Jindřich Spal

Fundamentals of a mathematical theory of fuzzy sets

Aplikace matematiky, Vol. 27 (1982), No. 5, 326–340

Persistent URL: http://dml.cz/dmlcz/103980

FUNDAMENTALS OF A MATHEMATICAL THEORY OF FUZZY SETS

JINDŘICH SPAL

(Received February 20, 1980)

A system of definitions and theorems, including informational entropy, is introduced in order to obtain a more adequate insight into the problems and nature of the fuzzy sets.

Key words: *Fuzzy sets, Fuzzy relations, Informational entropy, Random variable.*

1. INTRODUCTION

The fuzzy sets, introduced by Zadeh [1] in 1965, are penetrating into various fields of application [2]. Several versions of their mathematical theory have been given by Zadeh, as well as by other authors [3, 4, 5]. The formation of an integrated theory of fuzzy sets meets considerable difficulties, which have their reason mainly in the following facts:

1. The mutual relation between the *random variable* and the *fuzzy set* is not clear-cut in many cases. For example, Goguen [6] defines the fuzzy set as a mapping from a system of subsets of a fundamental (basic) set Z to a defined interval Q of real numbers, thus indicating the "degree of membership" of an element $z \in Z$ in regard to the fuzzy set F:

$$F: Z \to Q . \tag{1}$$

This does not comport exactly with the real meaning of the fuzzy set and is often the cause of various misunderstandings. The situation was mentioned by the author in a former paper [7]. There exists a formal similarity between these two mathematical objects, both starting from a function defined over all elements of the fundamental (basic) set Z. In the case of the *random variable* it is the probabilistic distribution or the probabilistic density. For the *fuzzy set* it is Zadeh's membership function. But at the same time there is a capital difference. The *random variable* assigns a real value of probability (or of another quantity expressing the probabilistic measure) from the space of the probabilistic measure M_P to any subset of a set system (field of events) defined on the basic set Z, thus producing the mapping:

$$(2) \qquad P: Z \to M_P .$$

On the contrary, the *fuzzy set* F attributes to any value m from the criterial interval Q a subset S of the basic set Z, thus representing a mapping from the interval Q to a set system $\mathscr{S}_S$, $S \subseteq Z$:

$$(3) \qquad F: Q \to \mathscr{S}_S .$$

Thus the fuzzy set is in a certain sense (not quite exactly) an inverse notion in regard to the random variable. It is interesting to note that inverse mappings do not generally exist in either case.

2. The values of the membership function are usually reduced to the interval $\langle 0; 1\rangle$ of real numbers, without taking full account of all consequences of such a reduction. This often leads to misunderstandings and misinterpretations in the application of fuzzy sets. Moreover, such a reduction need not be always meaningful or purposeful from the practical point of view. That is why the further exposé assumes for the generating function over the basic set Z admissible values from any arbitrary predefined interval of real numbers. This real function $f(z)$ $(z \in Z)$ with an arbitrary interval of values is called the *criterial function*, in contradistinction to the *membership function* of Zadeh, the values of which are strictly limited to the interval $\langle 0; 1\rangle$. In addition to this purely formal side, there is a conceptual difference between the criterial function and the membership function, too. The values of the criterial function are supposed to express numerically the degree of the property, forming the basis of the process of classification connected with the "verbal" definition of the fuzzy set. The values of the membership function indicate the consequences of the process of classification, as realized by the system of subsets on the basic set. Thus the criterial function formulates the presumptions, the membership function indicates the consequences of the process of classification, underlying the definition of the fuzzy set.

3. The importance of the level value of the criterial function or of the membership function is often underestimated.

The system of definitions and theorems, as formulated further, is aimed at putting right the mentioned inconsequences. It is concentrated on conceptual and methodological aspects of the problem, not claiming to form an exhaustive integrated mathematical theory.

2. BASIC DEFINITIONS

The non-empty closed *basic set* Z of real numbers $z \in Z$ may be supposed to be obtained by a bijectional isomorphic mapping from a *fundamental set* B of objects of arbitrary nature.

