Igor Bock; Ján Lovíšek

An analysis of a contact problem for a cylindrical shell: A primary and dual formulation

Aplikace matematiky, Vol. 28 (1983), No. 6, 408–429

Persistent URL: http://dml.cz/dmlcz/104054

AN ANALYSIS OF A CONTACT PROBLEM FOR A CYLINDRICAL SHELL: A PRIMARY AND DUAL FORMULATION

IGOR BOCK, JÁN LOVÍŠEK

(Received June 15, 1982)

1. INTRODUCTION

For many applications it is very important to know the behaviour of complicated structures consisting of elementary substructures (for example of various components of a mashine). It is natural that the behaviour of each element of the structure influences the behaviour of the others and vice versa it is influenced by them. Such problems are called the contact problems.

Evaluation of the contact forces of (deformed) bodies is of great importance for the stability estimates of constructions. The first solution of this problem in the three dimensional formulation is due to G. Hertz who introduced the hypothesis (Hertz's hypothesis) about a small contact area as compared with the linear measures of contacting bodies. Contact problems are nonclassical in the sense that the initial contact area or the contact stress are unknown. The intensity of the contact forces of two bodies in a contact assumes a great value on a certain subdomain of the contact area. This is a disturbing phenomenon, because great values of the normal contact stress of a structure causes weariness and even the collapse of the element of the structure. For this reason it is inevitable to correct the contact area in such a way that the points of the normal stress graph are small. It is possible to assume that the function expressing the dependence of pressure on the vector displacement has a known form.

The mathematical model of contact problems does not directly lead to a classical boundary value problem of the elasticity theory. It is because of the unknown zone of contact and of the unknown contact stress along that zone. Unlike the classical problems of the elasticity theory, the set of admissible functions is not a linear space, but only its convex subset. Hence the problem is nonlinear as a consequence of the contact boundary conditions. These boundary conditions are in a form of inequalities. Problems of that form in the elasticity theory were for the first time examined by Signorini. The theory of variational inequalities is a simple but strong device for the study of the above problems. We can find a detailed analysis of the variational inequalities in the monographs Kinderlehrer-Stampacchia [12], Duvaut-Lions [3]. The books Hlaváček, Haslinger, Nečas, Lovíšek [8], Kikuchi-Oden [10], Glowinski, Lions, Trèmolières [4], give their numerical solutions by the method of finite elements.

In this paper we analyze the primary and the dual variational formulation of the contact problem for a cylindrical shell and a stiff punch. We use the method of penalization and duality (Lagrange multipliers). A numerical realization using finite elements is presented for the primary formulation.

2. THE GEOMETRY OF A CYLINDRICAL SHELL

Let $\Omega \subset R^2$ be bounded. $\partial\Omega$ is the boundary of Ω. The middle surface of a cylindrical shell can be considered as the image of $\bar{\Omega}$ with respect to a function $\boldsymbol{\Phi} : \bar{\Omega} \to R^3$. We assume that the boundary $\partial\Omega$ and the function $\boldsymbol{\Phi}$ are sufficiently smooth.

A cylindrical shell is an elastic body T defined in the space R^3 by

$$(2.1) \qquad T = \{M \in R^3 \colon \mathbf{OM} = \boldsymbol{\Phi}(x, \varphi) + z\, \boldsymbol{\nu}(\varphi),\ (x, \varphi) \in \Omega\,,\ -\tfrac{1}{2}e(x, \varphi) \leqq z \leqq \tfrac{1}{2}e(x, \varphi)\}\,,$$

where e is the thickness of the shell, $\boldsymbol{\nu}$ is the normal vector for the middle surface $\mathscr{S}$ and we assume

$$(2.2) \qquad \Omega = [-H, H] \times [\alpha, \beta]\,,\quad \boldsymbol{\Phi}(x, \varphi) = x\mathbf{e}_x + a \cos \varphi \mathbf{e}_y + a \sin \varphi \mathbf{e}_z\,.$$

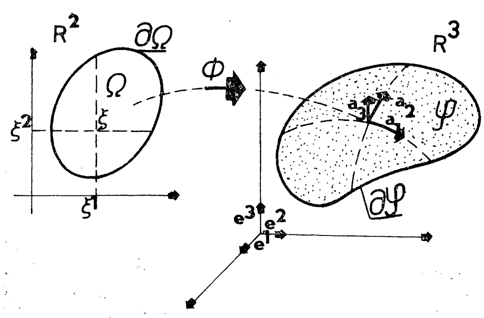

Fig. 1.

For simplicity we consider only kinematic homogeneous boundary conditions on $\partial\Omega$:

$$(2.3) \qquad u = v = w = \frac{\partial w}{\partial n} = 0\,,$$

where $\boldsymbol{n}$ is the normal vector to the surface $\partial\Omega \times [-e; e]$ and $\boldsymbol{u} = \langle u, v, w\rangle$ is the displacement vector of the points on the shell middle surface.

3. FUNCTION SPACES

We denote by $L_2(\Omega)$ the space of all measurable square integrable functions with respect to the Lebesgue measure $\mathrm{d}\Omega = a \, \mathrm{d}x \, \mathrm{d}\varphi$.

Let

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \, \partial \varphi^{\alpha_2}}, \quad |\alpha| = \alpha_1 + \alpha_2 .$$

We introduce the Sobolev spaces

$$H^k(\Omega) = \{v | \, D^\alpha v \in L^2(\Omega); \, |\alpha| \leqq k\} ,$$

$$H^1_0(\Omega) = \{v | \, v \in H^1(\Omega), \, v|_{\partial\Omega} = 0\} ,$$

$$H^2_0(\Omega) = \left\{v | \, v \in H^2(\Omega), \, v|_{\partial\Omega} = \frac{\partial v}{\partial n}\bigg|_{\partial\Omega} = 0\right\}$$

$H^k(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leqq k} \int_\Omega D^\alpha u \, D^\alpha v \, \mathrm{d}\Omega .$$

Further we consider $C^n(\bar{\Omega})$ – the space of n-times continuously differentiable functions defined on $\bar{\Omega}$, $\mathscr{E}(\bar{\Omega})$ – the space of arbitrary differentiable functions on $\bar{\Omega}$.

We denote the indicatrix function of the set $K(\Omega)$ by

$$X_{K(\Omega)}(v) = \begin{cases} 0 & \text{for} \quad v \in K(\Omega) \\ +\infty & \text{for} \quad v \notin K(\Omega) ; \end{cases}$$

$(a \, . \, b)_{R^3} = a \, . \, b = \sum_{i=1}^{3} a_i b_i$ denotes the usual scalar product in R^3.

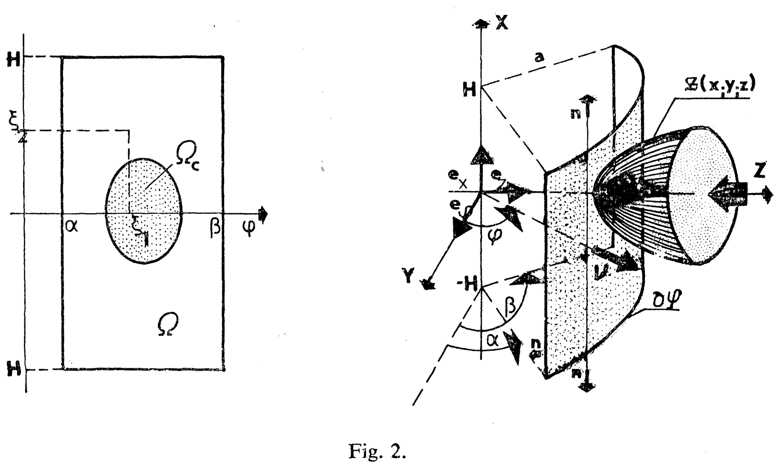

Fig. 2.

