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Ivan Hlaváček

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CONVERGENCE OF AN EQUILIBRIUM FINITE ELEMENT MODEL FOR PLANE ELASTOSTATICS

IVAN HLAVÁČEK

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INTRODUCTION

In a recent study [1], an analysis of a dual variational procedure for a scalar second order elliptic equation has been presented. Using equilibrium finite elements of Veubeke and Hogge [2] with linear functions on triangles, we have proven some a priori error estimates, provided the solution was sufficiently smooth.

It is the aim of the present paper to extend the main idea of the article [1] to boundary value problems of plane elastostatics. A weak version of the Castigliano principle is established in Section 1 and an approximate variational problem defined, using equilibrium stress fields.

There exists a rich variety of equilibrium stress models, based on the triangular or quadrilateral elements (see [3], [5], [6] a.o.). To the author's knowledge, the only theoretical convergence analysis concerning equilibrium finite elements has been given recently by Johnson and Mercier in [8]. They apply a mixed variational formulation of Reissner's type.

In Section 2 we choose the triangular self-equilibriated "building block" element of Watwood and Hartz [3] and investigate its approximating properties. By means of a projection mapping, a quasi-optimal a priori error estimate $O(h^2)$ is obtained in L_2-norm, provided the solution is smooth enough. On the basis of some density theorems, presented in Section 3, the convergence of the proposed finite element procedure is justified even in the general case, i.e., without any regularity assumption.

For the algorithm and the computational point of view, we refer the reader to the paper [3].

1. PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY

In the present section we introduce a weak form of the well-known Castigliano-Menabrea principle in plane elastostatics. Then a corresponding approximate problem will be defined, which enables us to employ finite element procedures.

Let us consider a bounded polygonal domain $\Omega \subset R^2$, with Cartesian coordinate system $\mathbf{x} = (x_1, x_2)$. Let the stress-strain relations be

$$e_{ij} = b_{ijkl}\sigma_{kl}\,, \quad i, j = 1, 2\,,$$

where e_{ij} and σ_{ij} are components of the strain tensor and stress tensor, respectively, b_{ijkl} are bounded measurable functions in Ω and a repeated index means summation over the range 1, 2. Assume that

$$b_{ijkl} = b_{klij} = b_{jikl}$$

and a constant $c_0 > 0$ exists such that

$$b_{ijkl}(\mathbf{x})\, s_{ij}s_{kl} \geqq c_0 s_{ij}s_{ij} \quad \forall s_{ij} = s_{ji}$$

holds almost everywhere in Ω.

Let the boundary $\partial\Omega \equiv \Gamma$ consist of two mutually disjoint parts,

$$\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_\sigma\,, \quad \Gamma_u \cap \Gamma_\sigma = \emptyset\,,$$

where Γ_u and Γ_σ are either open in Γ or empty. On Γ_u and Γ_σ the displacements and the surface tractions will be given, respectively.

Henceforth $L_2(M)$ denotes the space of square-integrable functions in the set M, $W^{j,2}(\Omega)$ the Sobolev space of functions, the derivatives of which (in the sense of distributions) exist up to the order j and belong to $L_2(\Omega)$. Let body force vector $F_i \in L_2(\Omega)$, a surface load vector $T_i \in L_2(\Gamma_\sigma)$ and a displacement vector $u_{0i} \in W^{1,2}(\Omega)$ be given. We define the space of symmetric stress fields

$$H = \{\sigma \in [L_2(\Omega)]^4 \mid \sigma_{ij} = \sigma_{ji}\}$$

and the set of statically admissible stress fields

$$\Lambda_{F,T} = \left\{\sigma \in H \,\middle|\, \int_\Omega \sigma_{ij} e_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega F_i v_i \,\mathrm{d}\mathbf{x} + \int_{\Gamma\sigma} T_i v_i \,\mathrm{d}s \quad \forall \mathbf{v} \in V\right\},$$

where

$$V = \{\mathbf{v} \in [W^{1,2}(\Omega)]^2 \mid \mathbf{v} = 0 \text{ on } \Gamma_u\}$$

is the space of virtual displacements and

$$e_{ij}(\mathbf{v}) = \tfrac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)\,.$$

Theorem 1.1. (Principle of minimum complementary energy.) *Let there exist a weak solution* $\mathbf{u}$ *of the mixed boundary value problem under consideration, i.e.,* $\mathbf{u} \in [W^{1,2}(\Omega)]^2$ *such that* $\mathbf{u} - \mathbf{u}_0 \in V$ *and*

$$\int_\Omega c_{ijkl}\, e_{ij}(\mathbf{u})\, e_{kl}(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega F_i v_i \,\mathrm{d}\mathbf{x} + \int_{\Gamma\sigma} T_i v_i \,\mathrm{d}s \quad \forall \mathbf{v} \in V,$$

(*where* $[c_{ijkl}]$ *is the matrix inverse to* $[b_{ijkl}]$).

Then the functional (complementary energy)

$$\mathscr{S}(\sigma) = \frac{1}{2} \int_\Omega b_{ijkl} \sigma_{ij} \sigma_{kl} \, \mathrm{d}\boldsymbol{x} - \int_\Omega \sigma_{ij} \, e_{ij}(\boldsymbol{u}_0) \, \mathrm{d}\boldsymbol{x}$$

attains its minimum over the set $\Lambda_{F,T}$, if and only if

$$\sigma_{ij} = \sigma_{ij}(\boldsymbol{u}) = c_{ijkl} \, e_{kl}(\boldsymbol{u}) \, .$$

For the proof – see e.g. [4] or [1], where an analogous theorem is proven in detail.

Next we transform the variational problem by shifting the affine hyperplane $\Lambda_{F,T} \subset H$ into a linear space $\Lambda_{0,0} \subset H$. To this end, let us have a fixed stress field $\bar{\sigma} \in \Lambda_{F,T}$. Then

$$\Lambda_{F,T} = \bar{\sigma} + \Lambda_{0,0} \, , \quad \Lambda_{0,0} = \left\{ \tau \in H \,\middle|\, \int_\Omega \tau_{ij} e_{ij}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} = 0 \ \ \forall \boldsymbol{v} \in V \right\} ,$$

i.e., every $\sigma \in \Lambda_{F,T}$ can be written in the form $\sigma = \bar{\sigma} + \tau$, where $\tau \in \Lambda_{0,0}$. Consequently,

$$\mathscr{S}(\sigma) = \frac{1}{2} \int_\Omega b_{ijkl} \tau_{ij} \tau_{kl} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \tau_{ij} (b_{ijkl} \bar{\sigma}_{kl} - e_{ij}(\boldsymbol{u}_0)) \, \mathrm{d}\boldsymbol{x} + N(\bar{\sigma}, \boldsymbol{u}_0) \, ,$$

where $N(\bar{\sigma}, \boldsymbol{u}_0)$ does not depend on τ. Let us introduce the scalar product

$$(\tau', \tau'')_H = \int_\Omega b_{ijkl} \tau'_{ij} \tau''_{kl} \, \mathrm{d}\boldsymbol{x}$$

and the functional

$$\Phi(\tau) = \tfrac{1}{2} (\tau, \tau)_H - f(\tau) \, ,$$

where

$$f(\tau) = \int_\Omega \tau_{ij} (e_{ij}(u_0) - b_{ijkl} \bar{\sigma}_{kl}) \, \mathrm{d}\boldsymbol{x} \, .$$

Then we may replace the minimum problem of Theorem 1.1 by an equivalent problem: to find $\tau^0 \in \Lambda_{0,0}$ such that

$$\Phi(\tau^0) \leqq \Phi(\tau) \quad \forall \tau \in \Lambda_{0,0} \, . \tag{1.1}$$

Let $h \in (0, 1\rangle$ be a parameter and let $\{S_h\}$ be a family of finite-dimensional subspaces of $\Lambda_{0,0}$. We define the following approximate problem:

to find $\tau_h^0 \in S_h$ such that

$$\Phi(\tau_h^0) \leqq \Phi(\tau) \quad \forall \tau \in S_h \, . \tag{1.2}$$

Theorem 1.2. *For any $h \in (0, 1\rangle$ there exists precisely one solution of the problem (1.2). It holds*

$$\|\tau^0 - \tau_h^0\|_H \leqq \inf_{\tau \in S_h} \|\tau^0 - \tau\|_H \, . \tag{1.3}$$

Proof. The existence and uniqueness of τ_h^0 is obvious. Moreover, from the conditions

$$(\tau^0, \tau)_H = f(\tau) \quad \forall \tau \in \Lambda_{0,0} ,$$
$$(\tau_h^0, \tau)_H = f(\tau) \quad \forall \tau \in S_h$$

we obtain

$$(\tau^0 - \tau_h^0, \tau)_H = 0 \quad \forall \tau \in S_h .$$

Consequently, τ_h^0 is the orthogonal projection of τ^0 onto the subspace S_h in the Hilbert space H and the assertion (1.3) follows.

2. AN EQUILIBRIUM STRESS FIELD MODEL

The crucial point of the dual variational approach is a proper choice of the finite element with a self-equilibriated stress field, i.e., the construction of subspaces $S_h \subset \subset \Lambda_{0,0}$. Several studies have been accomplished (see e.g. [3], [5]), where criteria for suitable finite elements have been proposed.

In the present paper we restrict ourselves to one of the simplest elements, namely to the triangular "building block" element consisting of three subtriangles (see Fig. 1), with piecewise linear stress field, which was proposed by Watwood and Hartz in [3]. Let us emphasize that the single triangle with linear stress components cannot be employed, in contrary to the problems for scalar second order elliptic equations (cf. [1], [2]). In fact, the single triangular element violates an important criterion (see [3]), as follows.

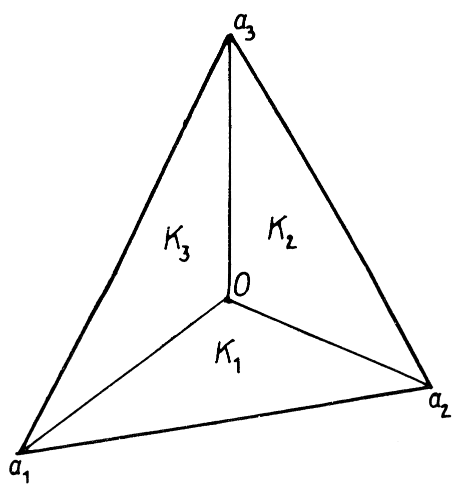

Fig. 1.

Let us define a set of self-equilibriated linear stress fields over the triangle K:

$$\mathscr{M}(K) = \{\tau \mid \tau \in P_1^s(K),\ \partial\tau_{kj}/\partial x_j = 0,\ k = 1, 2\} , \tag{2.1}$$

where

$$P_1^s(K) = \{\tau \in [P_1(K)]^4 |\ \tau_{ij} = \tau_{ji}\}$$

and $P_1(K)$ is the set of linear functions defined on K.

It is easy to derive that $\tau \in \mathcal{M}(K)$ if and only if

$$\begin{aligned} \tau_{11} &= \beta_1 + \beta_2 x_1 + \beta_3 x_2\,, \\ \tau_{22} &= \beta_4 + \beta_5 x_1 + \beta_6 x_2\,, \\ \tau_{12} &= \tau_{21} = \beta_7 - \beta_6 x_1 - \beta_2 x_2\,, \end{aligned} \tag{2.2}$$

where β_m, $m = 1, \ldots, 7$, are arbitrary real coefficients. Thus $\mathcal{M}(K)$ is a seven-dimensional linear set.

Obviously, the stress vectors $t_k(\tau) = \tau_{kj} n_j$ for $\tau \in \mathcal{M}(K)$ are linear on every side of the triangle K. They satisfy three overall equilibrium conditions

$$\int_{\partial K} t_k(\tau)\, \mathrm{d}s = 0\,, \quad k = 1, 2\,, \tag{2.3}$$

$$\int_{\partial K} [x_1\, t_2(\tau) - x_2\, t_1(\tau)]\, \mathrm{d}s = 0\,, \tag{2.4}$$

as a consequence of the equilibrium equations and of the symmetry of the tensor τ. The stress vectors, however, are constrained by two more (redundant) conditions, which follow from the symmetry and continuity of the stress field at the vertices (cf. the Lemmas 2.1 and 2.3 in what follows). Hence the element has not enough independent stress modes on each side to balance an arbitrary self-equilibriated loading which is linear on every side, thus violating a criterion, established in [3]. (The same requirement is necessary for the existence of a proper projection mapping, as we shall see later – cf. Theorem 2.2).

The above defect can be overcome by bisecting the vertex with a "cut" across which the continuity of the stress vector only is maintained (instead of the continuity of the stress tensor). Thus the triangular "building block" is generated. It is worth of remark that this element is dual of the triangular element of Clough and Tocher, if the duality is considered in the sense of the so called "slab analogy" (see e.g. [5], [6]), using the Airy stress function.