Definition 1. *The criterial function $f(z)$ is a real function of bounded variation defined for all elements $z \in Z$ with values from a given interval of real numbers*

$$Q = \langle q_{\min}; q_{\max} \rangle . \tag{4}$$

For any level value $m \in Q$ there exists a set of real numbers

$$M(m) = \{q: q \geqq m\} \tag{5}$$

and another set of real numbers

$$\overline{M}(m) = \{q: q < m\} , \tag{6}$$

$$M(m) \cup \overline{M}(m) = Q ; \quad M(m) \cap \overline{M}(m) = \emptyset . \tag{7}$$

Definition 2. *A realization $R(m)$ at the level m is the set*

$$R(m) = \{z \in Z; f(z) \geqq m\} . \tag{8}$$

Definition 3. *A complementary realization $\overline{R}(m)$ at the level m is the set:*

$$\overline{R}(m) = \{z \in Z; f(z) < m\} . \tag{9}$$

Evidently:

$$\forall m \in Q \quad R(m) \cup \overline{R}(m) = Z . \tag{10}$$

The conditions (5), (6) define on Q two systems of subsets:

$$\mathcal{S}_M = \{M(m); \forall m \in Q\} , \tag{11}$$

$$\mathcal{S}_{\overline{M}} = \{\overline{M}(m); \forall m \in Q\} . \tag{12}$$

Similarly, the conditions (8), (9) define on Z two systems of non-empty or empty subsets:

$$\mathcal{S}_R = \{R(m); \forall m \in Q\} , \tag{13}$$

$$\mathcal{S}_{\overline{R}} = \{\overline{R}(m); \forall m \in Q\} . \tag{14}$$

Thus there exists a mapping between the systems $\mathcal{S}_M$ and $\mathcal{S}_R$, as well as between the systems $\mathcal{S}_{\overline{M}}$ and $\mathcal{S}_{\overline{R}}$, mediated through the values of the level parameter $m \in Q$.

Definition 4. *A fuzzy set F is the mapping from the system $\mathcal{S}_M$ to the system $\mathcal{S}_R$, induced by the criterial function $f(z)$*:

$$F: \mathcal{S}_M \to \mathcal{S}_R . \tag{15}$$

Definition 5. *A complementary fuzzy set $\overline{F}$ is the mapping from the system $\mathcal{S}_{\overline{M}}$ to the system $\mathcal{S}_{\overline{R}}$ induced by the critical function $f(z)$*:

$$\overline{F}: \mathcal{S}_{\overline{M}} \to \mathcal{S}_{\overline{R}} . \tag{16}$$

Remark. Accordingly, a fuzzy set is not a set in the sense of the classical set theory. It is a quintuple (Z, Q, M, R, f), where Z is the basic set, Q is an interval of real values, M is an element of the system of subsets $M \subseteq Q$, R is an element of the system of subsets $R \subseteq Z$, f is a function defined for all elements of Z, mapping any $m \in M$ onto some $R(m) \subseteq Z$.

The procedure, which was taken advantage of in Definition 4, consisted in transferring to the whole system of subsets $\mathscr{S}_{R(m)}$ a property valid for all realizations $R(m)$ at any value $m \in Q$. This principle can be generalized by the following definition:

Definition 6. *Any property, which is valid for all realizations $R(m)$ (or for all complementary realizations $\bar{R}(m)$, respectively) for all $m \in Q$, is considered to be the respective property of the fuzzy set F (of the complementary fuzzy set $\bar{F}$, respectively) itself.*

Let us define special cases of fuzzy sets:

Definition 7. *The fuzzy set, the realizations $R(m)$ of which are empty sets for all $m \in Q$, is called the empty fuzzy set E.*

Definition 8. *The complementary fuzzy set, the complementary realizations $\bar{R}(m)$ of which are empty sets for all $m \in Q$, is called the empty complementary fuzzy set $\bar{E}_c$.*

Theorem 1. *The fuzzy set, defined by the criterial function of the empty complementary fuzzy set $\bar{E}_c$, consists of realizations, equal to the basic set Z for all $m \in Q$ (universal fuzzy set U). The complementary fuzzy set, defined by the criterial function of the empty fuzzy set E, consists of complementary realizations, equal to the basic set Z for all $m \in Q$ (universal complementary fuzzy set $\bar{U}_c$).*

The proof results directly from the preceding definitions.