The initial configuration of a body T is determined by the position vector $\mathbf{OM} = \mathbf{x} = \langle x, y, z \rangle$, where the components are the Lagrange coordinates of a material point. Let $\mathbf{u}(M)$ be the displacement vector of the point M for the given load. Then the new position of the point M, after the deformation, is described in terms of the new coordinates by

$$\mathbf{OM}^* = \zeta = \mathbf{x} + \mathbf{u}(M), \tag{4.1}$$

where the coordinates of the vector $\zeta = \langle \zeta_1, \zeta_2, \zeta_3 \rangle$ are Euler's coordinates of the material point M^*.

The surface of the punch is described by the equation

$$Z(x, y, z) = 0 \tag{4.2}$$

We further define the contact area $\mathscr{S}_c$ of the shell by

$$\mathscr{S}_c = \{M \in R^3 | \mathbf{OM} = \boldsymbol{\Phi}(x, \varphi) + e\,\mathbf{v}(\varphi);\ (x, \varphi) \in \Omega_c\}.$$

Then we have $Z(M) < 0$ for the inner points of the punch and $Z(M) > 0$ for the points outside the punch. Moreover, we assume $Z(M) < 0$ on a set of positive measure contained in Ω_c, hence the contact of the shell is active.

The stress vector $\mathbf{p}_c$ on the area $\mathscr{S}_c$ of the contact has the form

$$\mathbf{p}^c = \mathbf{p}_T^c + p_N^c \mathbf{v}; \quad (\mathbf{p}_T^c)_i = p_{ij}^c v_j - p_N^c v_i, \tag{4.3}$$

where

$$p_N^c = p_{ij}^c v_i v_j = p_i^c v_i, \tag{4.4}$$

p_{ij}^c are the components of the stress tensor and p_c^i the components of the stress vector on the contact surface $\mathscr{S}_c$. We formulate the contact conditions – the conditions of nonpenetrating (a kinematic restriction on the field of the displacement vectors $\mathbf{u}(M)$ for the contact problem) in the following way:

1°. If after the deformation of the elastic cylindrical shell the points of the surface $\mathscr{S}_c$ are also points of the surface (4.2), hence

$$Z(M + \mathbf{u}(M)) = 0, \quad M \in \mathscr{S}_c, \tag{4.5}$$

then the tangential component $\mathbf{p}_T^c$ of the stress vector $\mathbf{p}_c$ vanishes and the normal component p_N^c is nonpositive.

2°. In the case that

$$Z(M + \mathbf{u}(M)) > 0, \quad M \in \mathscr{S}_c, \tag{4.6}$$

the vector $\mathbf{p}^c$ vanishes at M. However, it is not known when the point $(M + \mathbf{u}(M))$ is at the same time a point of the area (4.2), which is the kernel of our problem. We further assume a sufficiently small value of the displacement vector $\mathbf{u}(M)$,

$|\text{grad } Z(M)| \geqq 1$ for every $M \in \mathcal{S}_c$ and the boundedness of the first and the second derivative of the function $Z(M)$. Hence, making use of the Taylor formula for the function $Z(M) + \mathbf{u}(M))$:

$$Z(M + \mathbf{u}(M)) = Z(M) + \text{grad } Z(M) \,.\, \mathbf{u}(M) + o(\|\mathbf{u}\|)\,,$$

we can linearize the contact conditions and write

$$(4.7) \qquad \begin{aligned} &Z(M) + \mathbf{u}(M) \,.\, \text{grad } Z(M) = 0 \quad \boldsymbol{p}_T^c = \mathbf{O}\,; \quad p_N^c \leqq 0\,, \\ &Z(M) + \mathbf{u}(M) \,.\, \text{grad } Z(M) > 0 \quad p_{ij}^c \nu_j = 0 = p_i^c\,, \quad M \in \mathcal{S}_c\,. \end{aligned}$$

Hence a displacement vector field $\mathbf{u}(M)$ is kinematically admissible for the contact problem if it satisfies the following conditions:

$$(4.8) \qquad \begin{aligned} &1^\circ \quad \Pi(M\,;\, \mathbf{u}(M)) \equiv Z(M) + \mathbf{u}(M) \,.\, \text{grad } Z(M) \geqq 0\,, \\ &2^\circ \quad p_N^c \leqq 0\,, \\ &3^\circ \quad p_N^c\, \Pi(M\,;\, \mathbf{u}(M)) = 0 \end{aligned}$$

for every $M \in \mathcal{S}_c$.

The identity (4.8,3°) expresses the complementary condition for a variational formulation of the problem. The condition (4.8,1°) means that the rigid punch can not penetrate into the body (the cylindrical shell) and the complementary condition, that a nonzero contact stress exists only at such points at which the punch is in accordance with the shell. Due to the Kirchhoff hypothesis about normal element conservation for $M \in \mathcal{S}_c$ we put

$$(4.9) \qquad \mathbf{u}(M) = \left\langle u - e\frac{\partial w}{\partial x}\,,\quad v - \frac{e}{a}\frac{\partial w}{\partial \varphi}\,,\ w \right\rangle,$$

where $\langle u, v, w\rangle$ is the displacement vector of the middle surface $\mathcal{S}_c$.

5. DISPLACEMENTS VARIATIONAL FORMULATION

We search a solution of the contact problem on a convex subset of the space

$$(5.1) \qquad W(\Omega) = H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)\,.$$

Let us define the subspace of $W(\Omega)$ by

$$(5.2) \qquad V(\Omega) = \left\{ \mathbf{u} = \langle u, v, w\rangle \in W(\Omega) \,\middle|\, u = v = w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\},$$

where we consider boundary conditions in the sense of traces. Further, we introduce the set of kinematically admissible displacements by

$$(5.3) \qquad K(\Omega) = \{\mathbf{u} = \langle \psi, \lambda, \omega\rangle \in V(\Omega) |\ \Pi(M;\ \mathbf{u}(M)) \geqq 0 \text{ for a.e. } M \in \mathcal{S}_c\}\,.$$

Lemma 1. *The set $K(\Omega)$ is convex and closed in $V(\Omega)$.*

Proof. Convexity follows directly from the form of the set $K(\Omega)$. Let $(\boldsymbol{u}^k) \in K(\Omega)$ be convergent in $V(\Omega)$. Then the sequence $\Pi(M; \boldsymbol{u}^k(M))$ is convergent in $[L_2(\Omega)]^3$ and $\Pi(M; \boldsymbol{u}^k(M)) \geqq 0$ for every k and a.e. $M \in \mathscr{S}_c$. Then also the limit element $\boldsymbol{u} \in V(\Omega)$ must satisfy the last inequality and hence $\boldsymbol{u} \in K(\Omega)$ and $K(\Omega)$ is closed.