Let K be a triangle with vertices a_1, a_2, a_3 and set $a_4 \equiv a_1$. We shall use the following notation:

$$C^j(\bar{K}) = \{\tau \in [C^{(j)}(\bar{K})]^4 |\ \tau_{12} = \tau_{21}\}\,, \quad j = 0, 1, 2\,,$$

where $C^{(j)}(\bar{K})$ is the space of functions, the derivatives of which up to the order j are continuous in K and have continnous extensions to $\bar{K}$. Further

$$W^j(K) = \{\tau \in [W^{j,2}(K)]^4 |\ \tau_{12} = \tau_{21}\}\,.$$

We introduce the norms

$$\|\tau\|_{C^j(\bar{K})} = \max_{r,s=1,2} \max_{\substack{\mathbf{x}\in\bar{K} \\ |\alpha|\leq j}} \left|D^\alpha \tau_{rs}(\mathbf{x})\right| ,$$

$$\|\tau\|_{j,K} = \Big(\sum_{r,s=1}^{2} \|\tau_{rs}\|^2_{W^{j,2}(K)}\Big)^{1/2} .$$

Moreover, on every side $a_i a_{i+1}$ we introduce the basic linear functions $\lambda_k^i \in P_1(a_i a_{i+1})$, $k = 1, 2$, such that

$$\lambda_1^i(a_i) = 1 , \quad \lambda_1^i(a_{i+1}) = 0 ,$$

$$\lambda_2^i(a_i) = 0 , \quad \lambda_2^i(a_{i+1}) = 1 .$$

Let n be the outward unit normal to the boundary ∂K. Thus $\mathbf{n} = \mathbf{n}(x_1, x_2) = \mathbf{n}^i \in R^2$ is constant along the side $a_i a_{i+1}$, $i = 1, 2, 3$. Let l_i denote the length of $a_i a_{i+1}$, $h = \max l_i$ for $i = 1, 2, 3$. Denote $\mathbf{a} \cdot \mathbf{b}$ the scalar product $a_i b_i$ of any two vectors $\mathbf{a}, \mathbf{b} \in R^2$.

For the stress field $\tau \in W^1(K)$ we define the stress vector on $a_i a_{i+1}$

$$t_k^i(\tau) = \tau_{kj} n_j^i , \quad k = 1, 2 . \tag{2.5}$$

Lemma 2.1. *Let $\tau \in C^0(\bar{K})$, (i.e. continuous on the closed triangle $\bar{K}$). Then for any $i = 1, 2, 3$*

$$\tau_{12}(a_i) = \tau_{21}(a_i) \tag{2.6}$$

holds if and only if

$$\mathbf{t}^i(\tau)(a_i) \cdot \mathbf{n}^{i-1} = \mathbf{t}^{i-1}(\tau)(a_i) \cdot \mathbf{n}^i \tag{2.7}$$

(where we set $i - 1 = 3$ for $i = 1$).

Proof. Let $\tau_{12} = \tau_{21}$ at the vertex a_i. By virtue of the definition (2.5),

$$\mathbf{t}^i \cdot \mathbf{n}^{i-1} = \tau_{kj} n_j^i n_k^{i-1} = \tau_{jk} n_k^i n_j^{i-1} = \tau_{kj} n_k^i n_j^{i-1} = \mathbf{t}^{i-1} \cdot \mathbf{n}^i .$$

On the other hand, let (2.7) hold at the vertex a_i. Then

$$0 = \mathbf{t}^i \cdot \mathbf{n}^{i-1} - \mathbf{t}^{i-1} \cdot \mathbf{n}^i = \tau_{kj} n_j^i n_k^{i-1} - \tau_{kj} n_j^{i-1} n_k^i = (\tau_{12} - \tau_{21}) D_{i-1,i} ,$$

where

$$D_{i-1,i} = \det \begin{vmatrix} n_1^{i-1}, & n_2^{i-1} \\ n_1^i, & n_2^i \end{vmatrix} = \sin \alpha_i \neq 0 .$$

Hence (2.6) follows.

Lemma 2.2. *Let twelve "external" parameters $T_k^{i,i}$, $T_k^{i,i+1}$, $(i = 1, 2, 3; i + 1 = 1$ for $i = 3$; $k = 1, 2)$ be given, which satisfy the following three conditions*

$$T_k^{i,i} n_k^{i-1} - T_k^{i-1,i} n_k^i = 0 \quad \text{for} \quad i = 1, 2, 3 . \tag{2.8}$$

Then there exists precisely one tensor $\tau \in P_1^S(K)$ such that

$$T_k^{i,i} = t_k^i(\tau)(a_i), \quad T_k^{i,i+1} = t_k^i(\tau)(a_{i+1}), \tag{2.9}$$
$$i = 1, 2, 3, \quad k = 1, 2.$$

Moreover, it holds

$$\|\tau\|_{C^0(K)} \leqq \frac{6\sqrt{2}}{\sin\alpha} \max_{i,k} \{|T_k^{i,i}|, |T_k^{i,i+1}|\}, \tag{2.10}$$

where α is the minimal angle of the triangle K.

Proof. Using (2.5), we write the equations (2.9) for a vertex a_i:

$$\tau_{kj}(a_i)\, n_j^i = T_k^{i,i},$$
$$\tau_{kj}(a_i)\, n_j^{i-1} = T_k^{i-1,i},$$

$(k = 1, 2)$. Inserting $\tau_{12} = \tau_{21}$ (and omitting the argument a_i), we obtain the system

$$\begin{bmatrix} n_1^i, & 0, & n_2^i \\ 0, & n_2^i, & n_1^i \\ n_1^{i-1}, & 0, & n_2^{i-1} \\ 0, & n_2^{i-1}, & n_1^{i-1} \end{bmatrix} \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} T_1^{i,i} \\ T_2^{i,i} \\ T_1^{i-1,i} \\ T_2^{i-1,i} \end{bmatrix} \tag{2.11}$$

Denote

$$|n_k^i| = \max\{|n_1^i|, |n_2^i|\}.$$

As $2\,|n_k^i|^2 \geqq 1$, we have $|n_k^i| \geqq \sqrt{2}/2$.

1°. Suppose $n_k^i = n_1^i$. From (2.8) it follows that the third equation in (2.11) can be omitted. For the corresponding determinant of the remaining system we obtain

$$\left| n_1^i \cdot \begin{vmatrix} n_2^i, & n_1^i \\ n_2^{i-1}, & n_1^{i-1} \end{vmatrix} \right| \geqq \tfrac{1}{2}\sqrt{2}\sin\alpha_i, \tag{2.12}$$

where α_i is the angle at the vertex a_i.

2°. Suppose $n_k^i = n_2^i$. Then (2.8) implies that the fourth equation in (2.11) can be omitted. For the determinant of the remaining system it holds

$$\left| n_2^i \cdot \begin{vmatrix} n_1^i, & n_2^i \\ n_1^{i-1}, & n_2^{i-1} \end{vmatrix} \right| \geqq \tfrac{1}{2}\sqrt{2}\sin\alpha_i. \tag{2.13}$$

From (2.12), (2.13) and (2.11) we conclude that there exists a unique array $\{\tau_{11}(a_i), \tau_{22}(a_i), \tau_{12}(a_i)\}$, satisfying (2.9) and for any $r, s = 1, 2$ we obtain

$$|\tau_{rs}(a_i)| \leqq \frac{6\sqrt{2}}{\sin\alpha_i} \max_{k=1,2} \{|T_k^{i,i}|, |T_k^{i-1,i}|\}.$$

Since every component $\tau_{rs} \in P_1(K)$ is uniquely determined by its values at the vertices, and

$$\max_{x \in K} |\tau_{rs}(x)| \leqq \max_{i=1,2,3} |\tau_{rs}(a_i)|, \quad r, s = 1, 2,$$

the assertion and the estimate of the lemma follow.

Lemma 2.3. *The stress field τ belongs to $\mathcal{M}(K)$ (see (2.1)), if and only if the following conditions hold simultaneously*:

$$\tau_{rs} \in P_1(K), \quad r, s = 1, 2,$$

$$\mathbf{t}^i(\tau)(a_i) \,.\, \boldsymbol{n}^{i-1} = \mathbf{t}^{i-1}(\tau)(a_i) \,.\, \boldsymbol{n}^i, \quad i = 1, 2, 3, \tag{2.14}$$

$$\int_{\partial K} t_k(\tau)\, \mathrm{d}s = 0, \quad k = 1, 2. \tag{2.15}$$

Proof. Let $\tau \in \mathcal{M}(K)$. Then

$$0 = \int_K \partial\tau_{kj}/\partial x_j \,\mathrm{d}\mathbf{x} = \int_{\partial K} \tau_{kj} n_j \,\mathrm{d}s = \int_{\partial K} t_k(\tau)\,\mathrm{d}s.$$

From Lemma 2.1 the conditions (2.14) follow.

Let $\tau \in [P_1(K)]^4$. Using Lemma 2.1 and (2.14), we conclude that $\tau_{21}(a_i) = \tau_{12}(a_i)$, $i = 1, 2, 3$. Thus $\tau_{12} = \tau_{21}$ on K and we may write

$$\begin{aligned} \tau_{11} &= \beta_1 + \beta_2 x_1 + \beta_3 x_2, \\ \tau_{22} &= \beta_4 + \beta_5 x_1 + \beta_6 x_2, \\ \tau_{12} = \tau_{21} &= \beta_7 + \beta_8 x_1 + \beta_9 x_2. \end{aligned} \tag{2.16}$$

From (2.15) we obtain

$$0 = \int_{\partial K} t_1(\tau)\,\mathrm{d}s = \int_K \partial\tau_{1j}/\partial x_j \,\mathrm{d}\mathbf{x} = \int_K (\beta_2 + \beta_9)\,\mathrm{d}\mathbf{x},$$

$$0 = \int_{\partial K} t_2(\tau)\,\mathrm{d}s = \int_K \partial\tau_{2j}/\partial x_j \,\mathrm{d}\mathbf{x} = \int_K (\beta_8 + \beta_6)\,\mathrm{d}\mathbf{x}.$$

Consequently, $\beta_9 = -\beta_2$ and $\beta_8 = -\beta_6$ can be inserted into (2.16), thus obtaining $\tau \in \mathcal{M}(K)$ – cf. (2.2).

Let us divide the triangle K into three subtriangles K_i, connecting the center of gravity 0 with the vertices (Fig. 1). Consider the set $\mathcal{N}(K)$ of self-equilibriated, piecewise linear stress fields in every K_i, i.e. denote

$$\begin{aligned} \mathcal{N}(K) = \{\tau = (\tau^1, \tau^2, \tau^3)\, |\tau|_{K_i} \equiv \tau^i \in M(K_i), \quad i = 1, 2, 3, \\ \mathbf{t}(\tau^i) + \mathbf{t}(\tau^{i-1}) = 0 \quad \forall Oa_i, \quad i = 1, 2, 3\}. \end{aligned} \tag{2.17}$$

The last condition in the definition of $\mathcal{N}(K)$ means that the stress vectors are continuous across any side Oa_i.

Lemma 2.4. *Let* $\tau \in \mathcal{N}(K)$. *Define twelve "external stress vector parameters" by the relations*

$$T_k^{i,i} = t_k^i(\tau^i)(a_i), \quad T_k^{i,i+1} = t_k^i(\tau^i)(a_{i+1}), \quad i = 1, 2, 3, \quad k = 1, 2. \tag{2.18}$$

Then the following three conditions of overall equilibrium hold:

$$\sum_{i=1}^{3} l_i(T_k^{i,i} + T_k^{i,i+1}) = 0, \quad k = 1, 2 \tag{2.19}$$

(resultant forces vanish) and

$$\sum_{i=1}^{3} \int_{a_i a_{i+1}} [x_1(T_2^{i,i}\lambda_1^i(s) + T_2^{i,i+1}\lambda_2^i(s)) - - x_2(T_1^{i,i}\lambda_1^i(s) + T_1^{i,i+1}\lambda_2^i(s))] \,\mathrm{d}s = 0 \tag{2.20}$$

(resulting moment vanishes).

Proof. Using the definition of $\mathcal{N}(K)$ and $\mathcal{M}(K_i)$, we may write

$$0 = \sum_{i=1}^{3} \int_{K_i} \partial\tau_{kj}/\partial x_j \,\mathrm{d}\mathbf{x} = \sum_{i=1}^{3} \int_{\partial K_i} \tau_{kj} n_j \,\mathrm{d}s =$$

$$= \sum_{i=1}^{3} \int_{a_i a_{i+1}} t_k^i(\tau^i)\,\mathrm{d}s + \sum_{i=1}^{3} \int_{0 a_i} (t_k(\tau^i) + t_k(\tau^{i-1}))\,\mathrm{d}s = \sum_{i=1}^{3} \int_{a_i a_{i+1}} t_k^i(\tau^i)\,\mathrm{d}s, \quad k = 1, 2.$$

Inserting

$$t_k^i(\tau^i) = T_k^{i,i}\lambda_1^i + T_k^{i,i+1}\lambda_2^i,$$

we obtain (2.19). To derive (2.20), we write (using ε_{ijk} for the Levi-Cività tensor)

$$0 = \sum_{i=1}^{3} \int_{K_i} (\tau_{12}^i - \tau_{21}^i)\,\mathrm{d}\mathbf{x} = \sum_{i=1}^{3} \int_{K_i} \varepsilon_{3jk}(\delta_{km}\tau_{jm}^i + x_k\,\partial\tau_{jm}^i/\partial x_m)\,\mathrm{d}\mathbf{x} =$$

$$= \sum_{i=1}^{3} \int_{K_i} \varepsilon_{3jk} \frac{\partial}{\partial x_m}(\tau_{jm}^i x_k)\,\mathrm{d}\mathbf{x} = \sum_{i=1}^{3} \int_{\partial K_i} \varepsilon_{3jk}\tau_{jm}^i x_k n_m \,\mathrm{d}s =$$

$$= \sum_{i=1}^{3} \int_{a_i a_{i+1}} \varepsilon_{3jk}\, t_j^i(\tau^i)\, x_k \,\mathrm{d}s + \sum_{i=1}^{3} \int_{0 a_i} \varepsilon_{3jk}[t_j(\tau^i) + t_j(\tau^{i-1})] x_k \,\mathrm{d}s.$$

The last term vanishes because of (2.17) and (2.20) follows easily.