Definition 9. *A fuzzy set is called definite on the level m_0, if for all $m \geqq m_0$ all realizations $R(m)$, and consequently all complementary realizations $\bar{R}(m)$, too, are equal:*

$$\forall m \geqq m_0: \quad R(m) = R(m_0) = R_0 \qquad (17)$$
$$\bar{R}(m) = \bar{R}(m_0) = \bar{R}_0 \, .$$

Definition 10. *A complementary fuzzy set is called definite on the level m_0, if for all $m < m_0$ all complementary realizations $\bar{R}(m)$, and consequently all realizations $R(m)$, too, are equal:*

$$\forall m < m_0: \quad \bar{R}(m) = \bar{R}(m_0) = \bar{R}_0 \, , \qquad (18)$$
$$R(m) = R(m_0) = R_0 \, .$$

Definition 11. *A fuzzy set (complementary fuzzy set) is called completely definite, if for all $m \in Q$ all realizations, and consequently all complementary realizations, too, are equal:*

(19)
$$\begin{aligned} &\forall m_1 \in Q \\ &\forall m_2 \in Q: \quad R(m_1) = R(m_2) = R_0\,, \\ &\qquad\qquad\quad \bar{R}(m_1) = \bar{R}(m_2) = \bar{R}_0\,. \end{aligned}$$

3. OPERATIONS ON FUZZY SETS AND FUZZY RELATIONS

3.1. The set algebra of fuzzy sets defined on the same basic set

Two fuzzy sets are supposed to be defined on the same basic set Z, namely a fuzzy set F with the criterial function $f(z)$ and a fuzzy set G with the criterial function $g(z)$.

Theorem 2. *The criterial function $u(z)$ of the union of fuzzy sets*

(20)
$$U = F \cup G$$

is

(21)
$$u(z) = \operatorname{Max}\,(f(z), g(z))\,.$$

Theorem 3. *The criterial function $v(z)$ of the intersection of fuzzy sets*

(22)
$$V = F \cap G$$

is

(23)
$$v(z) = \operatorname{Min}\,(f(z), g(z))\,.$$

Theorem 4. *The criterial function $\bar{u}(z)$ of the union of complementary fuzzy sets*

(24)
$$\bar{U} = \bar{F} \cup \bar{G}$$

is

(25)
$$\bar{u}(z) = \operatorname{Min}\,(f(z), g(z))\,.$$

Theorem 5. *The criterial function $\bar{v}(z)$ of the intersection of complementary fuzzy sets*

(26)
$$\bar{V} = \bar{F} \cap \bar{G}$$

is

(27)
$$\bar{v}(z) = \operatorname{Max}\,(f(z), g(z))\,.$$

Proof of Theorems 2 to 5:

a) The meaning of the unions and intersections results from Definition 6.

b) The following, mutually disjoint, non-empty or empty subsets are defined on Z by means of the functions $f(z)$, $g(z)$ for all $m \in Q$:

$$R_1(m) = \{r_1 \colon f(r_1) \geqq m,\ g(r_1) < m\}\,, \quad r_1 \in Z \tag{28}$$

$$R_2(m) = \{r_2 \colon f(r_2) < m,\ g(r_2) \geqq m\}\,, \quad r_2 \in Z \tag{29}$$

$$R_3(m) = \{r_3 \colon f(r_3) \geqq m,\ g(r_3) \geqq m\}\,, \quad r_3 \in Z \tag{30}$$

$$R_4(m) = \{r_4 \colon f(r_4) < m,\ g(r_4) < m\}\,, \quad r_4 \in Z \tag{31}$$

These subsets form a complete system:

$$R_1(m) \cup R_4(m) \cup R_3(m) \cup R_4(m) = Z \tag{32}$$

and evidently

$$f(r_1) > g(r_1)\,, \quad f(r_2) < g(r_2)\,. \tag{33}$$

Further,

$$R_F(m) = R_1(m) \cup R_3(m) \tag{34}$$

$$R_G(m) = R_2(m) \cup R_3(m) \tag{35}$$

$$R_{\bar{F}}(m) = R_2(m) \cup R_4(m) \tag{36}$$

$$R_{\bar{G}}(m) = R_1(m) \cup R_4(m) \tag{37}$$

and thus

$$U(m) = R_1(m) \cup R_2(m) \cup R_3(m) \tag{38}$$

$$\bar{U}(m) = R_1(m) \cup R_2(m) \cup R_4(m) \tag{39}$$

$$V(m) = R_3(m) \tag{40}$$

$$\bar{V}(m) = R_4(m) \tag{41}$$

The relations (21), (23), (25), (27) result from (38), (39), (40), (41) as sufficient and necessary.