Following [16] we introduce the system of six deformation operators

$$\mathbf{N}(\boldsymbol{u})^{\mathsf{T}} = \langle N_1(\boldsymbol{u}), \ldots, N_6(\boldsymbol{u}) \rangle$$

for

$$(5.4) \qquad \begin{aligned} &N_1(\boldsymbol{u}) = \ell_{11}, \quad N_2(\boldsymbol{u}) = \ell_{22}, \quad N_3(\boldsymbol{u}) = \tfrac{1}{2}(\ell_{12} + \ell_{21}) \\ &N_4(\boldsymbol{u}) = X_{11}, \quad N_5(\boldsymbol{u}) = X_{22}, \quad N_6(\boldsymbol{u}) = X_{12}, \end{aligned}$$

where

$$\ell_{11} = \frac{\partial u}{\partial x}, \quad \ell_{12} = \frac{1}{a}\frac{\partial u}{\partial \varphi}, \quad \ell_{21} = \frac{\partial v}{\partial x}, \quad \ell_{22} = \frac{1}{a}\left(\frac{\partial v}{\partial \varphi} - w\right),$$

$$X_{11} = \frac{\partial^2 w}{\partial x^2}, \quad X_{22} = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right), \quad X_{12} = \frac{1}{2a}\left(2\frac{\partial^2 w}{\partial x\, \partial \varphi} + \frac{\partial v}{\partial x} - \frac{1}{a}\frac{\partial u}{\partial \varphi}\right).$$

Moreover we define the system of six operators of forces and moments

$$\mathbf{S}^{\mathsf{T}} = \langle S_1, S_2, \ldots, S_6 \rangle$$

with

$$(5.5) \qquad \begin{aligned} &S_1 = N_{11}, \quad S_2 = N_{22}, \quad S_3 = 2N_{12}, \\ &S_4 = -M_{11}, \quad S_5 = -M_{22}, \quad S_6 = -2M_{12}, \end{aligned}$$

where

$$\begin{aligned} N_{11} &= B(\ell_{11} + \mu\ell_{22}), & M_{11} &= -D(X_{11} + \mu X_{22}), \\ N_{12} &= \tfrac{1}{2}B(1-\mu)(\ell_{12} + \ell_{21}), & M_{12} &= -D(1-\mu) X_{12}, \\ N_{22} &= B(\ell_{22} + \mu\ell_{11}), & M_{22} &= -D(X_{22} + \mu X_{11}), \\ B &= 2E\frac{e}{1-\mu^2}, & D &= 2E\frac{e^3}{3}(1-\mu^2), \end{aligned}$$

μ – the Poisson number.

Now we have

$$(5.6) \qquad \mathbf{S} = [K]\, \mathbf{N}(\boldsymbol{u}),$$

where $[K] = [K_{ij}]$ is a constant, symmetric and positively definite matrix of the form

$$[K] = \begin{bmatrix} B & B & . & . & . & . \\ B & B & . & . & . & . \\ . & . & 2B(1-\mu) & . & . & . \\ . & . & . & D & D & . \\ . & . & . & D & D & . \\ . & . & . & . & . & 2D(1-\mu) \end{bmatrix}$$

We introduce a bilinear form $a(\mathbf{u}; \mathbf{v})$ on the space $V(\Omega)$ by

$$a(\mathbf{u}; \mathbf{v}) = \int_\Omega \mathbf{N}(\mathbf{u})^\mathrm{T} [K] \, \mathbf{N}(\mathbf{v}) \, \mathrm{d}\Omega = \int_\Omega \mathbf{V}^\mathrm{T} [A] \, \mathbf{U} \, \mathrm{d}\Omega \,, \quad \mathbf{u}, \mathbf{v} \in V(\Omega) \,,$$

where

$$\mathbf{V} = \left\langle \psi, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial \varphi}, \lambda, \frac{\partial \lambda}{\partial x}, \frac{\partial \lambda}{\partial \varphi}, \omega, \frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial \varphi}, \frac{\partial^2 \omega}{\partial x^2}, \frac{\partial^2 \omega}{\partial x \, \partial \varphi}, \frac{\partial^2 \omega}{\partial \varphi^2} \right\rangle^\mathrm{T},$$

$$\mathbf{U} = \left\langle u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial \varphi}, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial \varphi}, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial \varphi}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \, \partial \varphi}, \frac{\partial^2 w}{\partial \varphi^2} \right\rangle^\mathrm{T}$$

(continued p. 415).

Lemma 2. *The bilinear form $a(\cdot; \cdot)$ is continuous on $V(\Omega) \times V(\Omega)$.*

The proof can be found in [16].

An element $\mathbf{u} \in K(\Omega)$ satisfying the variational inequality

$$a(\mathbf{u}; \mathbf{v} - \mathbf{u}) \geqq 0 \quad \text{for all} \quad \mathbf{v} \in K(\Omega) \tag{5.7}$$

or equivalently

$$\mathscr{E}(\mathbf{u}) \leqq \mathscr{E}(\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in K(\Omega) \,, \tag{5.7$_1$}$$

where $\mathscr{E}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}; \mathbf{v})$, is called a weak variational solution of the contact problem. The variational inequality (5.7) corresponds to the principle of virtual work in the unilateral problem for a cylindrical shell.

We can express the bilinear form $a(\mathbf{u}; \mathbf{v})$ in the form

$$a(\mathbf{u}; \mathbf{v}) = \langle \mathscr{A}\mathbf{u}; \mathbf{v} \rangle_{V(\Omega)} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in V(\Omega) \,,$$

where $\mathscr{A} : V(\Omega) \to V^*(\Omega)$ is a linear bounded operator and $\langle \cdot, \cdot \rangle_{V(\Omega)}$ is the duality pairing between $V(\Omega)$ and $V^*(\Omega)$. The inequality (5.7) can be expressed in the form

$$\langle \mathscr{A}\mathbf{u}; \mathbf{v} - \mathbf{u} \rangle_{V(\Omega)} \geqq 0 \,. \tag{5.8}$$

Further we define the operator E as the restriction of the operator $\mathscr{A}$ on the set $D(E) = \{\mathbf{v} \in V(\Omega) | \, \mathscr{A}\mathbf{v} \in [L_2(\Omega)]^3\}$ by

$$E\mathbf{v} = \mathscr{A}\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in D(E) \,.$$

The set $D(E)$ is dense in $[L_2(\Omega)]^3$ and the operator E is unbounded and closed from $[L_2(\Omega)]^3 \to [L_2(\Omega)]^3$. Then for $\mathbf{u} \in V(\Omega) \cap [H^2(\Omega) \times H^2(\Omega) \times H^4(\Omega)] \subset D(E)$ we have

$$a(\mathbf{u}; \mathbf{v}) = (E\mathbf{u}; \mathbf{v})_{[L_2(\Omega)]^3}$$

and the variational equation

$$a(\mathbf{u}; \mathbf{v}) = 0 \quad \text{for all} \quad \mathbf{v} \in [D(\Omega)]^3 \tag{5.9}$$

$$
[A]_{(12,12)}=\frac{Eea}{(1-\mu^2)}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{a^4}+\frac{e^2}{3a^6} & 0 & 0 & 0 & \frac{\mu}{a^2} & \frac{1}{a^3}+\frac{e^2}{6a^5} & 0 & 0 & -\frac{e^2}{6a^5} & 0 & -\frac{\mu e^2}{6a^3}\\
0 & 0 & \frac{1-\mu}{2a^2}\left(1+\frac{e^2}{3a^2}\right) & 0 & \frac{1-\mu}{2a^2} & 0 & 0 & 0 & 0 & 0 & \frac{(1-\mu)\,e^2}{6a^3} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1-\mu}{2a^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{\mu}{a^3} & 0 & 0 & 0 & 1 & \frac{\mu}{a} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{a^3}+\frac{e^2}{6a^5} & 0 & 0 & 0 & \frac{\mu}{a} & \frac{1}{a^2}+\frac{e^2}{12a^4} & 0 & 0 & -\frac{e^2}{12a^4} & 0 & -\frac{e^2}{12a^2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{e^2}{6a^5} & 0 & 0 & 0 & 0 & -\frac{e^2}{12a^4} & 0 & 0 & \frac{e^2}{12a^4} & 0 & \frac{\mu e^2}{12a^2}\\
0 & 0 & -\frac{(1-\mu)\,e^2}{6a^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(1-\mu)\,e^2}{6a^3} & 0\\
0 & -\frac{\mu e^2}{6a^3} & 0 & 0 & 0 & 0 & -\frac{\mu e^2}{12a^2} & 0 & 0 & \frac{\mu e^2}{12a^2} & 0 & \frac{e^2}{12}
\end{bmatrix}
$$