Theorem 2.1. *Let twelve external parameters* $T_k^{i,i}$, $T_k^{i,i+1}$ *be given*, ($i = 1, 2, 3$; $k = 1, 2$), *which satisfy* (2.19) *and* (2.20).

Then there exists precisely one stress field $\tau \in \mathcal{N}(K)$ such that (2.18) *holds. Moreover, there is an estimate*

$$\max_{i=1,2,3} \|\tau^i\|_{C^0(K_i)} \leqq c(\alpha) \,.\, \max_{j,k} \{|T_k^{j,j}|, |T_k^{j,j+1}|\} , \tag{2.21}$$

where $c(\alpha) > 0$ *depends on the minimal angle* α *of* K *only.*

Proof. Denote $\boldsymbol{n}^4, \boldsymbol{n}^5, \boldsymbol{n}^6$ the unit normal vectors to the sides Oa_2, Oa_3, Oa_1. Introduce twelve auxiliary parameters S_k^{ii}, S_k^{i0} on Oa_i, $i = 1, 2, 3$, $k = 1, 2$, such that

$$S_k^{ii} = t_k(\tau^{i-1})(a_i) = \tau_{kj}^{i-1} n_j^{i+2} \,, \quad S_k^{i0} = t_k(\tau^{i-1})(O) = \tau_{kj}^{i-1}(O)\, n_j^{i+2} \,;$$

let $i + 2 = 6$ for $i = 1$.

Denote the length $|Oa_i| = d_i$, $i = 1, 2, 3$.

The "transversal" conditions of continuity (2.17) on Oa_i can easily be satisfied by changing only the sign of S_k^{ii}, S_k^{i0}. With respect to the conditions (2.14), (2.15), applied to K_1, we set

$$-d_1(S_k^{11} + S_k^{10}) + d_2(S_k^{22} + S_k^{20}) + l_1(T_k^{11} + T_k^{12}) = 0 \,, \quad k = 1, 2 \,, \tag{2.22}$$

$$-S_k^{11} n_k^1 = -T_k^{11} n_k^6 \,, \quad S_k^{22} n_k^1 = T_k^{12} n_k^4 \,, \tag{2.23}$$

$$-S_k^{20} n_k^6 = -S_k^{10} n_k^4 \,. \tag{2.24}$$

A similar set of five equations can be written for the triangle K_2 and K_3, respectively. Thus we obtain a system of 15 equations for 12 parameters S_k^{ii}, S_k^{i0}, $i = 1, 2, 3$, $k = 1, 2$

$$\mathscr{A}\mathbf{S} = \mathscr{F}\mathbf{T} \,,$$

where

$$\mathscr{A} = \begin{bmatrix}
-d_1, & 0, & -d_1, & 0, & d_2, & 0, & d_2, & 0, & 0, & \dots & \dots & 0 \\
0, & -d_1, & 0, & -d_1, & 0, & d_2, & 0, & d_2, & 0, & \dots & \dots & 0 \\
0, & \dots & \dots & 0, & -d_2, & 0, & -d_2, & 0, & d_3, & 0, & d_3, & 0 \\
0, & \dots & & \dots & 0, & -d_2, & 0, & -d_2, & 0, & d_3, & 0, & d_3 \\
d_1, & 0, & d_1, & 0, & \dots & & \dots & 0, & -d_3, & 0, & -d_3, & 0 \\
0, & d_1, & 0, & d_1, & 0, & \dots & & \dots & 0, & -d_3, & 0, & -d_3 \\
n_1^1, & n_2^1, & 0, & \dots & & & & & & & \dots & 0 \\
n_1^3, & n_2^3, & 0, & \dots & & & & & & & \dots & 0 \\
0, & \dots & \dots & 0, & n_1^1, & n_2^1, & 0, & \dots & & & \dots & 0 \\
0, & \dots & \dots & 0, & n_1^2, & n_2^2, & 0, & \dots & & & \dots & 0 \\
0, & \dots & & & & & \dots & 0, & n_1^2, & n_2^2, & 0, & 0 \\
0, & \dots & & & & & \dots & 0, & n_1^3, & n_2^3, & 0, & 0 \\
0, & 0, & n_1^4, & n_2^4, & 0, & 0, & -n_1^6, & -n_2^6, & 0, & \dots & \dots & 0 \\
0, & \dots & & & \dots & 0, & n_1^5, & n_2^5, & 0, & 0, & -n_1^4, & -n_2^4 \\
0, & 0, & n_1^5, & n_2^5, & 0, & \dots & & & \dots & 0, & -n_1^6, & -n_2^6
\end{bmatrix}$$

$$\mathbf{S} = (S_1^{11}, S_2^{11}, S_1^{10}, S_2^{10}, S_1^{22}, S_2^{22}, S_1^{20}, S_2^{20}, S_1^{33}, S_2^{33}, S_1^{30}, S_2^{30})^{\mathsf{T}} \,,$$

$$
\mathscr{F} = \begin{bmatrix}
l_1, & 0, & l_1, & 0, & \ldots & & & & & & \ldots & 0 \\
0, & l_1, & 0, & l_1 & 0, & \ldots & & & & & \ldots & 0 \\
0, & \ldots & \ldots & 0, & l_2, & 0, & l_2, & 0, & \ldots & & \ldots & 0 \\
0, & \ldots & & & \ldots & l_2, & 0, & l_2, & 0, & \ldots & \ldots & 0 \\
0, & \ldots & & & & & \ldots & 0, & l_3, & 0, & l_3, & 0 \\
0, & \ldots & & & & & & \ldots & 0, & l_3, & 0, & l_3 \\
-n_1^6, & -n_2^6, & 0, & \ldots & & & & & & & \ldots & 0 \\
0, & \ldots & & & & & & & \ldots & 0, & -n_1^6, & -n_2^6 \\
0, & 0, & -n_1^4, & -n_2^4, & 0, & \ldots & & & & & \ldots & 0 \\
0, & \ldots & \ldots & 0, & -n_1^4, & -n_2^4, & 0, & \ldots & & & \ldots & 0 \\
0, & \ldots & & & \ldots & 0, & -n_1^5, & -n_2^5, & 0, & \ldots & \ldots & 0 \\
0, & \ldots & & & & & \ldots & 0, & -n_1^5, & -n_2^5, & 0, & 0 \\
0, & \ldots & & & & & & & & & \ldots & 0 \\
0, & \ldots & & & & & & & & & \ldots & 0 \\
0, & \ldots & & & & & & & & & \ldots & 0
\end{bmatrix}
$$

$$
\mathbf{T} = (T_1^{11}, T_2^{11}, T_1^{12}, T_2^{12}, T_1^{22}, T_2^{22}, T_1^{23}, T_2^{23}, T_1^{33}, T_2^{33}, T_1^{31}, T_2^{31})^{\mathsf{T}}
$$

From the three conditions (2.19), (2.20), it follows that we can omit three equations of the system, namely e.g. (2.22) and the last equation for K_3, i.e.

$$
S_k^{10} n_k^5 - S_k^{30} n_k^6 = 0 .
$$

In fact, if the center of gravity O coincides with the origin, we may write

$$
\mathbf{n}^{i+2} = (-x_2(a_i)/d_i, x_1(a_i)/d_i) ,
$$

(where $i + 2 = 6$ for $i = 1$). Multiplying the equation of the type (2.22) for K_i by $-[x_2(a_i) + x_2(a_{i+1})]$ if $k = 1$, and by $[x_1(a_i) + x_1(a_{i+1})]$ if $k = 2$, equations of the type (2.23) by $(-d_i l_i)$ and $(-d_i l_j)$, respectively, and equations of the type (2.24) by $(\pm d_i d_j)$, we find the linear dependence of all the fifteen equations, using the moment equilibrium condition (2.20) for the right hand sides.

Finally, the sum of three equations of the type (2.22) for K_i, $i = 1, 2, 3$, $k = 1$ and $k = 2$, respectively, vanishes by virtue of the force equilibrium conditions (2.19).

To obtain dimensionless coefficients, we divide the remaining equations of the type (2.22) for K_i by l_i. Then the remaining system has the form

$$
\mathscr{B}\mathbf{S} = F(\mathbf{T}) , \tag{2.25}
$$

where

$$
|\det \mathscr{B}| = \sin^2 \alpha_1 \sin \alpha_2 \sin \alpha_3 \, d_1 d_3 l_1 l_2^{-2} l_3^{-1} > 0 . \tag{2.26}
$$

We can find a lower bound for d_j

$$
d_j \geqq \tfrac{2}{3} h \sin^2 \alpha , \quad j = 1, 2, 3 \tag{2.27}
$$

(where $h = \max_i l_i$). In fact, denoting t_j the length of the axis of center of gravity,

$$(2.28) \qquad d_j = \tfrac{2}{3} t_j \geqq \tfrac{2}{3} l_{\min} \sin \alpha \,,$$

$$l_{\min} \geqq l_{\max} \sin \alpha = h \sin \alpha$$

and (2.27) follows.

Inserting the estimates (2.27), (2.28) into (2.26), we obtain

$$(2.29) \qquad |\det \mathscr{B}| \geqq \sin^4 \alpha \, \tfrac{4}{9} h^2 \sin^4 \alpha l_{\min} h^{-3} = \tfrac{4}{9} \sin^9 \alpha \,.$$

Consequently, the system (2.25) has a unique solution $\boldsymbol{S} \in R^{12}$. The entries $\mathscr{B}_{ij}$ of the matrix $\mathscr{B}$ are bounded above, as follows

$$(2.30) \qquad |\mathscr{B}_{ij}| \leqq \sin^{-1} \alpha \,, \quad i, j = 1, 2, \ldots, 12 \,.$$

In fact, $|n_j^i| \leqq 1$ and $d_i / l_j \leqq h \, (h \sin \alpha)^{-1} = \sin^{-1} \alpha$. From (2.29) and (2.30) we obtain for the matrix $\mathscr{B}^{-1}$ inverse to $\mathscr{B}$:

$$|\mathscr{B}_{ij}^{-1}| \leqq \tfrac{9}{4} \, 11! \sin^{-20} \alpha \,, \quad i, j = 1, \ldots, 12 \,.$$

Moreover, $F_j(\boldsymbol{T})$ are linear forms in T_k^{ii}, $T_k^{i,i+1}$, two coefficients of which only are nonzero, being bounded by one. Consequently, we obtain

$$(2.31) \qquad \max_{i,k} \{|S_k^{ii}|, |S_k^{i0}|\} \leqq 24 \,. \tfrac{9}{4}(11!) \sin^{-20} \alpha \max_{i,k} \{|T_k^{ii}|, |T_k^{i,i+1}|\} \,.$$

Now Lemma 1.2 yields the existence of a unique stress field $\tau^i \in P_1^s(K_i)$ such that for any $i = 1, 2, 3$ (2.18) hold and

$$S_k^{ii} = \tau_{kj}^i(a_i) \, n_j^{i+2} \,, \quad S_k^{i0} = \tau_{kj}^i(O) \, n_j^{i+2} \,,$$

$$(i + 2 = 6 \quad \text{for} \quad i = 1) \,,$$

$$S_k^{i+1,i+1} = \tau_{kj}^i(s_{i+1}) \, n_j^{i+3} \,, \quad S_k^{i+1,0} = \tau_{kj}^i(O) \, n_j^{i+3} \,.$$

By virtue of Lemma 2.3 and the system of conditions of the type (2.22), (2.23), (2.24), we conclude that $\tau^i \in \mathscr{M}(K_i)$, $i = 1, 2, 3$, and $\tau = (\tau^1, \tau^2, \tau^3) \in \mathscr{N}(K)$.

Moreover, we deduce on the basis of (2.10) and (2.31)

$$(2.32) \qquad \|\tau^i\|_{C^0(K_i)} \leqq \frac{6\sqrt{2}}{\sin \alpha_0} \max_{k=1,2} \{|S_k^{ii}|, |S_k^{i0}|, |S_k^{i+1,i+1}|, |S_k^{i+1,0}|, |T_k^{ii}|, |T_k^{i,i+1}|\} \leqq$$

$$\leqq \frac{6\sqrt{2}}{\sin \alpha_0} 54(11!) \sin^{-20} \alpha \,. \max_{j,k} \{|T_k^{jj}|, |T_k^{j,j+1}|\} \,, \quad i = 1, 2, 3 \,,$$

where α_0 is the minimal angle of the subtriangles K_1, K_2, K_3.