(End of the proof).

In the preceding considerations, the set operations on fuzzy sets and complementary fuzzy sets have been expressed in terms of set relations on realizations and complementary realizations, which are sets in the sense of the classical set theory. Thus, *the axiomatic theory of the classical set theory remains valid for the fuzzy sets and complementary fuzzy sets, as defined by the above system of definitions.*

3.2. Operations in the space of values of criterial functions

Various operations may be defined in the space of real values of the criterial functions. A new criterial function $h(z)$ is derived in this way from the original criterial functions (operands) $f_1(z), f_2(z), \ldots$:

$$h(z) = \mathrm{Op}\left(f_1(z), f_2(z), \ldots\right). \tag{42}$$

Such operations are generally accompanied with changes of the variational span of the criterial functions and may lead to exceeding the limits of the predefined interval of the values $Q = \langle q_{\min}, q_{\max} \rangle$. It depends on the concrete problem which the fuzzy set is to serve for, if and how the limits of the interval are to be adapted when applying such operations.

Definition 12. *Operations in the space of real values of the criterial functions, mapping the operands $f_1(z), f_2(z), \ldots$ with a common interval of values $Q = \langle q_{\min}, q_{\max} \rangle$ onto the same interval, are called invariant operations.*

3.2.1. Examples of unary operations. The general formula of such operations is

$$h(z) = \mathrm{Op}\left(f(z)\right). \tag{43}$$

Let us introduce two examles:

a) Linear transformation

$$h(z) = A \,.\, f(z) + B\,, \tag{44}$$

A, B being real values. A special case is the reduction of the critical function to the unit interval $\langle 0; 1 \rangle$, applicable in transforming a general criterial function into the respective Zadeh's membership function:

$$h(z) = \frac{f(z) - q_{\min}}{q_{\max} - q_{\min}}\,. \tag{45}$$

b) Non-linear invariant operation

Let us formulate the following problem:

For $q_{\min} > 0$, $q_{\max} > 0$ the coefficients A, B are to be determined in such a way as to make the operation

$$h(z) = (A \,.\, f(z) + B)^2 \tag{46}$$

invariant. The conditions of invariance

$$\begin{aligned}(A \cdot q_{\max} + B)^2 &= q_{\max}\,,\\ (A \cdot q_{\min} + B)^2 &= q_{\min}\end{aligned} \tag{47}$$

lead to the equation:

$$q^2 + \left(2 \cdot \frac{B}{A} - A^{-2}\right) \cdot q + \frac{B^2}{A^2} = 0 \tag{48}$$

which should have the roots q_{min}, q_{max}. This yields

$$A = \left(q_{min} + q_{max} + 2 \cdot (q_{min} \cdot q_{max})^{+1/2}\right)^{-1/2}, \tag{49}$$
$$B = (q_{min} \cdot q_{max})^{1/2} \cdot A .$$

3.2.2. Examples of binary operations. The binary operations used in defining the unions and intersections of fuzzy sets and complementary fuzzy sets

$$h(z) = \text{Max}\,(f_1(z), f_2(z)), \tag{50}$$

$$h(z) = \text{Min}\,(f_1(z), f_2(z)) \tag{51}$$

may serve as an example.

Examples of binary invariant operations:

Generalized multiplication:

$$h_m(z) = f_1(z) \odot f_2(z) = \tag{52}$$
$$= (q_{max} - q_{min})^{-2} \cdot (f_1(z) - q_{min}) \cdot (f_2(z) - q_{min}) + q_{min} .$$

Generalized addition:

$$h_a(z) = f_1(z) \oplus f_2(z) = \tag{53}$$
$$= (q_{max} - q_{min}) \cdot (f_1(z) + f_2(z) - 2 \cdot q_{min}) - f_1(z) \odot f_2(z) .$$

In these operations the value q_{min} is the null element and the value

$$e = (q_{max} - q_{min})^2 + q_{min} \tag{54}$$

represents the unit.