is equivalent to

(5.10) $$Eu = 0$$

(the equilibrium equation in the sense of distribution), whose scalar form is

$$-\frac{\partial}{\partial x}\left\{\frac{Ee^2}{1-\mu}\left[\frac{1}{a^4}\left(1+\frac{e}{3a^2}\right)\frac{\partial u}{\partial x}+\frac{\mu}{a^2}\frac{\partial v}{\partial \varphi}+\frac{1}{a^3}\left(1+\frac{e^2}{6a^2}\right)w-\frac{e^2}{6a^5}\frac{\partial^2 w}{\partial x^2}-\right.\right.$$

$$\left.\left.-\frac{\mu e^2}{6a^3}\frac{\partial^2 w}{\partial \varphi^2}\right]-\frac{\partial}{\partial \varphi}\frac{Ee}{1+\mu}\left[\frac{1}{2a^2}\left(1+\frac{e^2}{3a^2}\right)\frac{\partial u}{\partial \varphi}+\frac{1}{2a^2}\frac{\partial v}{\partial x}-\frac{e^2}{6a^3}\frac{\partial^2 w}{\partial x\,\partial \varphi}\right]\right\}=0\,,$$

$$-\frac{\partial}{\partial x}\left[\frac{Ee}{1+\mu}\frac{1}{2a^2}\left(\frac{\partial u}{\partial \varphi}-\frac{\partial v}{\partial x}\right)\right]-\frac{\partial}{\partial \varphi}\left[\frac{Ee}{1-\mu^2}\left(\frac{\mu}{a^2}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial \varphi}+\frac{\mu}{a}w\right)\right]=0\,,$$

$$\frac{Ee}{1-\mu^2}\left[\frac{1}{a^3}\left(1+\frac{e^2}{6a^2}\right)\frac{\partial u}{\partial x}+\frac{\mu}{a}\frac{\partial v}{\partial \varphi}+\frac{1}{a^2}\left(1+\frac{e^2}{12a^2}\right)w-\frac{e^2}{12a^4}\frac{\partial^2 w}{\partial x^2}-\frac{\mu e^2}{12a^2}\frac{\partial^2 w}{\partial \varphi^2}\right]+$$

$$+\frac{\partial^2}{\partial x^2}\left[\frac{Ee^2}{12a^5(1-\mu^2)}\left(-2\frac{\partial u}{\partial x}-aw+a\frac{\partial^2 w}{\partial x^2}+\mu a^3\frac{\partial^2 w}{\partial \varphi^2}\right)\right]-$$

$$-\frac{\partial^2}{\partial x\,\partial \varphi}\left[\frac{Ee^3}{6a^3(1-\mu)}\left(-\frac{\partial w}{\partial \varphi}+a\frac{\partial^2 w}{\partial x\,\partial \varphi}\right)\right]+$$

$$+\frac{\partial^2}{\partial y^2}\left[\frac{Ee^2}{12a^3(1-\mu^2)}\left(-2\mu\frac{\partial u}{\partial x}-\mu a w+\mu a\frac{\partial^2 w}{\partial x^2}+a^3\frac{\partial^2 w}{\partial \varphi^2}\right)\right]=0\,.$$

We define the unilateral contact problem for a cylindrical shell in the following way:

Problem ($\mathscr{P}$). *To find such a sufficiently smooth vector function* $\mathbf{u}$ *that*

(5.11) $$p_N^c(M) = E\,\mathbf{u}(M) \quad \text{on} \quad \mathscr{S} = \{\mathbf{OM} = \boldsymbol{\Phi}(x,\varphi) + e\,\mathbf{v}(\varphi),\ (x,\varphi)\in\Omega\}$$

and

(5.12) $$\begin{aligned} &\mathbf{p}^c(M) = 0 \quad \text{on} \quad \mathscr{S}-\mathscr{S}_c\,,\\ &p_N^c(M) \leqq 0\,,\quad \Pi(M;\mathbf{u}(M)) \geqq 0 \quad \text{on} \quad \mathscr{S}_c\,,\\ &p_N^c(M)\,\Pi(M;\mathbf{u}(M)) = 0 \quad \text{on} \quad \mathscr{S}_c \quad \text{(a complementary condition)}\,,\\ &\mathbf{p}_T^c(M) = \mathbf{0} \quad \text{on} \quad \mathscr{S}_c\,,\\ &u = v = w = \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned}$$

Theorem 1. *Let* $\mathbf{u}\in K(\Omega)$ *be a solution of* (5.12). *Then*

(5.13) $$a(\mathbf{u};\mathbf{v}-\mathbf{u}) \geqq 0 \quad \text{for all} \quad \mathbf{v}\in K(\Omega)\,.$$

On the other hand, if $\boldsymbol{u} \in K(\Omega)$ *is a solution of* (5.13), *then*

$p_N^c(M) = E\boldsymbol{u}(M) \leqq 0$ (*in the sense of distributions*),

$\Pi(M; \boldsymbol{u}(M)) \geqq 0$ (*in the sense of* $L_2(\Omega_c)$).

Moreover, if $\boldsymbol{u} \in [H^2(\Omega) \times H^2(\Omega) \times H^4(\Omega)] \cap V(\Omega)$, *then*

$E\boldsymbol{u} = p_N^c \in L_2(\Omega_c)$ *and* $p_N^c \Pi(M; \boldsymbol{u}(M)) = 0$ *a.e. in* Ω_c *and* $p_N^c \leqq 0$ *in the sense of* $L_2(\Omega_c)$.

Proof. The conditions (5.12) imply the inequality

$$p_N^c(M)\left[\Pi(M; \boldsymbol{v}(M)) - \Pi(M; \boldsymbol{u}(M))\right] \leqq 0 \quad \text{for all} \quad \boldsymbol{v} \in K(\Omega)\,. \tag{5.14}$$

Multiplying the equation (5.11) by $(\boldsymbol{v} - \boldsymbol{u})$ and integrating by parts we obtain

$$a(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) = \int_{\Omega_c} \boldsymbol{p}^c \,.\, (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}\Omega\,. \tag{5.15}$$

For the active points M of the contact on Ω_c we can write

$$\operatorname{grad} Z(M) = -\gamma\, \varrho(M)\,, \tag{5.16}$$

where $\varrho(M) = |\operatorname{grad} Z(M)| > 0$ (a smooth convex surface $\equiv$ the boundary of the punch). Then we arrive at

$$p_N^c(\boldsymbol{v} - \boldsymbol{u}) \,.\, \operatorname{grad} Z(M) = p_N^c[(Z(M) + \boldsymbol{v} \,.\, \operatorname{grad} Z(M)) - \\ - (Z(M) + \boldsymbol{u} \,.\, \operatorname{grad} Z(M))] = p_N^c[\Pi(M; \boldsymbol{v}(M)) - \Pi(M; \boldsymbol{u}(M))] \leqq 0 \tag{5.17}$$

(due to (5.14)).

Using the condition (5.16) we obtain the estimate

$0 \geqq p_N^c(\boldsymbol{v} - \boldsymbol{u}) \,.\, \operatorname{grad} Z(M) = -p_N^c\, \varrho(M)\, (v_N - u_N)$, which holds for $p_N^c \leqq 0$, $\varrho(M) > 0$ if and only if $(v_N - u_N) \leqq 0$ for every $\boldsymbol{v} \in K(\Omega)$.

As

$$\int_{\Omega_c} \boldsymbol{p}^c \,.\, (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}\Omega = \int_{\Omega_c} (\langle p_N^c \boldsymbol{v} + \boldsymbol{p}_T^c \rangle \,.\, \langle (v_N - u_N)\, \boldsymbol{v} + (\boldsymbol{v}_T - \boldsymbol{u}_T) \rangle)\, \mathrm{d}\Omega =$$

$$= \int_{\Omega_c} p_N^c(v_N - u_N)\, \mathrm{d}\Omega \geqq 0 \ (\boldsymbol{p}_T^c \equiv \boldsymbol{0})\,,$$

we have, taking into account (5.14), the inequality (5.13).