It is easy to derive the following estimate

$$(2.33) \qquad \sin \alpha_0 \geqq \tfrac{1}{3} \sin^3 \alpha \,.$$

In fact, without any loss of generality we may write

$$\sin \alpha_0 = \frac{1}{2} \frac{d_3}{d_1} \sin \gamma ,$$

where γ is the angle between $a_1 a_2$ and the axis Oa_3. From the relations

$$\gamma > \alpha_1 \geqq \alpha , \quad \sin \gamma \geqq \sin \alpha , \quad d_1 \leqq h$$

and (2.27), we arrive at (2.33).

From (2.33) and (2.32), the estimate (2.21) follows. Q.E.D.

Let us introduce the set

$$U(K) = \{ \tau \in W^1(K) | \; \partial \tau_{ij} / \partial x_j = 0, \; i = 1, 2 \} .$$

Theorem 2.2. *Let $\tau \in U(K)$ and let the array of twelve external parameters $T_k^{i,i}$, $T_k^{i,i+1}$ be determined by the conditions*

$$\int_{a_i a_{i+1}} (T_k^{ii} \lambda_1^i + T_k^{i,i+1} \lambda_2^i) \, \lambda_m^i \, \mathrm{d}s = \int_{a_i a_{i+1}} t_k^i(\tau) \, \lambda_m^i \, \mathrm{d}s , \tag{2.34}$$

$$k, m = 1, 2 , \quad i = 1, 2, 3 .$$

Then the external parameters satisfy the overall equilibrium conditions (2.19), (2.20) and there exists a unique stress field $\Pi\tau \in \mathcal{N}(K)$ such that (2.18) holds for $(\Pi\tau)^i$ instead of τ^i.

The mapping $\Pi : U(K) \to \mathcal{N}(K)$ is linear and continuous. Moreover,

$$\max_{i=1,2,3} \|(\Pi\tau)^i\|_{C^0(K_i)} \leqq C_0(\alpha) \max_{i=1,2,3} \|\tau^i\|_{C^0(K_i)} \tag{2.35}$$

holds for any $\tau \in U(K) \cap \prod_{i=1}^{3} C^0(\bar{K}_i)$, where $C_0(\alpha) > 0$ depends on the angle α only and

$$\Pi\tau = \tau \quad \forall \tau \in \mathcal{M}(K) . \tag{2.36}$$

Proof. By conditions (2.34), the parameters $T_k^{i,i}$, $T_k^{i,i+1}$ are uniquely determined, the matrix

$$A_{jm}^i = \int_{a_i a_{i+1}} \lambda_j^i \lambda_m^i \, \mathrm{d}s , \quad j, m = 1, 2 ,$$

being regular ($\det A = l_i^2/12$). If $\tau \in U(K)$, for $k = 1, 2$ we have

$$0 = \int_{\partial K} \tau_{kj} n_j \, \mathrm{d}s = \sum_{i=1}^{3} \int_{a_i a_{i+1}} t_k^i(\tau) \, \mathrm{d}s = \tfrac{1}{2} \sum_{i=1}^{3} l_i (T_k^{ii} + T_k^{i,i+1}) , \tag{2.37}$$

where (2.34) has been used. Furthermore, (cf. the proof of Lemma 2.4),

(2.38)
$$0 = \int_K (\tau_{12} - \tau_{21})\,\mathrm{d}\boldsymbol{x} = \int_{\partial K} \varepsilon_{3jk}\tau_{jm}n_m x_k\,\mathrm{d}s = \sum_{i=1}^{3} \int_{a_i a_{i+1}} (t_1^i(\tau)\,x_2 - t_2^i(\tau)\,x_1)\,\mathrm{d}s\,.$$

Since we may insert

$$x_k = x_k(a_i)\,\lambda_i^i + x_k(a_{i+1})\,\lambda_2^i\,, \quad k = 1, 2\,,$$

from (2.34) we deduce that

$$\int_{a_i a_{i+1}} t_j^i(\tau)\,x_k\,\mathrm{d}s = \int_{a_i a_{i+1}} (T_j^{ii}\lambda_1^i + T_j^{i,i+1}\lambda_2^i)\,x_k\,\mathrm{d}s\,, \quad j, k = 1, 2\,. \tag{2.39}$$

From (2.38) and (2.39) the condition (2.20) follows. Theorem 1.1 implies the existence and uniqueness of the stress field $\Pi\tau \in \mathcal{N}(K)$, satisfying (2.18).

The linearity of Π follows from the linearity of the mapping $W^{1,2}(K) \to L_2(a_i a_{i+1})$, (2.5), (2.34), (2.25) and (2.11).

To prove the boundedness of Π, we estimate the right-hand sides of (2.34). If $\tau \in U(K) \cap \prod_{i=1}^{3} C^0(\overline{K}_i)$, then the upper bound is

$$l_i \sqrt{\tfrac{2}{3}} \max_{i=1,2,3} \|\tau^i\|_{C^0(K_i)}\,. \tag{2.41}$$

From (2.34) we deduce easily

$$\max_{j,k} \{|T_k^{jj}|, |T_k^{j,j+1}|\} \leqq 2\sqrt{6} \max_i \|\tau^i\|_{C^0(K_i)}\,. \tag{2.42}$$

Inserting (2.42) into (2.21), (where τ^i is replaced by $(\Pi\tau)^i$), we obtain the boundedness of Π and the estimate (2.35), respectively.

To prove (2.36), we first realize that for $\tau \in \mathcal{M}(K)$ the stress vectors $t_k^i(\tau)$ are linear along $a_i a_{i+1}$, consequently $T_k^{ii} = t_k^i(\tau)\,(a_i)$, $T_k^{i,i,+1} = t_k^i(\tau)\,(a_{i+1})$. Next defining $\tau|_{K_i} = \tau^i$, $i = 1, 2, 3$, we conclude that $\tau^i \in \mathcal{M}(K_i)$ and verify the conditions (2.17). Then (2.36) follows from the "uniqueness assertion" involved in Theorem 2.1.

Theorem 2.3. *Let $\tau \in U(K) \cap C^2(\overline{K})$. Then*

$$\max_{i=1,2,3} \|\tau^i - (\Pi\tau)^i\|_{C^0(K_i)} \leqq c_1(\alpha)\,h^2\,\|\tau\|_{C^2(K)}\,, \tag{2.44}$$

where $c_1(\alpha)$ depends on the minimal angle α of K only and h is the maximal side of the triangle K.

Proof. Let $\boldsymbol{x}_0 \in K$ be an arbitrary point. Taylor's theorem implies for $\boldsymbol{x} \in \overline{K}$

$$\tau_{ij}(\boldsymbol{x}) = \tau_{ij}(\boldsymbol{x}_0) + D\,\tau_{ij}(\boldsymbol{x}_0)\,(\boldsymbol{x} - \boldsymbol{x}_0) + \tfrac{1}{2}D^2\,\tau_{ij}(\vartheta)\,(\boldsymbol{x} - \boldsymbol{x}_0)^2\,, \tag{2.45}$$

$i, j = 1, 2$, where $\vartheta \in \boldsymbol{x}_0\boldsymbol{x}$.

Denote $\tau_{ij}(\mathbf{x}_0) + D\,\tau_{ij}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = L_{ij}(\mathbf{x})$,

$$\tfrac{1}{2}D^2\,\tau_{ij}(\vartheta)(\mathbf{x} - \mathbf{x}_0)^2 = Q_{ij}(\mathbf{x}) .$$

Since we have $L \in \mathcal{M}(K) \subset U(K) \cap C^2(\bar{K})$,

$$Q = \tau - L \in U(K) \cap C^2(\bar{K}) .$$

Applying the mapping Π to (2.45) (i.e., $\tau = L + Q$), using its linearity and (2.36), we obtain

$$\Pi\tau = L + \Pi Q . \tag{2.46'}$$

Consequently, by virtue of (2.45), (2.46') and (2.35), we may write $\tau - \Pi\tau = Q - \Pi Q$,

$$\max_{i=1,2,3} \|\tau^i - (\Pi\tau)^i\|_{C^0(K_i)} = \max_i \|Q^i - (\Pi Q)^i\|_{C^0(K_i)} \leqq$$
$$\leqq \max_i \|Q^i\|_{C^0(K_i)} + \max_i \|(\Pi Q)^i\|_{C^0(K_i)} \leqq$$
$$\leqq (1 + c_0(\alpha)) \max_i \|Q^i\|_{C^0(K_i)} = (1 + c_0(\alpha))\,\|Q\|_{C^0(K)} .$$

Moreover, on the basis of (2.45) we obtain

$$\|Q\|_{C^0(\bar{K})} \leqq 2\,h^2 \|\tau\|_{C^2(\bar{K})}$$

and the estimate (2.44) follows. Q.E.D.

Theorem 2.4. *Let $\tau \in U(K) \cap W^2(K)$. Then it holds*

$$\|\tau - \Pi\tau\|_{0,K} \leqq C\,h^2 \|\tau\|_{2,K} , \tag{2.46}$$

where $h =$ diam K, C depends on the minimal angle α only.

Proof. We shall need the following result on the equivalence of norms.

Lemma 2.5. *Let $\hat{\Omega}$ be a bounded domain with Lipchitz boundary, $\mathcal{M} = \mathcal{M}(\hat{\Omega})$ the subspace defined in (2.1), $\tilde{q}$ the class of equivalence from the quotient space $W^2(\hat{\Omega})/\mathcal{M}$ with the usual norm*

$$\|\tilde{q}\|_{W^2(\hat{\Omega})/\mathcal{M}} = \inf_{q \in \tilde{q}} \|q\|_{W^2(\hat{\Omega})}$$

and

$$|q|_{2,\hat{\Omega}} = \Big(\sum_{|\alpha|=2} \sum_{i,j=1}^{2} \|D^\alpha q_{ij}\|_{0,\hat{\Omega}}^2\Big)^{1/2} .$$

Then a positive constant C exists such that

$$\|\tilde{q}\|_{W^2(\hat{\Omega})/\mathcal{M}} \leqq C|q|_{2,\hat{\Omega}} \tag{2.47}$$

holds for all $q \in \tilde{q} \in H/\mathcal{M}$, where

$$H = W^2(\hat{\Omega}) \cap U(\hat{\Omega}) .$$

Proof. Let us introduce the functionals $g_i\colon W^2(\hat{\Omega}) \to R^1$

$$g_i(q) = \int_{\hat{\Omega}} \partial q_{ij}/\partial \hat{x}_j \,\mathrm{d}\hat{\mathbf{x}}\,, \quad i = 1, 2\,.$$

We shall prove that the space $W^2(\hat{\Omega})/\mathcal{M}$ is complete with the following norm

$$\|q\|' = (|q|^2_{2,\hat{\Omega}} + \sum_{i=1}^{2} g_i^2(q))^{1/2}\,. \tag{2.48}$$

It is readily seen that

$$q \in W^2(\hat{\Omega})\,, \quad \|q\|' = 0 \Leftrightarrow q \in \mathcal{M}\,.$$

Let $\{\tilde{q}_n\}$ be a Cauchy sequence (with the norm (2.48)). Hence it is a Cauchy sequence with the seminorm $|\cdot|_{2,\hat{\Omega}}$, as well. It holds

$$\|\tilde{q}\|_{W^2(\hat{\Omega})/P_1{}^s(\hat{\Omega})} \leqq C|q|_{2,\hat{\Omega}} \quad \forall q \in \tilde{q} \in W^2(\hat{\Omega})/P_1^s\,. \tag{2.49}$$

(The proof of (2.49) is parallel to that of Theorem 7.2 in [7].) Consequently, to any $q_n \in \tilde{q}_n$ there exists $p_n \in P_1^s(\hat{\Omega})$ such that

$$r_n = q_n + p_n \to q \quad \text{in} \quad W^2(\hat{\Omega})\,.$$

Then for $\tilde{p}_n \in W^2(\hat{\Omega})/\mathcal{M}$ it holds

$$\|\tilde{p}_n - \tilde{p}_m\|' = \|r_n - r_m - (q_n - q_m)\|' \leqq \|r_n - r_m\|' + \|\tilde{q}_n - \tilde{q}_m\|'\,,$$

which implies that $\{\tilde{p}_n\}$ is a Cauchy sequence. Since

$$(\sum_{i=1}^{2} g_i^2(p))^{1/2}$$

is a norm in a finite-dimensional space $P_1^s/\mathcal{M}$, we have $\tilde{p}_n \to \tilde{p}$ in $P_1^s/\mathcal{M}$ and in $W^2(\hat{\Omega})/\mathcal{M}$.