3.3. Fuzzy relations

3.3.1. Remarks on relations. The starting point in defining a *relation* is the Cartesian product of sets.

Definition 13. *A Cartesian product of the sets* $Z_1, Z_2, \ldots, Z_n$,

$$K = Z_1 \times Z_2 \times \ldots \times Z_n , \tag{55}$$

is an ordered progression of these sets (components).

A Cartesian product, all the sets $Z_1, Z_2, \ldots, Z_n$ of which are mutually different, determines a *vector space*. Several components on the same common basic set, i.e. $Z_i \equiv Z_j \equiv Z_k \equiv \ldots$, establish a *multicriterial evaluation* of this basic set.

Definition 14. *A subset of a Cartesian product, containing non-empty or empty subsets of the basic sets of all components in the given order, is called a relation* ϱ *on this Cartesian product*:

$$\varrho = S_1 \times S_2 \times \ldots \times S_n \tag{56}$$

$$S_1 \subseteq Z_1,\ S_2 \subseteq Z_2,\ \ldots,\ S_n \subseteq Z_n\,.$$

The most remarkable type of relations are *functions*. Here the components are divided into two groups:

1. The *independent variables*, for which the respective subsets are arbitrarily defined.

2. The *dependent variables*, the subsets of which are unambiguously established by the choice of the independent variables.

In real systems, functions serve as a means of description of events. The independent variables determine the conditions of the implementation of the event, while the dependent variables describe the resulting effect.

Often further *auxiliary variables* are introduced, serving the aims of evaluation (classification) of the events.

Some or all components can be *measurable*. An *additive measure* is attributed to the elements of these components. Its value for a union of disjoint subsets (parts) is equal to the sum of the measures of these parts.

Sometimes a subset is completely represented by the value of its measure. In this case the individuality of the elements gets lost and subsets of the same value of the measure are freely interchangeable. The respective variable can be substituted by intervals of real numbers, the subsets being represented by segments of the interval, which corresponds to the whole basic set.

A *random variable* can be given as an example of such a measurable functional relation. Here the space (field of events) is represented by an ensemble of measurable independent components. The additive probabilistic measure is established in such a way as to give the unit value of probability to the whole space, which corresponds to the Cartesian product of the basic sets of the independent variables.

3.3.2. Vector fuzzy sets and multicriterial fuzzy sets. A *vector fuzzy set* consists of n components, each of them being defined by an individual criterial function. In the case of a *monocriterial evaluation* any of these criterial functions has its own basic set. In a *multicriterial evaluation* of a basic set some or even all criterial functions are applied to this common basic set.

Moreover, a vector fuzzy set has an interval of real numbers Q, common for all components, on which the values of the critical functions are represented.

For any value $m \in Q$ the basic set Z_k is divided by the criterial function $f_k(z_k)$ into the realization $R_k(m)$ and the complementary realization $\bar{R}_k(m)$ in the same manner as in the case of one-dimensional fuzzy sets (Chap. 2).

Thus the space, in which a vector fuzzy set is established, is represented by the Cartesian product

$$K = Q \times Z_1 \times Z_2 \times \ldots \times Z_n \tag{57}$$

where some, or even all of the basic sets $Z_1, Z_2, \ldots, Z_n$ can be identical (if multicriterial components occur).

A relation is formed on this Cartesian product by the ensemble of *realizations*

$$\varrho(m) = M(m) \times R_1(m) \times R_2(m) \times \ldots \times R_n(m) \tag{58}$$

and another by the ensemble of *complementary realizations*

$$\bar{\varrho}(m) = \bar{M}(m) \times \bar{R}_1(m) \times \bar{R}_2(m) \times \ldots \times \bar{R}_n(m)\,. \tag{59}$$