On the other hand let $\boldsymbol{u} \in K(\Omega)$ be a solution of the inequality (5.13). Let $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x, \varphi) = \langle \varphi_1, \varphi_2, \varphi_3 \rangle$ be a smooth vector function that satisfies the homogeneous boundary conditions and its restriction on Ω_c is zero. Then $\boldsymbol{u} \pm \boldsymbol{\varphi} \in K(\Omega)$, and inserting $\mathbf{v} \equiv \mathbf{u} \pm \boldsymbol{\varphi}$ in (5.13) we have $a(\mathbf{u}; \boldsymbol{\varphi}) = 0$. Hence the homogeneous equilibrium equations (the equations (5.10) on $\Omega - \Omega_c$) are fulfilled in the sense of distributions. Then from the inequality (5.13) we obtain the estimate (for a smooth function $\mathbf{u}$)

$$(5.18) \qquad \langle \mathscr{A}\mathbf{u}; \mathbf{v} - \mathbf{u} \rangle_{V(\Omega)} = (E\mathbf{u}; \mathbf{v} - \mathbf{u})_{[L_2(\Omega_c)]^3} =$$

$$= \int_{\Omega_c} \mathbf{p}^c(M) \,.\, (\mathbf{v}(M) - \mathbf{v}(M))\, \mathrm{d}\Omega \geqq 0 \quad \text{for all} \quad \mathbf{v} \in K(\Omega)$$

$$\left(\int_{\Omega - \Omega_c} \mathbf{p}^c(M) \,.\, (\mathbf{v}(M) - \mathbf{u}(M))\, \mathrm{d}\Omega = 0 \quad \text{for all} \quad \mathbf{v} \in K(\Omega) \right).$$

Putting $\boldsymbol{\varphi}(M) = \mathbf{v}(M) - \mathbf{u}(M)$ on Ω and using $\boldsymbol{\varphi}(M) = \varphi_N\, \mathbf{v}(M) + \boldsymbol{\varphi}_T$ we obtain

$$(5.19) \qquad \int_{\Omega_c} p_N^c \varphi_N \, \mathrm{d}\Omega + \int_{\Omega_c} \mathbf{p}_T^c \,.\, \boldsymbol{\varphi}_T \, \mathrm{d}\Omega \geqq 0 \,.$$

If we have contact at a point $M \in \mathscr{S}_c$, then (5.16) yields $\varphi_N(M) \leqq 0$, $\boldsymbol{\varphi}_T(M)$ arbitrary. The set

$$(5.20) \qquad W(\Omega_c) = \{\boldsymbol{\varphi} \in V(\Omega) |\ \varphi_N(M) = 0;\ M \in \mathscr{S}_c\}$$

is a linear subspace of $[L_2(\Omega_c)]^3$. Analyzing (5.19) we have

$$\int_{\Omega_c} \mathbf{p}_T^c(\mathbf{u}) \,.\, \boldsymbol{\varphi}_T \, \mathrm{d}\Omega = 0 \quad \text{for any} \quad \boldsymbol{\varphi} \in W(\Omega_c)\,.$$

Then we have

$$(5.21) \qquad \mathbf{p}_T^c(\mathbf{u}(M)) = \mathbf{0} \quad \text{for any} \quad M \in \mathscr{S}_c$$

as an element of the space $[L_2(\Omega_c)]^3$, which due to (5.19) implies

$$(5.22) \qquad \int_{\Omega_c} p_N^c(\mathbf{u})\,(v_N - u_N)\, \mathrm{d}\Omega \geqq 0 \quad \text{for all} \quad \mathbf{v} \in K(\Omega)\,.$$

Let us denote by $\tilde{\mathscr{S}}_c$ the zone of contact, i.e. the set of such points $M \in \mathscr{S}_c$ that the relation (4.8) is an equality. We put $\varphi_N = v_N - u_N$ on $\mathscr{S}_c$, assuming additionally $\varphi_N = 0$ on $\tilde{\mathscr{S}}_c$. Putting $\mathbf{v} = \mathbf{u} + \boldsymbol{\varphi} \in K(\Omega)$ for $|\varphi|$ sufficiently small with a suitable support, we obtain $p_N^c(M) = 0$ for every $M \in \mathscr{S}_c - \tilde{\mathscr{S}}_c$ as an element of $L_2(\Omega_c)$. The inequality (5.22) now implies

$$\int_{\Omega_c} p_N^c(\mathbf{u})\, \varphi_N \, \mathrm{d}\Omega \geqq 0 \quad \text{for all} \quad \varphi_N \in L_2(\Omega_c)\,, \quad \varphi_N \leqq 0\,,$$

because $\mathbf{v} = \mathbf{u} + \boldsymbol{\varphi} \in K(\Omega)$ for $\varphi_N \leqq 0$, i.e. $p_N^c(\mathbf{u}(M)) \leqq 0$ in the sense of $L_2(\Omega_c)$ for every $M \in \tilde{\mathscr{S}}_c$.

Remark 1. For $p_N^c \leqq 0$ in the sense of distributions we can show the following interpretation. Let $\bar{D}(\Omega) = \{\varphi \in D(\Omega) \mid \varphi \leqq 0\}$. Then $\bar{D}(\Omega)$ is a negative cone of test functions. Hence we define a nonpositive distribution $p_N^c \leqq 0$ if $\langle p_N^c, \varphi \rangle_{D(\Omega_c)} \geqq 0$ for every $\varphi \in D(\Omega_c)$, where $\langle \cdot, \cdot \rangle_{D(\Omega_c)}$ is the pairing between $D^*(\Omega)$ and $D(\Omega)$. The complementary condition $p_N^c \Pi(M; \boldsymbol{u}(M)) = 0$ a.e. on in (5.12) will be verified in Section 7 after using a Lagrange multiplier.

6. EXISTENCE AND UNICITY

We use a technique of Hlaváček, Nečas [8] for the proof of existence of the above problem.

The matrix $[K]$ is positively definite, symmetric and hence

$$a(\boldsymbol{v}; \boldsymbol{v}) \geqq \alpha \int_\Omega |\boldsymbol{N}(\boldsymbol{v})|^2 \, d\Omega = \alpha \sum_{\ell=1}^{6} \|N_\ell(\boldsymbol{v})\|^2_{L_2(\Omega)}, \quad \alpha > 0, \tag{6.1}$$

$$a(\boldsymbol{u}; \boldsymbol{v}) = a(\boldsymbol{v}; \boldsymbol{u}) \text{ for any } \boldsymbol{u}, \ \boldsymbol{v} \in K(\Omega). \tag{6.2}$$

Let $P_{V(\Omega)}$ be the subspace of possible virtual displacements of the middle surface of the shell as of a solid body, i.e.

$$P_{V(\Omega)} = \{\boldsymbol{v} \in V(\Omega) | \sum_{\ell=1}^{6} \|N_\ell(\boldsymbol{v})\|^2_{L_2(\Omega)} = 0\}. \tag{6.3}$$

Lemma 3. *The system of operators* $\{N_\ell(\boldsymbol{u})\}_{\ell=1}^6$ *is coercive on* $V(\Omega)$, *i.e. there exists such a* $c > 0$ *that*

$$\sum_{\ell=1}^{6} \|N_\ell(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{[L_2(\Omega)]^3} \geqq c\|\boldsymbol{u}\|^2_{V(\Omega)} \quad \textit{for all} \quad \boldsymbol{u} \in V(\Omega). \tag{6.4}$$