Then $\tilde{q}_n \to \tilde{q} - \tilde{p}$. In fact $\tilde{q}_n = \tilde{r}_n - \tilde{p}_n$ and

$$\|\tilde{r}_n - \tilde{p}_n - (\tilde{q} - \tilde{p})\|' \leqq \|r_n - q\|' + \|\tilde{p}_n - \tilde{p}\|' \leqq \|r_n - q\|_{2,\hat{\Omega}} + \|\tilde{p}_n - \tilde{p}\|' \to 0\,.$$

(Note that

$$\|p\|' \leqq C\|\tilde{p}\|_{W^2(\hat{\Omega})/\mathcal{M}} \quad \forall p \in \tilde{p} \in W^2(\hat{\Omega})/\mathcal{M})\,. \tag{2.50}$$

Hence the space $W^2(\hat{\Omega})/\mathcal{M}$ with the norm (2.48) is complete.

Consider the identical mapping from the space $W^2(\hat{\Omega})/\mathcal{M}$ with the usual norm onto the same space with the norm (2.48). By virtue of the Banach theorem on isomorphism and (2.50) we obtain that

$$c\|\tilde{q}\|_{W^2(\hat{\Omega})/\mathcal{M}} \leqq \|q\|'\,.$$

Since $g_i(q) = 0$, $i = 1, 2$ for all $q \in H$, the assertion (2.47) follows.

We also employ a modification of the Bramble-Hilbert lemma.

Lemma 2.6. *Let $\hat{\Omega}$, H, $\mathcal{M}$ and $|\cdot|_{2,\hat{\Omega}}$ be the same as in Lemma 2.5. Let a linear functional $F \in H'$ be given such that*

$$|F(q)| \leqq C_1 \|q\|_{2,\hat{\Omega}}, \tag{2.51}$$

$$F(p) = 0 \quad \forall p \in \mathcal{M}(\hat{\Omega}). \tag{2.52}$$

Then there exists a constant C_2 such that

$$|F(q)| \leqq C_1 C_2 |q|_{2,\hat{\Omega}} \quad \forall q \in H. \tag{2.53}$$

Proof. From (2.51), (2.52) and Lemma 2.5 we obtain

$$|F(q)| = |F(\tilde{q})| \leqq C_1 \|\tilde{q}\|_{W^2/\mathcal{M}} \leqq C_1 C_2 |q|_{2,\hat{\Omega}} \quad \forall q \in \tilde{q} \in H/\mathcal{M}.$$

Let us choose a reference triangle $\hat{K}$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ in the $(\hat{x}_1, \hat{x}_2)$-plane and introduce a linear one-to-one mapping $\mathcal{F} : \hat{K} \to K$,

$$\boldsymbol{x} \equiv \mathcal{F}(\hat{\boldsymbol{x}}) = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{b}, \tag{2.54}$$

where

$$\boldsymbol{A} = \begin{bmatrix} x_1^2 - x_1^1, & x_1^3 - x_1^1 \\ x_2^2 - x_2^1, & x_2^3 - x_2^1 \end{bmatrix},$$

$$\boldsymbol{b}^{\mathrm{T}} = (x_1^1, x_2^1)$$

and (x_1^i, x_2^i), $i = 1, 2, 3$ are vertices of K.

If the stress tensor τ is defined on K, then the corresponding tensor defined on $\hat{K}$ is

$$\hat{\tau}(\hat{\boldsymbol{x}}) = \boldsymbol{A}^{-1} \tau(\mathcal{F}(\hat{x})) (\boldsymbol{A}^{-1})^{\mathrm{T}}, \tag{2.55}$$

(i.e. the correspondence between contravariant tensors).

Making use of (2.55), the relation

$$\tau \in U(K) \Leftrightarrow \hat{\tau} \in U(\hat{K})$$

can be verified by direct calculation.

Next let us set in Lemma 2.6 $\hat{\Omega} = \hat{K}$,

$$F(\hat{q}) = (\hat{q} - \widehat{\Pi q}, r)_{0,\hat{K}} \tag{2.56}$$

where $r \in W^0(\hat{K})$,

$$(q, r)_{0,\hat{K}} = \int_{\hat{K}} q_{ij} r_{ij} \, \mathrm{d}\boldsymbol{x}.$$

It is easy to verify (2.52). In fact, if $\hat{p} \in \mathcal{M}(\hat{K})$, then $p \in \mathcal{M}(K)$ and $p = \Pi p$ by virtue of (2.36). Consequently, we have $\hat{p} = \widehat{\Pi p}$ and $F(\hat{p}) = 0$.

We may write

$$(2.57)\qquad |F(\hat{q})| \leqq \|r\|_{0,\hat{K}}(\|\hat{q}\|_{0,\hat{K}} + \|\widehat{\Pi q}\|_{0,\hat{K}}) .$$

Let us show that if $\hat{q} \in W^2(\hat{K})$, then

$$(2.58)\qquad \|\widehat{\Pi q}\|_{0,\hat{K}} \leqq C\|\hat{q}\|_{2,\hat{K}} .$$

In fact, we have $\hat{q} \in C^0(\hat{K})$, $q \in C^0(\bar{K})$. Then (2.35) implies

$$M = \max_{i=1,2,3} \|(\widehat{\Pi q})^i\|_{C^0(\hat{K}_i)} = \max_{i=1,2,3} \|(\Pi q)^i\|_{C^0(K_i)} \leqq$$
$$\leqq C_0\|q\|_{C^0(K)} = C_0\|\hat{q}\|_{C^0(\hat{K})} \leqq C_1\|\hat{q}\|_{2,\hat{K}} ,$$
$$\|\widehat{\Pi q}\|^2_{0,\hat{K}} = \sum_{j,k=1}^{2} \int_{\hat{K}} (\widehat{\Pi q})^2_{jk} \,\mathrm{d}\mathbf{x} \leqq 4M^2 \operatorname{mes} \hat{K}$$

and (2.58) follows.

The functional F is defined on H. In fact, $\hat{q} \in H \Rightarrow q \in U(K)$ and Π is defined on $U(K)$. Obviously, F is linear and (2.51) holds, as a consequence of (2.57) and (2.58), with $C_1 = (1 + C)\,\|r\|_{0,\hat{K}}$. From Lemma 2.6 we obtain that

$$(2.59)\qquad |F(\hat{q})| \leqq C_2(1 + C)\,\|r\|_{0,\hat{K}}\,|\hat{q}|_{2,\hat{K}} \quad \forall \hat{q} \in H .$$

Inserting $r = \hat{q} - \widehat{\Pi q}$ into (2.56), from (2.59) it follows

$$(2.60)\qquad \|\hat{q} - \widehat{\Pi q}\|_{0,\hat{K}} \leqq C|\hat{q}|_{2,\hat{K}} \quad \forall \hat{q} \in H .$$

It holds

$$\|w\|_{0,K} = |A|^{1/2}\,\|\hat{w}\|_{0,\hat{K}} \quad \forall w \in L_2(K) ,$$
$$|\hat{w}|_{2,\hat{K}} \leqq C\,h^2|A|^{-1/2}\,|w|_{2,K} \quad \forall w \in W^{2,2}(K)$$

(see e.g. [8]), where $|A|$ is the Jacobian of the mapping (2.54).

Using these relations, the estimate (2.60) leads to the assertion (2.46) for $q = \tau$.

Q.E.D.

Let $\Omega \subset R^2$ be a bounded polygonal domain, $h \in (0, 1\rangle$, $\mathscr{T}_h$ a triangulation of $\bar{\Omega}$. Suppose that

$$h = \max_{K \in \mathscr{T}_h} \operatorname{diam} K .$$

Let $\mathbf{t}(\tau)|_K$ denote the stress vector defined in (2.5) by means of the stress field $\tau \in W^1(K)$. Let K, K' be two adjacent triangles in $\mathscr{T}_h$ with a common side a_ia_{i+1}. We say that *the condition* (R) *is satisfied, if*

$$(2.61)\qquad \mathbf{t}(\tau)|_K + \mathbf{t}(\tau)|_{K'} = 0 \quad \text{on} \quad \bar{K} \cap \bar{K}' = a_ia_{i+1}$$

for any interelement side $a_ia_{i+1} \in \Omega$.

Let us define

$$U(\Omega) = \{\tau \in [W^{1,2}(\Omega)]^4 | \tau_{12} = \tau_{21},\ \partial\tau_{ij}/\partial x_j = 0,\ i = 1, 2\}\,, \tag{2.62}$$

$$\mathcal{N}_h(\Omega) = \{\tau | \tau|_K \in \mathcal{N}(K)\ \forall K \in \mathcal{T}_h,\ \tau \text{ satisfies } (R)\}\,. \tag{2.63}$$

We say that a family $\{\mathcal{T}_h\}$, $h \in (0, 1\rangle$ of triangulations of Ω is regular, if there exists a constant $\alpha_0 > 0$, independent of h and such that all interior angles of the triangles of $\mathcal{T}_h \in \{\mathcal{T}_h\}$ are not less than α_0.

For $\tau \in U(\Omega)$ we define a mapping r_h by the relation

$$r_h\tau|_K = \Pi_K\tau \quad \forall K \in \mathcal{T}_h\,, \tag{2.64}$$

where Π_K denotes the mapping defined in Theorem 2.2.

Theorem 2.5. *Let $\{\mathcal{T}_h\}$, $h \in (0, 1\rangle$, be a regular family of triangulations of Ω. Then r_h maps $U(\Omega)$ into $\mathcal{N}_h(\Omega)$, being linear and continuous, and it holds*

$$\|\tau - r_h\tau\|_{0,\Omega} \leqq C\, h^2 \|\tau\|_{[C^2(\bar{\Omega})]^4} \quad \forall\tau \in U(\Omega) \cap [C^2(\bar{\Omega})]^4\,, \tag{2.65}$$

$$\|\tau - r_h\tau\|_{0,\Omega} \leqq Ch^2 \|\tau\|_{2,\Omega} \quad \forall\tau \in U(\Omega) \cap W^2(\Omega)\,, \tag{2.66}$$

where C is independent of h and τ.

Proof. Since

$$\tau \in U(\Omega) \Rightarrow \tau|_K \in U(K) \quad \forall K \in \mathcal{T}_h\,,$$

from Theorem 2.2 it follows that

$$r_h\tau|_K = \Pi_K\tau|_K \in \mathcal{N}(K)\,.$$

Since the traces of τ_{ij} from both sides of the interelement boundary coincide, it holds

$$\mathbf{t}(\tau|_K) + \mathbf{t}(\tau|_{K'}) = 0\,.$$

Consequently, the right-hand sides of (2.34) change the sign only, when K is replaced by K'. With regard to (2.18), the same is true for $\mathbf{t}(\Pi_K\tau)$ and $\mathbf{t}(\Pi_K, \tau)$ and the condition (R) follows. Hence $r_h : U(\Omega) \to \mathcal{N}_h(\Omega)$. The linearity and boundedness of r_h is a consequence of the analogous properties of the "local" mappings Π_K.

To verify (2.65), by virtue of (2.44) we may write

$$\|\tau - r_h\tau\|^2_{0,\Omega} = \sum_{K\in\mathcal{T}_h} \|\tau - \Pi_K\tau\|^2_{0,K} \leqq$$

$$\leqq \sum_{K\in\mathcal{T}_h} (\text{mes } K)\, C_1^2 h^4 \|\tau\|^2_{C^2(K)} \leqq C_1^2 h^4 (\text{mes } \Omega)\, \|\tau\|^2_{C^2(\bar{\Omega})}\,,$$

where $C_1 = 2(1 + c_0 \sin^{-23} \alpha)$. The estimate (2.66) is a consequence of Theorem 2.4.

Remark 2.1. Any field $\tau \in \mathcal{N}_h(\Omega)$ satisfies the equation div $\tau = 0$ in the sense of distributions. In fact, let $\tau \in \mathcal{N}_h(\Omega)$, $\varphi \in [C_0^\infty(\Omega)]^2$. Then

$$\langle \operatorname{div} \tau, \varphi \rangle \equiv - \int_\Omega \tau_{kj} \frac{\partial \varphi_k}{\partial x_j} \, \mathrm{d}\mathbf{x} = - \int_\Omega \tau_{kj} e_{ij}(\varphi) \, \mathrm{d}\mathbf{x} = - \sum_{K \in \mathcal{T}_h} \int_K \tau_{kj} \frac{\partial \varphi_k}{\partial x_j} \, \mathrm{d}\mathbf{x} =$$

$$= \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{3} \left[\int_{K_i} \varphi_k \frac{\partial \tau_{kj}}{\partial x_j} \, \mathrm{d}\mathbf{x} - \int_{\partial K_i} \tau_{kj} n_j \varphi_k \, \mathrm{d}s \right].$$

Using the conditions (2.17) and (R), the sum of all integrals over ∂K_i vanishes. Since $\partial \tau_{kj} / \partial x_j = 0$ in every K_i, we obtain $\langle \operatorname{div} \tau, \varphi \rangle = 0$.

3. APPLICATION OF THE SUBSPACE $\mathcal{N}_h(\Omega)$ TO THE DUAL VARIATIONAL SOLUTION

Let $\{\mathcal{T}_h\}$, $h \in (0, 1\rangle$ be a regular family of triangulations of Ω, satisfying moreover the following requirement: the "endpoints" of $\bar{\Gamma}_\sigma$ coincide with some vertices of $\mathcal{T}_h$. Defining

$$S_h = \mathcal{N}_h(\Omega) \cap \Lambda_{0,0},$$

it is easy to show that

$$S_h = \{\tau \in \mathcal{N}_h(\Omega) \mid \mathbf{t}(\tau) = 0 \text{ on } \Gamma_\sigma\}.$$

Recalling the definitions (1.1) and (1.2), we can establish the following

Theorem 3.1. *Let $\tau^0 \in W^2(\Omega)$. Then for any regular family of triangulations it holds*

$$\|\tau^0 - \tau_h^0\|_{0,\Omega} \leqq C\, h^2 \|\tau^0\|_{2,\Omega},$$

where C is independent of h and τ^0.