By the procedure of generalization for all $m \in Q$, in the same way as in (11), (12), (13), (14), we get set systems:

$$\mathscr{S}_M, \mathscr{S}_{R1}, \mathscr{S}_{R2}, \ldots, \mathscr{S}_{Rn}\,, \tag{60}$$

$$\mathscr{S}_{\bar{M}}, \mathscr{S}_{\bar{R}1}, \mathscr{S}_{\bar{R}2}, \ldots, \mathscr{S}_{\bar{R}n}\,, \tag{61}$$

defining *vector fuzzy sets* as ensembles of ordinary (one-dimensional) fuzzy sets:

$$\phi = (\mathscr{S}_M \rightarrow \mathscr{S}_M) \times (\mathscr{S}_M \rightarrow \mathscr{S}_{R1}) \times (\mathscr{S}_M \rightarrow \mathscr{S}_{R2}) \times \ldots \times (\mathscr{S}_M \rightarrow \mathscr{S}_{R_n}) \tag{62}$$

and *vector complementary fuzzy sets* as ensembles of ordinary (one-dimensional) complementary fuzzy sets:

$$\bar{\phi} = (\mathscr{S}_{\bar{M}} \rightarrow \mathscr{S}_{\bar{M}}) \times (\mathscr{S}_{\bar{M}} \rightarrow \mathscr{S}_{\bar{R}1}) \times (\mathscr{S}_{\bar{M}} \rightarrow \mathscr{S}_{\bar{R}2}) \times \ldots \times (\mathscr{S}_{\bar{M}} \rightarrow \mathscr{S}_{\bar{R}n})\,, \tag{63}$$

where the identities $(\mathscr{S}_M \rightarrow \mathscr{S}_M)$, $(\mathscr{S}_{\bar{M}} \rightarrow \mathscr{S}_{\bar{M}})$ are introduced only in order to obtain formal completeness as to the number of the components.

The fuzzy sets (complementary fuzzy sets) of the components are in either case interconnected through the common real interval Q. The formulae (62), (63) may be considered as a kind of "generalized relations", in which the respective components of the vector fuzzy sets (of the vector complementary fuzzy sets, resp.) are represented by the "partial" fuzzy sets (complementary fuzzy sets) of the components.

Remark. The following important fact is worth mentioning. While the definition of the *random variable* is based on the existence of a measure on the basic set (on the field of events), the definition of the *fuzzy set* does not make any use of the notion of measurability. The existence of a measure is not necessary on the sets participating in the specification of a fuzzy set (realizations, basic sets).

3.3.3. Binary relations. Let us pay some attention to *binary relations*, which are the simplest case of relations. Moreover, relations of higher orders can be decomposed into an ensemble of binary relations.

A *binary relation* is defined by the indication of two criterial functions $f_1(z_1)$, $f_2(z_2)$ with a common interval of values Q. Their basic sets Z_1, Z_2 can be different (*vector fuzzy sets*) or identical (*bicriterial fuzzy sets*).

Two functions can be defined by means of these binary relations:

a) The relation ϱ_{12}, for which the component derived from $f_1(z_1)$ is the independent variable and that derived from $f_2(z_2)$ is the dependent variable.

b) The relation ϱ_{21} with interchanged components, where $f_2(z_2)$ represents the independent variable, $f_1(z_1)$ is the dependent variable.

The *spans* (intervals of values) of the criterial functions have an important role in the valuation of the functional mappings:

$$\Delta M_i = m_{i\max} - m_{i\min} = \operatorname{Max} f_i(z)_i - \operatorname{Min} f_i(z_i)\,, \quad z_i \in Z_i\,. \tag{64}$$

The non-empty realization of the independent variable

$$R_1(m) = \{z_1 \colon z_1 \in Z_1,\ f_1(z_1) \geqq m\} \tag{65}$$

is mapped onto a non-empty realization $R_2(m)$, if

$$m \in (\Delta M_1 \cap \Delta M_2)\,. \tag{66}$$

For

$$m \in (\Delta M_1 \cap \overline{\Delta M_2}) \tag{67}$$

the image $R_2(m)$ is an empty set.

The interval of

$$m \in (\overline{\Delta M_1} \cap \Delta M_2) \tag{68}$$

is not affected by the mapping ϱ_{12}.

The situation is quite similar for complementary realizations.

Remark. The symbol $\overline{\ldots\ldots}$ indicates the "outside" of the respective interval.