Further, due to ([8] – 10.4.5) *we have*

$$P_{V(\Omega)} = \{0\}.$$

Then due to (*Lemma* 11.3.2. – [8]), *Lemma* 3 *and the inequality* (6.1) *we obtain*

$$a(\boldsymbol{v}; \boldsymbol{v}) \geqq \alpha_1 \|\boldsymbol{v}\|^2_{V(\Omega)} \quad \text{for all} \quad \boldsymbol{v} \in V(\Omega), \quad \alpha_1 > 0. \tag{6.5}$$

Theorem 2. *There exists a unique solution* $\boldsymbol{u} \in K(\Omega)$ *of the variational inequality*

$$\langle \mathscr{A}\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u} \rangle_{V(\Omega)} = a(\boldsymbol{u}; \boldsymbol{v} - \boldsymbol{u}) \geqq 0 \quad \textit{for any} \quad \boldsymbol{v} \in K(\Omega). \tag{6.6}$$

Proof. We formulate the penalized equation

$$a(\boldsymbol{u}_\varepsilon; \boldsymbol{v}) + \frac{1}{\varepsilon} \langle \beta(\boldsymbol{u}_\varepsilon), \boldsymbol{v} \rangle_{V(\Omega)} = 0 \quad \text{for all} \quad \boldsymbol{v} \in V(\Omega), \tag{6.7}$$

where the penalizator $\beta : V(\Omega) \to V^*(\Omega)$ is of the form

$$(6.8)\quad \langle \beta(\mathbf{u}); \mathbf{v} \rangle_{V(\Omega)} = - \int_{\Omega} [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^- \operatorname{grad} Z(M) \,.\, \mathbf{v}(M) \, d\Omega$$

with $v^+ = \sup(v, 0)$, $v^- = \sup(-v, 0)$, $v = v^+ - v^-$ the linear form $\mathbf{v} \to \int_{\Omega_c} [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^- \operatorname{grad} Z(M) \,.\, \mathbf{v} \, d\Omega$ is continuous on $V(\Omega)$ and defines the functional $\beta(\mathbf{u}) \in V^*(\Omega)$.

The properties of β:

1° Lipschitz continuity: Using the inequality $((a + c)^- - (b + c)^-) \, d \leqq |a - b| \, |d|$ for arbitrary real numbers a, b, c, d, we can write

$$\langle \beta(\mathbf{v}) - \beta(\mathbf{u}); \mathbf{w} \rangle_{V(\Omega)} = \int_{\Omega_c} \{[Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^- -$$

$$- [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{v}(M)]^-\} \operatorname{grad} Z(M) \,.\, \mathbf{w}(M) \, d\Omega \leqq$$

$$\leqq \int_{\Omega_c} |\operatorname{grad} Z(M) \,.\, (\mathbf{u}(M) - \mathbf{v}(M))| \, |\operatorname{grad} Z(M) \,.\, \mathbf{w}(M)| \, d\Omega \leqq$$

$$\leqq \text{const.} \, \|\mathbf{v}(M) - \mathbf{u}(M)\|_{V(\Omega)} \, \|\mathbf{w}(M)\|_{V(\Omega)}$$

(using the theorem on traces [15]);

2° $\beta(\mathbf{u}) = 0 \Leftrightarrow [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^- = 0 \Leftrightarrow \mathbf{u} \in K(\Omega)$ (considering the estimate $|\operatorname{grad} Z(M)| \geqq 1$);

3° monotonicity on $V(\Omega)$:

$$\langle \beta(\mathbf{v}) - \beta(\mathbf{u}); \mathbf{v} - \mathbf{u} \rangle_{V(\Omega)} =$$

$$= - \int_{\Omega} \{[Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{v}(M)]^- - [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^-\}$$

$$[\operatorname{grad} Z(M) \,.\, (\mathbf{v}(M) - \mathbf{u}(M)] \, d\Omega =$$

$$= - \int_{\Omega} \{[Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{v}(M)]^- - [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^-\}$$

$$\{[Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{v}(M)]^- - [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]\} \, d\Omega \geqq$$

$$\geqq \int_{\Omega} \{[Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{v}(M)]^- - [Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}(M)]^-\}^2 \, d\Omega \geqq 0 \,,$$

where we have used the relations: $-(a^- - b^-)(a - b) = -(a^- - b^-) \,.\, [a^+ - a^-) - (b^+ - b^-)] = -(a^- - b^-)(a^+ - b^+) + (a^- - b^-)^2 \geqq \geqq (a^- - b^-)^2$ for any $a, b \in R$;

4° hemicontinuity: It is a consequence of Lipschitz continuity of the operator β, i.e. for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V(\Omega)$ the function $\lambda \to \langle \beta(\mathbf{u} + \lambda \mathbf{v}); \mathbf{w} \rangle_{V(\Omega)}$ is continuous on R.

Applying the theory of monotone operators [14] we obtain a unique solution $\boldsymbol{u} \in V(\Omega)$ of the penalized equation (6.7).

Now we verify that the solutions $\boldsymbol{u}_\varepsilon \in V(\Omega)$ are bounded with respect to ε. Let $\boldsymbol{v}_0 \in K(\Omega)$ be an arbitrary element. Inserting $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \boldsymbol{v}_0$ in the equation (6.7) and using $\beta(\boldsymbol{v}_0) = 0$ we arrive at

$$a(\boldsymbol{u}_\varepsilon;\ \boldsymbol{u}_\varepsilon - \boldsymbol{v}_0) + \frac{1}{\varepsilon}\langle\beta(\boldsymbol{u}_\varepsilon) - \beta(\boldsymbol{v}_0);\ \boldsymbol{u}_\varepsilon - \boldsymbol{v}_0\rangle_{V(\Omega)} = 0\,. \tag{6.9}$$

The monotonicity of β implies

$$a(\boldsymbol{u}_\varepsilon;\ \boldsymbol{u}_\varepsilon - \boldsymbol{v}_0) \leqq 0\,, \tag{6.10}$$

while the ellipticity of the form $a(\cdot;\ \cdot)$ yields

$$C_1\|\boldsymbol{u}_\varepsilon\|^2_{V(\Omega)} \leqq C_2\|\boldsymbol{u}_\varepsilon\|_{V(\Omega)}\ \|\boldsymbol{v}_0\|_{V(\Omega)}$$

and

$$\|\boldsymbol{u}_\varepsilon\|_{V(\Omega)} \leqq C\,, \tag{6.11}$$

where C does not depend on ε.

We can extract such a subsequence (denoting it again by $\boldsymbol{u}_\varepsilon$) that

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \quad \text{(weakly) in } V(\Omega) \text{ for } \quad \varepsilon \to 0\,. \tag{6.12}$$

Further the equation (6.7) and the estimate (6.11) imply

$$\|\beta(\boldsymbol{u}_\varepsilon)\|_{V^*(\Omega)} = \sup_{\boldsymbol{v}\neq 0}\frac{-a(\boldsymbol{u}_\varepsilon;\ \boldsymbol{v})}{\|\boldsymbol{v}\|_{V(\Omega)}} = O(\varepsilon)\,, \tag{6.13}$$

and, due to (6.11), (6.13) we have

$$\lim_{\varepsilon\to 0}\langle\beta(\boldsymbol{u}_\varepsilon);\ \boldsymbol{u}_\varepsilon - \boldsymbol{v}\rangle_{V(\Omega)} = 0\,.$$

Using the monotonicity of β and the weak convergence (6.12) we arrive at

$$\langle-\beta(\boldsymbol{v});\ \boldsymbol{u} - \boldsymbol{v}\rangle_{V(\Omega)} \geqq 0 \quad \text{for all} \quad \boldsymbol{v} \in V(\Omega)\,. \tag{6.14}$$

Let $\boldsymbol{v} = \boldsymbol{u} + \lambda\boldsymbol{w}$; $\lambda > 0$; $\boldsymbol{w} \in V(\Omega)$. Then the inequality (6.14) implies

$$\langle\beta(\boldsymbol{u} + \lambda\boldsymbol{w});\ \boldsymbol{w}\rangle_{V(\Omega)} \geqq 0 \quad \text{for any} \quad \lambda > 0\,. \tag{6.15}$$

Using the hemicontinuity of β we obtain after $\lambda \to 0$

$$\langle\beta(\boldsymbol{u});\ \boldsymbol{w}\rangle_{V(\Omega)} \geqq 0 \quad \text{for all} \quad \boldsymbol{w} \in V(\Omega) \tag{6.16}$$

which implies $\beta(\boldsymbol{u}) = 0$ and $\boldsymbol{u} \in K(\Omega)$.