Proof. 1° We can show that $r_h \tau^0 \in S_h$. In fact,

$$\tau^0 \in [C^2(\bar{\Omega})]^4 \cap \Lambda_{0,0} \Leftrightarrow \int_\Omega \tau_{ij}^0 e_{ij}(\mathbf{v}) \, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{v} \in V \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \partial \tau_{ij}^0 / \partial x_j = 0 & \text{in} \quad \Omega, \\ \tau_{ij}^0 n_j = 0 & \text{on} \quad \Gamma_\sigma \end{cases}$$

Consequently, $\tau^0 \in U(\Omega)$, Theorem 2.4 implies that $r_h \tau^0 \in \mathcal{N}_h(\Omega)$ and it remains to verify that $\mathbf{t}(r_h \tau^0) = 0$ on Γ_σ.

Let $a_i a_{i+1} \in \Gamma_\sigma$ be a side of a boundary triangle $K \in \mathcal{T}_h$. Since $\mathbf{t}(\tau^0) = 0$ on $a_i a_{i+1}$, from (2.34) we obtain $T_k^{ii} = T_k^{i,i+1} = 0$, $k = 1, 2$, which results in $\mathbf{t}(\Pi_K \tau^0) = 0$ on $a_i a_{i+1}$. Consequently, $\mathbf{t}(r_h \tau^0) = 0$ on Γ_σ.

2° Using Theorems 1.2 and 2.5, we obtain

$$C_1\|\tau^0 - \tau_h^0\|_{0,\Omega} \leqq \|\tau^0 - \tau_h^0\|_H \leqq \|\tau^0 - r_h\tau^0\|_H \leqq$$
$$\leqq C_2\|\tau^0 - r_h\tau^0\|_{0,\Omega} \leqq C_3\, h^2\|\tau^0\|_{2,\Omega}\,. \qquad \text{Q.E.D.}$$

Corollary 3.1. *Let the assumptions of Theorem* 3.1 *hold. Then for* $\sigma(\mathbf{u}) = \bar{\sigma} + \tau^0$, $\sigma^h = \bar{\sigma} + \tau_h^0$ *we have the estimate*

$$\|\sigma(\mathbf{u}) - \sigma^h\|_{0,\Omega} = O(h^2)\,.$$

Let us recall the transformation of the problem $\mathcal{S}(\sigma) = \min$ over the set $\Lambda_{F,T}$ into the equivalent problem (1.1). We supposed that a stress-field $\bar{\sigma} \in \Lambda_{F,T}$ was available. In praxis, however, this requirement may be difficult to satisfy. Therefore, suppose that we can find a $\tilde{\sigma} \in \Lambda_{F,\mathcal{T}}$ where $\mathcal{T}$ is close to $\mathbf{T}$ in some sense. For example, let us have a $\sigma^1 \in H$ such that

$$\partial\sigma_{ij}^1/\partial x_j + F_i = 0\,, \quad i = 1, 2$$

(note that σ_{ij}^1 can be found by integrations of F_i only). Let us set $\tilde{\mathbf{T}} = \mathbf{T} - \mathbf{t}(\sigma^1)$ and suppose that $\mathbf{t}(\sigma^1) \in [L_2(\Gamma_\sigma)]^2$.

Let $\bar{\Gamma}_\sigma = \bigcup_{j=1}^{m} a_j a_{j+1}$. Assume that we can find a $\sigma^2 \in \mathcal{N}_h(\Omega)$ such that

$$\int_{a_j a_{j+1}} (t_k(\sigma^2) - \tilde{T}_k)\, \lambda_n^j\, \mathrm{d}s = 0\,, \quad j = 1, \ldots, m\,; \quad k, n = 1, 2\,,$$

(i.e., $t_k(\sigma^2)$ are orthogonal projections $\hat{\Pi}\,\tilde{T}_k$ of $\tilde{T}_k$ into $P_1(a_j a_{j+1})$).
Let us define $\tilde{\sigma} = \sigma^1 + \sigma^2$, $\mathcal{T} = \mathbf{t}(\sigma^1 + \sigma^2)$. Then $\tilde{\sigma} \in \Lambda_{F,\mathcal{T}}$,

$$\|\mathcal{T}_k - T_k\| = \|t_k(\sigma^1) + \hat{\Pi} T_k - \hat{\Pi} t_k(\sigma^1) - T_k\| \leqq$$
$$\leqq \|t_k(\sigma^1) - \Pi t_k(\sigma^1)\| + \|\hat{\Pi} T_k - T_k\|\,.$$

Consequently, $\mathcal{T}$ is an approximation of $\mathbf{T}$.
Define the problem to find $\tau_{\mathcal{T}}^0 \in \Lambda_{0,0}$ such that

$$\Phi_{\mathcal{T}}(\tau_{\mathcal{T}}^0) \leqq \Phi_{\mathcal{T}}(\tau) \quad \forall \tau \in \Lambda_{0,0}$$

and the approximate problem to find $\tau_{\mathcal{T}}^h \in \mathcal{N}_h(\Omega)$ such that

$$\Phi_{\mathcal{T}}(\tau_{\mathcal{T}}^h) \leqq \Phi_{\mathcal{T}}(\tau) \quad \forall \tau \in S_h\,,$$

where

$$\Phi_{\mathcal{T}}(\tau) = \tfrac{1}{2}(\tau, \tau)_H - \int_\Omega \tau_{ij}[e_{ij}(\mathbf{u}_0) - b_{ijkl}\tilde{\sigma}_{kl}]\,\mathrm{d}\mathbf{x}\,,$$
$$\sigma_{\mathcal{T}}^0 = \tilde{\sigma} + \tau_{\mathcal{T}}^0\,, \quad \sigma_{\mathcal{T}}^h = \tilde{\sigma} + \tau_{\mathcal{T}}^h\,.$$

Then we have the following

Theorem 3.2. *Let $\tau^0_{\mathscr{T}} \in W^2(\Omega)$ and $\mathbf{T}$, $\mathbf{t}(\sigma^1) \in [W^{2,2}(\Gamma_m)]^2$ for any side Γ_m of the polygonal Γ_σ. Then for any regular family of triangulations it holds*

$$\|\sigma(\mathbf{u}) - \sigma^h_{\mathscr{T}}\|_{0,\Omega} \leqq Ch^2 ,$$

where C is independent of h.

Proof is based on the inequality

$$\|\sigma(\mathbf{u}) - \sigma^h_{\mathscr{T}}\|_H \leqq \|\sigma(\mathbf{u}) - \sigma^0_{\mathscr{T}}\|_H + \|\sigma^0_{\mathscr{T}} - \sigma^h_{\mathscr{T}}\|_H .$$

The last term can be estimated using Corollary 3.1. The term $\sigma(\mathbf{u}) - \sigma^0_{\mathscr{T}}$ can be treated like an analogous term in Section 3 of [1].

4. CONVERGENCE OF THE EQUILIBRIUM FINITE ELEMENT MODEL IN A GENERAL CASE

In Theorems 3.1, 3.2 strong regularity assumptions were imposed upon the solution of the dual variational problem. A question arises about the convergence of the method in a general case, when the regularity of τ^0 cannot be justified. The main point of the following convergence analysis will be a proper density theorem. *We shall distinguish the cases*: (i) $\Gamma = \Gamma_u$, (ii) $\Gamma = \Gamma_\sigma$ (iii) $\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_\sigma$. In what follows, we use the notations:

$$\|v\|_{1,\Omega} = \left(\sum_{k=1}^{2} \|v_k\|^2_{1,\Omega}\right)^{1/2} ,$$

$$\|v\|_{1/2,\Gamma} = \left(\sum_{k=1}^{2} \|v_k\|^2_{W^{1/2,2}(\Gamma)}\right)^{1/2} .$$

(i) Let $\Gamma = \Gamma_u$. We have $V = [W^{1,2}_0(\Omega)]^2$,

$$\Lambda_{0,0}(\Omega) = \left\{\tau \in [L_2(\Omega)]^4 \mid \tau_{ij} = \tau_{ji}, \int_\Omega \tau_{ij} e_{ij}(\mathbf{v}) \, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in [W^{1,2}_0(\Omega)]^2\right\} .$$

Theorem 4.1. *The set*

$$\Lambda_{0,0}(\Omega) \cap [C^\infty(\bar{\Omega})]^4$$

is dense in $\Lambda_{0,0}(\Omega)$ (with the topology of $[L_2(\Omega)]^4$).

Proof. Let $\Omega^* \subset R^2$ be a bounded domain with a Lipschitz boundary such that $\Omega^* \supset \bar{\Omega}$. In case that Ω is a domain of connectivity m, we choose Ω^* of the same connectivity. Then $\Omega^* \dot{-} \Omega = \bigcup_{j=1}^{m} G_j$, where G_j are doubly-connected domains. Let $\tau \in \Lambda_{0,0}(\Omega)$ be given. We construct an extension $E\tau \in \Lambda_{0,0}(\Omega^*)$, $E\tau|_\Omega = \tau$ as follows.

In every G_j let us consider the following auxiliary problem: to find

$$\mathbf{w} \in V(G_j) = \{\mathbf{v} \in [W^{1,2}(G_j)]^2 \mid \mathbf{v} = 0 \text{ on } \partial G_j \dot{-} \partial\Omega\}$$

such that

$$\int_{G_j} e_{ik}(\mathbf{w})\, e_{ik}(\mathbf{v})\, \mathrm{d}\mathbf{x} = -\int_{\Omega} \tau_{ik} e_{ik}(P\mathbf{v})\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in V(G_j)\,, \tag{4.1}$$

where $P\mathbf{v}$ is any extension of $\mathbf{v} \in V(G_j)$ such that

$$P\mathbf{v} \in V_j(\Omega) = \{\mathbf{v} \in [W^{1,2}(\Omega)]^2 \mid \mathbf{v} = 0 \text{ on } \partial\Omega \dot{-} \partial G_j\}\,,$$
$$P\mathbf{v} = \mathbf{v} \quad \text{on} \quad \partial\Omega \cap \partial G_j\,.$$

The right-hand side of (4.1) is independent of the kind of extension from $V(G_j)$ into $V_j(\Omega)$. In fact, since $\tilde{P}\mathbf{v} - P\mathbf{v} = 0$ on $\partial\Omega \cap \partial G_j$, $\tilde{P}\mathbf{v} - P\mathbf{v} \in [W_0^{1,2}(\Omega)]^2$ and

$$\int_{\Omega} \tau_{ik} e_{ik}(\tilde{P}\mathbf{v} - P\mathbf{v})\, \mathrm{d}\mathbf{x} = 0$$

follows from the definition of $\tau \in \Lambda_{0,0}(\Omega)$.

There exists a linear mapping of $[W^{1/2,2}(\partial\Omega \cap \partial G_j)]^2$ into $V_j(\Omega)$ such that (cf. [7] – chpt. 2, § 5)

$$\|P\mathbf{v}\|_{1,\Omega} \leqq C\|\mathbf{v}\|_{1/2,\partial\Omega\cap\partial G_j} \leqq C\, C_1\|\mathbf{v}\|_{1,G_j}\,.$$

Consequently,

$$\left|\int_{\Omega} \tau_{ik} e_{ik}(P\mathbf{v})\, \mathrm{d}\mathbf{x}\right| \leqq C\|\tau\|_{0,\Omega}\, \|P\mathbf{v}\|_{1,\Omega} \leqq C_2\|\tau\|_{0,\Omega}\, \|\mathbf{v}\|_{1,G_j}$$

and the right-hand side of (4.1) is a linear bounded functional on $[W^{1,2}(G_j)]^2$. Using the Korn's inequality for $\mathbf{v} \in V(G_j)$ and Lax-Milgram's theorem, we arrive at the existence and uniqueness of the solution $\mathbf{w}$ of (4.1).