Definition 15. *The mappings ϱ_{ij}, ϱ_{ji} are inverse to each other, if*

$$\forall m \in Q\colon \quad \varrho_{ji}(\varrho_{ij}(R_i(m))) = R_i(m)\,, \quad \varrho_{ij}(\varrho_{ji}(R_j(m))) = R_j(m)\,. \tag{69}$$

Theorem 6. *Two mappings ϱ_{ij}, ϱ_{ji}, established by the criterial functions $f_i(z_i)$, $f_j(z_j)$, are inverse to each other, if and only if*

$$\Delta M_i = \Delta M_j\,. \tag{70}$$

Proof. 1. For any $m \in \Delta M_i$, $m \in \Delta M_j$ there exist non-empty realizations $R_i(m)$, $R_j(m)$, making both the mappings ϱ_{ij}, ϱ_{ji} possible.

2. For any $m \in \Delta M_i$, $m \notin \Delta M_j$ the realization $R_i(m)$ is non-empty, the realization $R_j(m)$ is empty. Thus there exists a mapping ϱ_{ij}, but the inverse mapping ϱ_{ji} does not exist, the definitory set of this mapping being empty.

3. Similarly, for any $m \notin \Delta M_i$, $m \in \Delta M_j$ there exists a mapping ϱ_{ji}, but the inverse mapping ϱ_{ij} does not exist.

4. For $m \notin \Delta M_i$, $m \notin \Delta M_j$ neither of the mappings ϱ_{ij}, ϱ_{ji} exists.

(End of the proof).

Let us define three fuzzy sets:

F_i with a criterial function $f_i(z_i)$ and with a span ΔM_i,
F_j with a criterial function $f_j(z_j)$ and with a span ΔM_j,
F_k with a criterial function $f_k(z_k)$ and with a span ΔM_k.

There exist non-empty realizations:

$$R_i(m) \quad \text{for any} \quad m \in \Delta M_i\,,$$
$$R_j(m) \quad \text{for any} \quad m \in \Delta M_j\,,$$
$$R_k(m) \quad \text{for any} \quad m \in \Delta M_k\,.$$

Definition 16. *The mapping*

$$\varrho^*_{\text{ik}} = \varrho_{ij} \circ \varrho_{jk} \tag{71}$$

is called the composition of ϱ_{ij}, ϱ_{jk}, *if*

$$\forall m \in \Delta M_i : \varrho^*_{\text{ik}}(m) : R_i(m) \to R_k(m)\,. \tag{72}$$

Theorem 7. *A composition* (71) *exists, if and only if*

$$\Delta M_i \subseteq \Delta M_j\,. \tag{73}$$

Proof. According to (73) there exist non-empty realizations $R_i(m)$, $R_j(m)$ for any $m \in \Delta M_i$. But if there exists $m \in \Delta M_i$, $m \notin \Delta M_j$, and consequently, if (73) is not fulfilled, the realization $R_j(m)$ is empty for this m and therefore the subsequent mapping

$$\varrho_{jk} : R_j(m) \to R_k(m) \tag{74}$$

is not defined in this case.

(End of the proof).

4. INFORMATIVENESS OF FUZZY SETS

The main field of application of the fuzzy sets are processes of classification, namely in connection with decision-making. The informational entropy is a suitable means of evaluation of their informative effect.

Definition 17. *The informational entropy of a complete system of disjoint subsets S_k $(k = 1, \ldots, n)$ on a basic set Z is*

$$E = \sum_{k=1}^{n} p_k \lg p_k \,, \tag{75}$$

where p_k are measures of the respective subsets S_k, such that

$$\sum_{k=1}^{n} p_k = 1 \,, \tag{76}$$

$$\bigcup_{k=1}^{n} S_k = Z \,. \tag{77}$$

As a rule binary logarithms are used in expressing the informational entropy.

In the classical information theory (Shannon) the measure p is interpreted in terms of probability of the respective event, described by the subset S_k.

This kind of interpretation is not applicable to fuzzy sets, because no such probability (except the "*subjective probability*", which consists in a quite arbitrary selection of the respective parameters) can be derived from a single step of the process of classification.