Inserting in (6.7) $\mathbf{v} - \mathbf{u}_\varepsilon$ instead of $\mathbf{v}$; $\mathbf{v} \in K(\Omega)$, we have

$$a(\mathbf{u}_\varepsilon; \mathbf{v} - \mathbf{u}_\varepsilon) = \frac{1}{\varepsilon} \langle \beta(\mathbf{v}) - \beta(\mathbf{u}_\varepsilon); \mathbf{v} - \mathbf{u}_\varepsilon \rangle_{V(\Omega)} \geqq 0 . \tag{6.17}$$

The bilinear form $a(\mathbf{v}; \mathbf{v})$ is weakly lower semicontinuous on $V(\Omega)$ and (6.12) implies

$$a(\mathbf{u}; \mathbf{u}) \leqq \liminf_{\varepsilon \to 0} a(\mathbf{u}_\varepsilon; \mathbf{u}_\varepsilon) \tag{6.18}$$

and the inequality (6.6) follows immediately from (6.17).

It remains to verify the unicity of a solution of (6.6). If $\mathbf{u}_1, \mathbf{u}_2 \in K(\Omega)$ are two solutions of (6.6), then $a(\mathbf{u}_1 - \mathbf{u}_2; \mathbf{u}_1 - \mathbf{u}_2) \leqq 0$ and $\mathbf{u}_1 = \mathbf{u}_2$, because the form $a(\mathbf{v}; \mathbf{v})$ is coercive – (6.5).

We have verified that a sequence of solutions $(\mathbf{u}_\varepsilon)$ of penalized equations converges to a solution $\mathbf{u} \in K(\Omega)$ of the variational inequality (6.6), if $\varepsilon \to 0$. We shall verify, using the Lagrange multiplier, that the approximate force $p^c_{N\varepsilon} = -1/\varepsilon\, (\Pi(M; \mathbf{u}_\varepsilon))^- \,.$ $.\, \operatorname{grad} Z(M)$ converges to the contact forces for the unilateral problem, if $\varepsilon \to 0$.

The physical meaning of the penalized member is described in [11] and [16]. The physical idea is rather simple: the solid punch is approximated by continuously distributed feathers with rigidity $1/\varepsilon$ for sufficiently small $\varepsilon > 0$. If $\varepsilon \to 0$ the feather of foundation turns into the rigid one and the condition (4.8) can be replaced by the relation

$$p^c_{N\varepsilon} = \frac{1}{\varepsilon} \left[Z(M) + \operatorname{grad} Z(M) \,.\, \mathbf{u}_\varepsilon(M) \right]^- \operatorname{grad} Z(M) , \tag{6.19}$$

which represents a boundary condition for a unilateral feather supporting. Replacement of the condition (4.8) by the relation (6.9) means that the intensive contact stress is determined on the region Ω_c as the penalization, if the obstacle $\Pi(M; \mathbf{u}) \geqq 0$ generated by the solid punch is abolished. Replacement of the solid punch by a system of solid feathers is called the penalization of the obstacle (4.8) with ε as a parameter of penalization.

7. A DUAL FORMULATION OF THE PROBLEM ($\mathscr{P}$)

We introduce a real function $\omega(\mathbf{v}, \mu)$ by

$$\omega(\mathbf{v}, \mu) = \int_{\Omega_c} \mu \Pi(M; \mathbf{v})\, \mathrm{d}\Omega ; \quad \mathbf{v} \in V(\Omega) , \quad \mu \in \Lambda^-(\Omega_c) , \tag{7.1}$$

where

$$\Lambda^-(\Omega_c) = \{\mu \in L_2(\Omega_c) |\ \mu \leqq 0\}\text{-convex cone} . \tag{7.2}$$

A function $\omega(\mathbf{v}, \mu)$ is homogeneous with respect to μ:

$$\omega(\mathbf{v}; k\mu) = k\, \omega(\mathbf{v}; \mu) \quad \text{for any} \quad \mu \leqq 0 , \quad k \in R . \tag{7.3}$$

The convex set $K(\Omega)$ can be characterized by

$$\mathbf{v} \in K(\Omega) \Leftrightarrow \int_{\Omega_c} \mu \Pi(M, \mathbf{v}) \, \mathrm{d}\Omega \leqq 0 \quad \text{for any} \quad \mu \in \Lambda^-(\Omega_c) . \tag{7.4}$$

We introduce the Lagrangian by

$$\mathscr{L}(\mathbf{v}, \mu) = \mathscr{E}(\mathbf{v}) + \int_{\Omega_c} \mu \Pi(M, \mathbf{v}) \, \mathrm{d}\Omega \quad \text{on} \quad V(\Omega) \times L_2(\Omega_c) , \tag{7.5}$$
$$\mathscr{E}(\mathbf{v}) = \tfrac{1}{2} a(\mathbf{v}; \mathbf{v}) .$$

Let us denote

$$\mathscr{E}_{K(\Omega)}(\mathbf{v}) = \begin{cases} \mathscr{E}(\mathbf{v}); & \mathbf{v} \in K(\Omega) , \\ +\infty; & \mathbf{v} \notin K(\Omega) . \end{cases} \tag{7.6}$$

Lemma 4. *Let* (7.2) *hold. Then*

$$\mathscr{E}_{K(\Omega)}(\mathbf{v}) = \sup_{\mu \in \Lambda^-(\Omega_c)} \mathscr{L}(\mathbf{v}; \mu) . \tag{7.7}$$

Proof. We have $\int_{\Omega_c} \mu \Pi(M; \mathbf{v}) \, \mathrm{d}\Omega \leqq 0$ for all $\mathbf{v} \in K(\Omega)$ and $\mu \in \Lambda^-(\Omega_c)$. Moreover, $0 \in \Lambda^-(\Omega_c)$ which implies

$$\sup_{\mu \in \Lambda^-(\Omega_c)} \mathscr{L}(\mathbf{v}, \mu) = \mathscr{E}(\mathbf{v}) + \sup_{\mu \in \Lambda^-(\Omega_c)} \int_{\Omega_c} \mu \Pi(M, \mathbf{v}) \, \mathrm{d}\Omega = \mathscr{E}(\mathbf{v}) = \mathscr{E}_{K(\Omega)}(\mathbf{v}) .$$

Next, if $\mathbf{v} \notin K(\Omega)$, there exists such an element $\mu^* \in \Lambda^-(\Omega_c)$ that $\int_{\Omega_c} \mu^* \Pi(M, \mathbf{v}) \, \mathrm{d}\Omega = \Theta > 0$. However, $\Lambda^-(\Omega_c)$ is a cone and $k\mu \in \Lambda^-(\Omega_c)$ for $k > 0$. Moreover, $\int_{\Omega_c} k\mu^* \Pi(M, \mathbf{v}) \, \mathrm{d}\Omega = k\Theta > 0$. Consequently,

$$\sup_{\mu \in \Lambda^-(\Omega_c)} \mathscr{L}(\mathbf{v}, \mu) \geqq \sup_{k \geqq 0} \mathscr{L}(\mathbf{v}; k\mu) = \mathscr{E}(\mathbf{v}) + \sup_{k \geqq 0} k\Theta = +\infty = \mathscr{E}_{K(\Omega)}(\mathbf{v}) .$$

If an element $\lambda \in \Lambda^-(\Omega_c)$ satisfies the relations

$$\inf_{\mathbf{v} \in K(\Omega)} \mathscr{E}(\mathbf{v}) = \inf_{\mathbf{v} \in V(\Omega)} \sup_{\mu \in \Lambda^-(\Omega_c)} \mathscr{L}(\mathbf{v}; \mu) = \inf_{\mathbf{v} \in V(\Omega)} \mathscr{L}(\mathbf{v}; \lambda) \tag{7.8}$$

then it is called the Lagrange multiplier of the original problem and λ is a solution of the dual problem.