Setting $E\tau = e(\mathbf{w})$ in G_j $\forall_j$, $E\tau|_{\Omega} = \tau$, we show that $E\tau \in \Lambda_{0,0}(\Omega^*)$. In fact, let $\mathbf{v} \in [W_0^{1,2}(\Omega^*)]^2$. Then

$$\int_{\Omega} (E\tau)_{ik}\, e_{ik}(\mathbf{v})\, \mathrm{d}\mathbf{x} = \int_{\Omega} \tau_{ik}\, e_{ik}(\mathbf{v})\, \mathrm{d}\mathbf{x} + \sum_{j=1}^{m} \int_{G_j} e_{ik}(\mathbf{w})\, e_{ik}(\mathbf{v})\, \mathrm{d}\mathbf{x}\,.$$

Since

$$[W_0^{1,2}(\Omega^*)]^2 = \{\mathbf{v} \in [W^{1,2}(\Omega^*)]^2 \mid \mathbf{v} = 0 \text{ on } \partial G_j \dot{-} \partial\Omega \ \forall j\}\,,$$
$$\mathbf{v}|_{G_j} \in V(G_j) \quad \forall j\,,$$
$$\mathbf{v} \in [W^{1,2}(\Omega)]^2 \Rightarrow \mathbf{v} = \sum_{j=1}^{m} \mathbf{w}_j\,, \quad \mathbf{w}_j \in V_j(\Omega)\,,$$

we may write

$$\int_{\Omega^*} (E\tau)_{ik}\, e_{ik}(\mathbf{v})\, \mathrm{d}\mathbf{x} = \sum_{j=1}^{m} \left[\int_{\Omega} \tau_{ik}\, e_{ik}(\mathbf{w})\, \mathrm{d}\mathbf{x} + \int_{G_j} e_{ik}(\mathbf{w})\, e_{ik}(\mathbf{v})\, \mathrm{d}\mathbf{x} \right] = 0\,,$$

because $\mathbf{w}_j = P\mathbf{v}$, $\mathbf{w}_j = \mathbf{v}$ on $\partial\Omega \cap \partial G_j\ \forall j$.

Let us regularize $(E\tau)_{ik}$ by means of a kernel $\omega_\varkappa(\mathbf{x} - \mathbf{y})$, where

$$A^{-1}\varkappa^2\, \omega_\varkappa(\mathbf{z}) = \begin{cases} \exp\left(|\mathbf{z}|^2/(|\mathbf{z}|^2 - \varkappa^2)\right) & \text{for } |\mathbf{z}| < \varkappa \\ 0 & \text{for } |\mathbf{z}| \geqq \varkappa\,, \end{cases} \tag{4.2}$$

$A = \text{const} > 0$, $\varkappa < \operatorname{dist}(\partial\Omega^*, \partial\Omega)$. We obtain $\tau^\varkappa_{ij} = R_\varkappa(E\tau)_{ij} \in C^\infty(\bar{\Omega})$,

$$\tau^\varkappa_{ij}(\mathbf{x}) = \int_{\Omega} \omega_\varkappa(\mathbf{x} - \mathbf{y})\, (E\tau)_{ij}(\mathbf{y})\, \mathrm{d}\mathbf{y}\,, \quad i, j = 1, 2\,, \tag{4.3}$$

$$\partial\tau_{ij}/\partial x_j(\mathbf{x}) = -\int_{\Omega} \frac{\partial}{\partial y_j}\, \omega_\varkappa(\mathbf{y} - \mathbf{x})\, (E\tau)_{ij}(\mathbf{y})\, \mathrm{d}\mathbf{y} \quad \forall \mathbf{x} \in \Omega\,, \quad i = 1, 2\,. \tag{4.4}$$

By virtue of the fact that $\omega_\varkappa \in C_0^\infty(\Omega^*) \subset W_0^{1,2}(\Omega^*)$ and

$$\int_{\Omega^*} (E\tau)_{ij} \frac{\partial \omega_\varkappa}{\partial y_j}\, \mathrm{d}\mathbf{y} = \int_{\Omega^*} (E\tau)_{ij} \frac{\partial \hat{\omega}_{\varkappa i}}{\partial y_j}\, \mathrm{d}\mathbf{y} = \int_{\Omega^*} (E\tau)_{ij}\, e_{ij}(\hat{\omega}_\varkappa)\, \mathrm{d}\mathbf{y} \tag{4.5}$$

where

$$\hat{\omega}_\varkappa \equiv (\omega_\varkappa, 0) \quad \text{for} \quad i = 1\,,$$

$$\hat{\omega}_\varkappa \equiv (0, \omega_\varkappa) \quad \text{for} \quad i = 2$$

has been defined, and using the definition of $\Lambda_{0,0}$, we are led to the equations

$$\frac{\partial \tau^\varkappa_{ij}}{\partial x_j} = 0 \quad \text{in} \quad \Omega\,, \quad i = 1, 2 \Rightarrow \tau^\varkappa \in \Lambda_{0,0}\,. \tag{4.6}$$

Moreover, we have for $\varkappa \to 0$

$$\|\tau^\varkappa - \tau\|_{0,\Omega} \leqq \|\tau^\varkappa - E\tau\|_{0,\Omega^*} \to 0\,. \tag{4.7}$$

Q.E.D.

(ii) Let $\Gamma = \Gamma_\sigma$. Assume that Ω is a starlike domain, i.e., a point $A \in \Omega$ exists such that each ray from A intersects the boundary Γ in one and only one point.

Theorem 4.2. *If the domain Ω is starlike, then the set*

$$\Lambda_{0,0}(\Omega) \cap [C^\infty(\bar{\Omega})]^4$$

is dense in $\Lambda_{0,0}(\Omega)$.

Proof. We have $V = [W^{1,2}(\Omega)]^2$,

$$\Lambda_{0,0}(\Omega) = \left\{\tau \in [L_2(\Omega)]^4 \mid \tau_{ij} = \tau_{ji}, \int_\Omega \tau_{ij}\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{v} \in [W^{1,2}(\Omega)]^2\right\}.$$

Let $\tau \in \Lambda_{0,0}(\Omega)$ be given. We extend it onto $R^2 \dot{-} \Omega$ by zero function. The extended function will be denoted by $E\tau$. Let us put the origin into the point A and define

$$\tau^\varepsilon_{ij}(\mathbf{x}) = E\tau_{ij}((1 + \varepsilon)\, \mathbf{x}), \quad \varepsilon > 0.$$

Lemma 4.1. *For any* $\sigma \in C_0^\infty(\Omega)$ *it holds*

$$\|\sigma^\varepsilon - \sigma\|_{0,\Omega} \to 0 \quad for \quad \varepsilon \to 0.$$

Proof. Using the mean value theorem we may write

$$\int_\Omega (\sigma^\varepsilon - \sigma)^2\, \mathrm{d}\mathbf{x} = \int_\Omega [\sigma((1 + \varepsilon)\, \mathbf{x}) - \sigma(\mathbf{x})]^2\, \mathrm{d}\mathbf{x} =$$

$$= \int_\Omega \left[\sum_{m=1}^2 \frac{\partial \sigma}{\partial x_m}(\vartheta \mathbf{x})\, \varepsilon x_m\right]^2 \mathrm{d}\mathbf{x} \leqq \varepsilon^2 \int_\Omega |\mathrm{grad}\, \sigma(\vartheta \mathbf{x})|^2\, |\mathbf{x}|^2\, \mathrm{d}\mathbf{x} \leqq \varepsilon^2\, C(\Omega)\, \|\sigma\|^2_{C^1(\bar{\Omega})},$$

where the constant $C(\Omega)$ depends on the domain only.

Lemma 4.2. *If* $\tau = 0$ *outside* Ω, $\tau \in L_2(\Omega)$, *there exists a sequence* $\tau^n \in C_0^\infty(\Omega)$, *such that*

$$\|\tau^\varepsilon - (\tau^n)^\varepsilon\|_{0,\Omega} \to 0$$

for $n \to \infty$ *uniformly with respect to* ε.

Proof. Denoting $1 + \varepsilon = k$, we have for the sequence $\tau^n \to \tau$ in $L_2(\Omega)$:

$$\int_\Omega [\tau(k\mathbf{x}) - \tau^n(k\mathbf{x})]^2\, \mathrm{d}\mathbf{x} = \frac{1}{k^2} \int_{k\Omega} [\tau(\mathbf{y}) - \tau^n(\mathbf{y})]^2\, \mathrm{d}\mathbf{y} \leqq$$

$$\leqq \int_\Omega [\tau(\mathbf{y}) - \tau^n(\mathbf{y})]^2\, \mathrm{d}\mathbf{y} \to 0 \quad \text{if} \quad n \to \infty,$$

because both τ and τ^n vanishes outside Ω.

Lemma 4.3. *If* $\tau = 0$ *outside* Ω, $\tau \in L_2(\Omega)$, *then*

$$\|\tau^\varepsilon - \tau\|_{0,\Omega} \to 0 \quad for \quad \varepsilon \to 0.$$

Proof. Using Lemma 4.2 and 4.1 (for $\sigma \equiv \tau^n$), we have for $\varepsilon \to 0$

$$\|\tau^\varepsilon - \tau\|_0 \leqq \|\tau^\varepsilon - (\tau^n)^\varepsilon\|_0 + \|(\tau^n)^\varepsilon - \tau^n\|_0 + \|\tau^n - \tau\|_0 \to 0.$$

Lemma 4.4. *The function* τ^ε *belongs to* $\Lambda_{0,0}(\Omega)$ *and* supp $\tau^\varepsilon \subset \Omega$.

Proof. Let us consider an arbitrary $v \in [W^{1,2}(\Omega)]^2$. Then $\bar{\mathbf{v}}(\mathbf{y}) = \mathbf{v}(\mathbf{y}/k)$ belongs to $[W^{1,2}(k\Omega)]^2$ and we may write

$$\int_\Omega \tau^\varepsilon_{ij}\, e_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x} = \int_\Omega E\,\tau_{ij}(k\mathbf{x})\, e_{ij}(\mathbf{v}(\mathbf{x}))\,\mathrm{d}\mathbf{x} =$$

$$= \frac{1}{k^2}\int_{k\Omega} E\,\tau_{ij}(\mathbf{y})\, e_{ij}(\bar{\mathbf{v}}(\mathbf{y}))\,\mathrm{d}\mathbf{y} = \frac{1}{k^2}\int_\Omega \tau_{ij}(\mathbf{y})\, e_{ij}(\bar{\mathbf{v}}(\mathbf{y}))\,\mathrm{d}\mathbf{y} = 0\,,$$

which yields that $\tau^\varepsilon \in \Lambda_{0,0}(\Omega)$.

Since the domain Ω is starlike, the function τ^ε vanishes in the "boundary layer" $\Omega \dot{-} k^{-1}\Omega \equiv \Omega^\varepsilon$. Q.E.D.

Now we are able to finish the proof of Theorem 4.2. Let us regularize the function τ^ε, defining (cf. (4.2), (4.3))

$$R_\varkappa\, \tau^\varepsilon_{ij}(\mathbf{x}) = \int_\Omega \omega_\varkappa(\mathbf{x} - \mathbf{y})\, \tau^\varepsilon_{ij}(\mathbf{y})\,\mathrm{d}\mathbf{y}\,, \quad i, j = 1, 2\,.$$

By an argument similar to (4.4), (4.5), we deduce, using Lemma 4.4, that

$$\partial R_\varkappa\, \tau^\varepsilon_{ij}/\partial x_j = 0 \quad \text{in} \quad \Omega\,, \qquad i = 1, 2\,. \tag{4.8}$$

Moreover from Lemma 4.4 it follows that

$$R_\varkappa\, \tau^\varepsilon_{ij}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \Gamma\,, \qquad \forall \varkappa < \operatorname{dist}(\partial\Omega, \operatorname{supp} \tau^\varepsilon)\,. \tag{4.9}$$

From (4.8), (4.9) we obtain that $R_\varkappa \tau^\varepsilon \in \Lambda_{0,0}(\Omega)$, using integration by parts.

Finally, by virtue of Lemma 4.3

$$\|\tau - R_\varkappa \tau^\varepsilon\|_{0,\Omega} \leqq \|\tau - \tau^\varepsilon\|_{0,\Omega} + \|\tau^\varepsilon - R_\varkappa \tau^\varepsilon\|_{0,\Omega} \to 0$$

for $\varepsilon \to 0$, $\varkappa \to 0$. Q.E.D.

(iii) Let $\Gamma = \bar{\Gamma}_u \cup \bar{\Gamma}_\sigma$.

Theorem 4.3. *Assume that there exists a point* $A \in R^2$ *such that if* A *coincides with the origin, then for* $k = 1 + \varepsilon$ *and* $\varepsilon > 0$ *sufficiently small, either*

(I) $\qquad k\bar{\Gamma}_\sigma \subset R^2 \dot{-} \bar{\Omega}$ *or*

(II) $\qquad k\bar{\Gamma}_\sigma \subset \Omega\,,$

where kM *denotes the image of a set* M *by means of the "dilatation" mapping* $\mathbf{y} = k\mathbf{x}$.

Then the set

$$\Lambda_{0,0}(\Omega) \cap [C^\infty(\bar{\Omega})]^4$$

is dense in $\Lambda_{0,0}(\Omega)$.

Proof. Let $\tau \in \Lambda_{0,0}(\Omega)$ be given. First we extend it as follows.