But it is possible to proceed in the following manner:

a) If *a measure μ exists* "a priori" on the basic set Z, it can be used in defining the parameters p_k:

$$p_k = \frac{\mu_k}{\mu_Z} \,, \tag{78}$$

where μ_Z is the measure of the basic set Z,
μ_k is the measure of the event k.

b) If *no measure is defined* on the basic set Z, an auxiliary measure can be introduced for any event k:

$$\mu_k = w_k \,.\, h_k \,, \tag{79}$$

where w_k is an arbitrarily chosen weight coefficient,
h_k is the power of the subset S_k. Then

$$p_k = \frac{w_k \,.\, h_k}{\sum_{k=1}^{n} w_k \,.\, h_k} \,. \tag{80}$$

In *monocriterial classification* the basic set Z is divided at the level m into a realization $R(m)$ and a complementary realization $\bar{R}(m)$, so that the informational entropy of this partitioning at the level m is

$$e(m) = p_R(m) \lg p_R(m) + p_{\bar{R}}(m) \lg p_{\bar{R}}(m) \,. \tag{81}$$

In *multicriterial classification* with m criteria the number of subsets, which are formed on the basic set, is

$$N = 2^m \tag{82}$$

and

$$e(m) = \sum_{k=1}^{N} p_k(m) \lg p_k(m) . \tag{83}$$

Of course some of the subsets may be empty, thus giving a zero contribution to the entropy. Nevertheless the informational effect may be considerably raised in multicriterial classification.

Definition 18. *The informational entropy of realizations and complementary realizations at the level m is given by the formula (83), where p_k are measures or auxiliary measures of the segments induced on the basic set Z by the criterial function (in monocriterial case) or by the ensemble of criterial functions (in multicriterial case).*

Definition 19. *The informational entropy of the fuzzy set is the weighted average of the informational entropies of the realizations at all levels m of the interval Q. The weight $v(m)$ is supposed to be selected in such a way as to give*

$$\int_{q_{\min}}^{q_{\max}} v(m) \, \mathrm{d}m = 1 . \tag{84}$$

The formula of the informational entropy of the fuzzy set is then:

$$E = \int_{q_{\min}}^{q_{\max}} v(m) \, e(m) \, \mathrm{d}m . \tag{85}$$

References

[1] *L. A. Zadeh:* Fuzzy sets. Information and Control 8, 1965, pp. 338—353.

[2] Fuzzy množiny a možnosti jejich uplatnění při řešení systémových úloh (Fuzzy sets and the possibilities of their aplication in solving problems about systems) (Proceedings of a conference). Dům techniky ČsVTS, Praha, 1980.

[3] *A. Kaufmann:* Introduction à la théorie des sous-ensembles flous. Masson, Paris, 1973.

[4] *B. Kummer, B. Straube:* Eine Einführung in die Theorie unscharfer Mengen. Wiss. Zeitschr. der Techn. Universität, Dresden, 26, 1977, No. 2, pp. 363—369.

[5] *M. Peschel:* Modellbildung für Signale und Systeme. VEB Verlag Technik., Berlin 1978.

[6] *J. A. Goguen:* L — fuzzy sets. J. Math. Anal. Appl., 18, 1967, pp. 145—174.

[7] *J. Spal:* Informácia a riadenie (Information and control) Transactions of the 10th Symposium of the Slovak Cybernetic Society, Slovak Academy of Sciences, Bratislava, 1980.

[8] *J. Spal:* Neostrá klasifikácia (Fuzzy classification). Informačné systémy, 9, 1980, No. 5, pp. 413—429.

Souhrn

ZÁKLADY MATEMATICKÉ TEORIE NEOSTRÝCH MNOŽIN

JINDŘICH SPAL

Podstatou neostrých množin je zobrazení z intervalu hodnot kriteriální funkce na systém podmnožin základní množiny. Uvádí se systém definic a vět, přiměřeně vyjadřující tato hlediska. Použitá kriteriální funkce s libovolným intervalem hodnot při tom číselně vyjadřuje skutečnou objektivní vlastnost, která je podkladem při definování neostré množiny.

Věnuje se pozornost vztahu mezi neostrou množinou a náhodnou proměnnou.

Zavádí se pojem informační entropie neostré množiny, který slouží na vyjádření její informační účinnosti.

Author's address: Prof. Ing. *Jindřich Spal,* CSc., VŠT, katedra techn. kybernetiky, Švermova 9, 041 20 Košice.