2\pi kx$ [(25)]	sin $2\pi kx$ [(25)]	cos $2\pi kx$ [(26)]	sin $2\pi kx$ [(26)]
	14	0	0·00953 88951 0	0·00042 53214 8	0·01759 26864	0·00340 89218
		1	0·03164 19012 4	0·00974 95315 6	0·11558 38281	0·06791 12948
		2	0·03433 28249 3	0·03908 47416 4	−0·23908 85199	0·02938 80887
		3	−0·00259 54861 9	0·06998 31034 0	0·55371 24619	0·12552 18254
		4	−0·07251 00754 8	0·04441 12219 9	−1·43302 45614	−0·19145 28247
		5	−0·07939 03365 2	−0·05291 56611 6	2·38622 47043	0·11998 24660
		6	0·02647 77951 1	−0·09886 58150 6	−3·31173 91059	−0·18892 21243
		7	0·10500 89636 2	0·00000 00000 0	3·82147 70130	0·00000 00000
3	6	0	0·04554 24277 90038	0·03359 10154 95784	0·03153 94791 11069	0·03452 83204 98075
		1	−0·05460 61883 81815	0·04390 06958 63156	−0·05596 45134 27790	0·05796 75415 80292
		2	0·00716 65777 53488	−0·03231 30732 31687	0·04132 39253 53871	−0·06036 51015 62915
		3	0·00379 43656 76577	0·00000 00000 00000	−0·03379 77820 74302	0·00000 00000 00000
	10	0	0·01650 61274 8892	0·00358 26783 6603	0·02382 05296 6807	0·01088 91113 8640
		1	0·04426 02057 3382	0·04035 10625 2497	0·05025 19845 8487	0·12243 13383 4249
		2	−0·07058 82957 6127	0·07593 59195 3119	−0·25674 04601 1986	−0·12056 68708 4256
		3	−0·03257 93849 0922	−0·11828 02696 3289	0·41440 86666 3296	0·02654 84964 5669
		4	0·10461 44740 2308	0·06871 55979 7101	−0·46506 77881 7317	0·02969 09477 9879
		5	−0·12442 62531 5068	0·00000 00000 0000	0·46665 41348 1425	0·00000 00000 0000
	14	0	0·00951 44625 0	0·00065 19380 5	0·01703 06181	0·00505 79960
		1	0·02983 39185 5	0·01429 31563 4	0·08721 81535	0·08874 78683
		2	0·01523 24398 1	0·05000 83199 2	−0·28349 36094	−0·03221 44047
		3	−0·05207 35126 9	0·04613 10129 2	0·55683 74413	0·09178 09846
		4	−0·06291 63281 0	−0·05723 50085 6	−1·28178 71822	−0·30564 14632

		5	0·07422 70722 7	—0·06111 15790 3	2·38751 36999	0·32565 62343
		6	0·03825 60125 9	0·09483 54319 4	—3·25874 52490	—0·13697 16421
		7	—0·10414 81298 8	0·00000 00000 0	3·55085 22553	0·00000 00000
5	6	0	0·02172 94642 20928	0·03231 45687 78941	0·01418 19938 71935	0·02760 40821 20339
		1	—0·03324 90853 45545	0·00317 80674 59985	—0·03474 86308 74228	0·01342 44602 73374
		2	0·01977 73197 05532	—0·00497 39122 56641	0·04783 22047 60735	—0·01416 82010 52628
		3	—0·01651 53971 61828	0·00000 00000 00000	—0·05453 11355 16885	0·00000 00000 00000
	10	0	0·02248 83140 3657	0·00366 98190 0215	0·01806 07317 9591	0·01620 25515 8597
		1	—0·00765 95446 0985	0·05547 14490 2203	—0·02539 35373 5280	0·06859 30213 0617
		2	—0·03509 32782 6530	—0·03799 86597 9770	—0·01052 27255 9843	—0·14488 63232 6673
		3	0·03771 72688 2943	0·01421 84591 3807	0·07143 26951 3582	0·19005 80534 3666
		4	—0·03304 57474 2773	—0·00436 47429 1043	—0·13492 67637 6806	—0·14168 70371 3579
		5	0·03118 59748 7378	0·00000 00000 0000	0·16269 91995 7512	0·00000 00000 0000
	14	0	0·00870 59963 0	0·00358 21363 0	0·01518 42133	0·00804 70667
		1	0·02676 24615 2	0·01464 69179 9	0·02555 81803	0·09357 07789
		2	—0·03280 32936 3	0·05857 14407 2	—0·21064 26385	—0·19009 35170
		3	—0·03279 63061 3	—0·07406 26598 8	0·57760 44422	0·30588 41855
		4	0·06327 48747 7	0·04061 05658 2	—1·17733 91010	—0·42804 87269
		5	—0·06230 91438 7	—0·01478 66927 7	1·95019 04243	0·46789 66502
		6	0·05545 01907 3	0·00383 50598 7	—2·64584 63258	—0·31965 34973
		7	—0·05256 95593 6	0·00000 00000 0	2·93058 16106	0·00000 00000

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Súhrn

NUMERICKÁ INTEGRÁCIA S VÁHOVOU FUNKCIOU $\cos kx$, $\sin kx$ NA $\left\langle 0, \frac{2\pi}{t} \right\rangle$, $t = 1, 2, \ldots$

JOZEF MIKLOŠKO

Článok opisuje numerickú metódu výpočtu integrálov s váhovou funkciou $\cos kx$, $\sin kx$, (k celé), skúma jej konvergenciu a odhad zvyšku. Sú tabelované niektoré váhové koeficienty týchto formúl a ich použitie je demonštrované numerickými experimentami.

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