1° Let $\Omega^* \supset \bar{\Omega}$, Ω^* be a bounded domain with Lipschitz boundary. Let $0 < 2d < \operatorname{dist}(\partial\Omega, \partial\Omega^*)$ and denote (see Fig. 2)

$$G^* = \{\mathbf{x} \notin \Omega \mid \operatorname{dist}(\mathbf{x}, \bar{\Gamma}_\sigma) < d\} \dot{-} \bigcup_j S_j\,,\ ^{1})$$

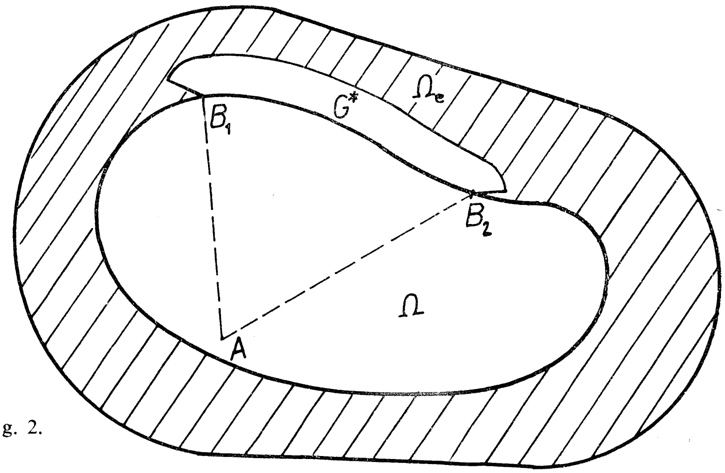

Fig. 2.

$$\Omega_1 = \Omega \cup G^*\,,$$
$$\Omega_e = \Omega^* \dot{-} \bar{\Omega}_1\,.$$

Consider the following auxiliary problem: to find $\mathbf{w} \in V_e$,

$$V_e = \{\mathbf{v} \in [W^{1,2}(\Omega_e)]^2 \mid \mathbf{v} = 0 \text{ on } \partial\Omega^*\}$$

such that

$$\int_{\Omega_e} e_{ij}(\mathbf{w})\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} = - \int_\Omega \tau_{ij}\, e_{ij}(P\mathbf{v})\, \mathrm{d}\mathbf{x} \quad \forall \mathbf{v} \in V_e\,, \tag{4.10}$$

where $P\mathbf{v}$ is (a restriction of) an arbitrary extension of $\mathbf{v} \in V_e$ into $[W_0^{1,2}(\Omega^*)]^2$.

The right-hand side of (4.10) is independent of the kind of extension P. In fact, $\tilde{P}\mathbf{v} - P\mathbf{v} = 0$ on $\partial\Omega_1$ for any two extensions $\tilde{P}$ and P.

Since $\Gamma_u \subset \partial\Omega_1$, it holds

$$\tilde{P}\mathbf{v} - P\mathbf{v} = 0 \quad \text{on} \quad \Gamma_u \Rightarrow (\tilde{P}\mathbf{v} - P\mathbf{v})|_\Omega \in V\,,$$

$$\int_\Omega \tau_{ij}\, e_{ij}(\tilde{P}\mathbf{v} - P\mathbf{v})\, \mathrm{d}\mathbf{x} = 0\,.$$

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[1]) S_j are sectors of sufficiently small angles with vertices at the points $\bar{\Gamma}_\tau \cap \bar{\Gamma}_u \equiv B_j$.

A linear extension $P : V_e \to [W_0^{1,2}(\Omega^*)]^2$ exists such that

$$\|P\mathbf{v}\|_{1,\Omega^*} \leqq C\|\mathbf{v}\|_{1,\Omega_e} \,.$$

(For the proof of this assertion see e.g. [7] – chpt. 2, Th. 3.9). Consequently, the problem (4.10) has a unique solution $\mathbf{w}$.

Let us define the extension $E\tau$ as follows:

$$(4.11) \qquad E\tau = \begin{cases} 0 & \text{in} \quad G^* \,, \\ e(\mathbf{w}) & \text{in} \quad \Omega_e \,, \\ \tau & \text{in} \quad \Omega \,. \end{cases}$$

By virtue of (4.10) we have for any $\mathbf{v} \in [W_0^{1,2}(\Omega^*)]^2$

$$(4.12) \qquad \int_{\Omega^*} (E\tau)_{ij}\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} = \int_{\Omega} \tau_{ij}\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} + \int_{\Omega_e} e_{ij}(\mathbf{w})\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} = 0 \,.$$

2° Let us transform $E\tau$, using the dilatation mapping

$$\tau^\varepsilon(\mathbf{x}) = E\,\tau(k\mathbf{x}) \quad \text{in case (I)} \,,$$
$$\tau^\varepsilon(\mathbf{x}) = E\,\tau(k^{-1}\mathbf{x}) \quad \text{in case (II)} \,.$$

It is easy to see that Lemma 4.1 remains valid in case (II), too.

Lemma 4.5. *Let $\tau \in L_2(\Omega)$, $E\tau \in L_2(\Omega^*)$, $0 < \varepsilon < \varepsilon_0 < \infty$. Then there exists a sequence $\tau^n \in C_0^\infty(\Omega)$ such that $\tau^n \to \tau$ in $L_2(\Omega)$ and*

$$\|\tau^\varepsilon - (\tau^n)^\varepsilon\|_{0,\Omega} \leqq C(\|\tau^n - \tau\|_{0,\Omega} + \|E\tau\|_{0,k\Omega \dot- \Omega})$$

for the case (I). In case (II) the last norm is to be replaced by $\|E\tau\|_{0,k^{-1}\Omega \dot- \Omega}$.

Proof. For the sequence, satisfying $\tau^n \to \tau$ in $L_2(\Omega)$, we obtain in case (I):

$$\int_\Omega [E\,\tau(k\mathbf{x}) - \tau^n(k\mathbf{x})]^2\, \mathrm{d}\mathbf{x} = k^{-2} \int_{k\Omega} [E\,\tau(\mathbf{y}) - \tau^n(\mathbf{y})]^2\, \mathrm{d}\mathbf{y} \leqq$$
$$\leqq \int_\Omega (\tau - \tau^n)^2\, \mathrm{d}\mathbf{y} + \int_{k\Omega \dot- \Omega} [E\tau]^2\, \mathrm{d}\mathbf{y} \,.$$

In case (II) it suffices to replace k by k^{-1} to obtain the same estimate.

Lemma 4.6. *Let $\tau \in L_2(\Omega)$, $E\tau \in L_2(\Omega^*)$, $0 < \varepsilon < \varepsilon_0$. Then*

$$\|\tau^\varepsilon - \tau\|_{0,\Omega} \to 0 \quad \textit{for} \quad \varepsilon \to 0 \,.$$

Proof. Using Lemma 4.5 and 4.1 (for $\sigma = \tau^n$), we may write

$$\|\tau^\varepsilon - \tau\| \leqq \|\tau^\varepsilon - (\tau^n)^\varepsilon\| + \|(\tau^n)^\varepsilon - \tau^n\| + \|\tau^n - \tau\| \to 0 \,.$$

Lemma 4.7. *There exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$\tau^\varepsilon = 0 \textit{ in a neighbourhood } \begin{Bmatrix} k^{-1}G^* \\ kG^* \end{Bmatrix} \textit{ of } \bar{\Gamma}_\sigma \textit{ in case } \begin{Bmatrix} (\mathrm{I}) \\ (\mathrm{II}) \end{Bmatrix}.$$

Proof. From the geometrical assumptions it follows that a positive ε_0 exists such that for $0 < \varepsilon < \varepsilon_0$

$$\bar{\Gamma}_\sigma \subset k^{-1}G^* \text{ in case (I)}, \quad (\bar{\Gamma}_\sigma \subset kG^* \text{ in case (II)}).$$

Then for $\mathbf{x} \in k^{-1}G^*$ ($\mathbf{x} \in kG^*$) we have $\tau^\varepsilon(\mathbf{x}) = 0$ by virtue of (4.11).

Lemma 4.8. *Let* $\Omega_d = \{\mathbf{x} \mid \operatorname{dist}(x, \bar{\Omega}) < d\}$. *Then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *one has*

$$\int_{\Omega_d} \tau^\varepsilon_{ij}\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{v} \in [W_0^{1,2}(\Omega_d)]^2.$$

Proof. Let $\mathbf{v} \in [W_0^{1,2}(\Omega_d)]^2$. Define in case (I) $\tilde{\mathbf{v}}(\mathbf{y}) = \mathbf{v}(\mathbf{y}/k)$. Since $k\bar{\Omega}_d \subset \Omega^*$ for sufficiently small ε and $\tilde{\mathbf{v}} \in [W_0^{1,2}(k\Omega_d)]^2$, we can extend $\tilde{\mathbf{v}}$ by zero to obtain $P\tilde{\mathbf{v}} \in$ $\in [W_0^{1,2}(\Omega^*)]^2$. Then

$$\int_{\Omega_d} \tau^\varepsilon_{ij}\, e_{ij}(\mathbf{v})\, \mathrm{d}\mathbf{x} = k^{-2} \int_{k\Omega_d} E\, \tau_{ij}(\mathbf{y})\, e_{ij}(\tilde{\mathbf{v}}(\mathbf{y}))\, \mathrm{d}\mathbf{y} = k^{-2} \int_{\Omega^*} E\tau_{ij}\, e_{ij}(P\tilde{\mathbf{v}})\, \mathrm{d}\mathbf{y} = 0$$

by virtue of (4.12). In case (II) the proof is parallel.

To finish the proof of Theorem 4.3, let us regularize τ^ε. By an argument similar to (4.4), (4.5), we obtain for $\varkappa < d$ that

$$\frac{\partial R_\varkappa\, \tau^\varepsilon_{ij}(\mathbf{x})}{\partial x_j} = -\int_{\Omega_d} \tau^\varepsilon_{ij}\, e_{ij}(\hat{\omega}_\varkappa)\, \mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{x} \in \Omega.$$

Using Lemmas 4.7 and 4.8 one deduce easily that for sufficiently small $\varkappa$

$$R_\varkappa \tau^\varepsilon_{ij} = 0 \quad \text{on} \quad \Gamma_\sigma.$$

Integrating by parts, we obtain that $R_\varkappa \tau^\varepsilon \in \Lambda_{0,0}(\Omega)$. The Lemma 4.6 and the well-known property of regularization yield that

$$\|R_\varkappa \tau^\varepsilon - \tau\|_{0,\Omega} \leqq \|R_\varkappa \tau^\varepsilon - \tau^\varepsilon\|_{0,\Omega} + \|\tau^\varepsilon - \tau\|_{0,\Omega} \to 0 \quad \text{for} \quad \varkappa \to 0, \quad \varepsilon \to 0.$$

Theorem 4.4. *Let us consider the cases*:

(i) $\Gamma = \Gamma_u$,
(ii) $\Gamma = \Gamma_\sigma$ *and the domain* Ω *is starlike* (*see Theorem* 4.2),
(iii) $\Gamma = \bar{\Gamma}_u \cup \Gamma_\sigma$ *and the assumptions of Theorem* 4.3 *hold.*

Then for any regular family of triangulations and for $\bar{\sigma} \in \Lambda_{F,T}$,

$$\sigma(\boldsymbol{u}) = \bar{\sigma} + \tau^0 , \quad \sigma^h = \bar{\sigma} + \tau_h^0 ,$$

one has

$$\|\sigma^h - \sigma(\boldsymbol{u})\|_{0,\Omega} \to 0 \quad \text{for} \quad h \to 0 . \tag{4.13}$$

Proof. On the basis of Theorem 1.2, we have

$$C_1\|\sigma^h - \sigma(\boldsymbol{u})\|_{0,\Omega} = C_1\|\tau_h^0 - \tau^0\|_{0,\Omega} \leqq \|\tau^0 - \tau_h^0\|_H \leqq \leqq \inf_{\tau \in S_h} \|\tau^0 - \tau\|_H . \tag{4.14}$$

Let an $\varepsilon_1 > 0$ be given. From the density Theorems 4.1, 4.2 and 4.3, there exists a $R\tau^0 \in [C^\infty(\bar{\Omega})]^4 \cap \Lambda_{0,0}(\Omega)$ such that

$$\|\tau^0 - R\tau^0\|_{0,\Omega} < \tfrac{1}{2}\varepsilon_1 .$$

Applying Theorem 2.4, we obtain

$$\|R\tau^0 - r_h(R\tau^0)\|_{0,\Omega} \leqq Ch^2\|R\tau^0\|_{[C^2(\bar{\Omega})]^4} .$$

Then $r_h(R\tau^0) \in S_h$, (see the proof of Theorem 3.1) and

$$\|\tau^0 - r_h(R\tau^0)\|_H \leqq C_2(\|\tau^0 - R\tau^0\|_{0,\Omega} + \|R\tau^0 - r_h(R\tau^0)\|_{0,\Omega}) < \varepsilon \tag{4.15}$$

follows for h sufficiently small. Finally, from (4.14), (4.15) we obtain (4.13).

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Souhrn

KONVERGENCE JEDNOHO ROVNOVÁŽNÉHO MODELU METODY KONEČNÝCH PRVKŮ V ROVINNÉ PRUŽNOSTI

IVAN HLAVÁČEK

Rovnovážný blokový trojúhelníkový prvek, navržený Watwoodem a Hartzem [3] je podroben analýze a dokázána jistá jeho aproximační vlastnost. Odtud plyne za předpokladu regularity řešení kvazi-optimální odhad chyby přibližného řešení kombinované úlohy pružnosti duální metodou (tj. na základě Castiglianova variačního principu).

Je podán důkaz konvergence i v obecném případě, kdy řešení není regulární.

Author's address: Ing. *Ivan Hlaváček*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.