Dual problem $(\mathscr{P}^*)$. To find the element $\lambda \in \Lambda^-(\Omega_c)$ realizing

$$\sup_{\mu \in \Lambda^-(\Omega_c)} \left\{ \inf_{\mathbf{v} \in V(\Omega)} \left[\mathscr{E}(\mathbf{v}) + \int_{\Omega_c} \mu \Pi(M; \mathbf{v}) \, \mathrm{d}\Omega \right] \right\} . \tag{7.9}$$

Lemma 5. *There exists at least one solution of the problem* $(\mathscr{P}^*)$.

Proof. For $\mu \in \Lambda^-(\Omega_c)$ there exists an element $\mathbf{u}_\mu \in K(\Omega)$ realizing

$$\inf_{\mathbf{v} \in V(\Omega)} \left\{ \mathscr{E}(\mathbf{v}) + \int_{\Omega_c} \mu \Pi(M; \mathbf{v}) \, \mathrm{d}\Omega \right\} .$$

Denote $M(\mu) = \inf_{\mathbf{v} \in V(\Omega)} \{E(\mathbf{v}) + \int_{\Omega_c} \mu \Pi(M; \mathbf{v})\, d\Omega\}$. Then the function $\mu \to M(\mu)$ is concave and weakly upper semicontinuous. As $\Lambda^-(\Omega_c)$ is convex and closed in $L_2(\Omega_c)$, there exists an element $\lambda \in \Lambda^-(\Omega_c)$ for which $M(\lambda) = \sup_{\mu \in \Lambda^-(\Omega_c)} M(\mu)$, which implies the proof of Lemma 5.

The next step is to verify the existence of an element $\lambda \in \Lambda^-(\Omega_c)$ satisfying the equation (7.8). Hence the original problem on $K(\Omega)$ is equivalent to the minimum problem on $V(\Omega)$ (the space without any obstacle).

Lemma 6. *Let the set $\Lambda^-(\Omega_c)$ be defined by* (7.2). *Then $\lambda \in \Lambda^-(\Omega_c)$ is theLagrange multiplier and $\mathbf{u} \in K(\Omega)$ minimizes $\mathscr{E}(\mathbf{v})$ on $K(\Omega)$ if and only if*

$$(7.10)\quad 1^\circ\ \mathscr{E}(\mathbf{u}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega \leqq \mathscr{E}(\mathbf{v}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{v})\, d\Omega \quad \text{for all} \quad \mathbf{v} \in K(\Omega),$$

$$2^\circ \int_{\Omega_c} \mu \Pi(M; \mathbf{u})\, d\Omega \leqq 0 \quad \text{for all} \quad \mu \in \Lambda^-(\Omega_c),$$

$$3^\circ \int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega = 0.$$

Proof. If the relations (7.10) hold, then $\mathbf{u} \in K(\Omega)$ due to (7.9,2°) and $\mathbf{u}$ minimizes $\mathscr{E}(\mathbf{v})$ on $K(\Omega)$ due to

$$\mathscr{E}(\mathbf{u}) = \mathscr{E}(\mathbf{u}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega \leqq \mathscr{E}(\mathbf{v}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{v})\, d\Omega \leqq \mathscr{E}(\mathbf{v})$$

and

$$(7.11)\qquad \inf_{\mathbf{v} \in K(\Omega)} \mathscr{E}(\mathbf{v}) = \inf_{\mathbf{v} \in K(\Omega)} \mathscr{L}(\mathbf{v}; \lambda).$$

Conversely, suppose that λ is the Lagrange multiplier and $\mathbf{u}$ minimizes $\mathscr{E}(\mathbf{v})$ on $K(\Omega)$. As $\mathbf{u} \in K(\Omega)$ we have $\int_{\Omega_c} \mu \Pi(M; \mathbf{u})\, d\Omega \leqq 0$ for all $\mu \in \Lambda^-(\Omega_c)$ and in particular $\int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega \leqq 0$. On the other hand, due to (7.8)

$$\mathscr{E}(\mathbf{u}) = \inf_{\mathbf{v} \in V(\Omega_c)} \mathscr{L}(\mathbf{v}; \lambda) \leqq \mathscr{L}(\mathbf{u}; \lambda) = \mathscr{E}(\mathbf{u}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega$$

which implies $\int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega \geqq 0$ and hence $\int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega = 0$.

We now have

$$(7.12)\qquad \mathscr{E}(\mathbf{u}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{u})\, d\Omega = \mathscr{E}(\mathbf{u}) = \inf_{\mathbf{v} \in V(\Omega)} \mathscr{L}(\mathbf{v}; \lambda) = = \inf_{\mathbf{v} \in V(\Omega)} \left[\mathscr{E}(\mathbf{v}) + \int_{\Omega_c} \lambda \Pi(M; \mathbf{v})\, d\Omega \right].$$

We can formulate the following theorem:

Theorem 3. *The element* $\langle \mathbf{u}; \lambda \rangle \in K(\Omega) \times \Lambda^{-}(\Omega_c)$ *is the saddle point of the functional* $\mathscr{L}(\mathbf{v}; \mu)$, *i.e.*

$$(7.13) \qquad \mathscr{L}(\mathbf{u}, \mu) \leqq \mathscr{L}(\mathbf{u}, \lambda) \leqq \mathscr{L}(\mathbf{v}, \lambda) \quad \textit{for any} \quad \mu \in \Lambda^{-}(\Omega_c), \quad \mathbf{v} \in V(\Omega).$$

Proof. The assertion of the theorem is a consequence of Lemmas 4, 5, 6.

Lemma 7. *The element* $\langle \mathbf{u}; \lambda \rangle$ *solves the problem* (7.13) *if and only if*

$$(7.14) \qquad a(\mathbf{u}; \mathbf{v}) + \int_{\Omega_c} \lambda \operatorname{grad} Z(M) \, . \, \mathbf{v} \, d\Omega = 0 \,,$$

$$\int_{\Omega_c} (\mu - \lambda) \, \Pi(M; \mathbf{u}) \, d\Omega \leqq 0 \quad \textit{for any} \quad \mathbf{v} \in V(\Omega), \quad \lambda \in \Lambda^{-}(\Omega_c).$$

Proof. The equality in (7.14) expresses the condition $\operatorname{grad}_{\mathbf{v}} \mathscr{L}(\mathbf{v}; \lambda) = 0$, which is equivalent to the right hand side of (7.13), the inequality in (7.14) is identified with the left hand side of (7.13).

Integrating the equation in (7.14) by parts we obtain (for a smooth $\mathbf{u}$ and vanishing tangential contact forces)

$$E\mathbf{u} = 0 \text{ — the equilibrium equation} \,,$$

$$-\int_{\Omega_c} p^c_{ij} v_j \, d\Omega + \int_{\Omega_c} \lambda \frac{\partial Z}{\partial x_i} \, d\Omega = 0 \,,$$

and hence we have

$p^c_{ij} v_j = \lambda(\partial Z / \partial x_i)$ — the equilibrium equation on the contact area of the cylinder.

A physical interpretation of the Lagrange multiplier λ can be given in the following way. Due to the vanishing forces we have the relation

$$p^c_N = \lambda \frac{\partial Z}{\partial \mathbf{v}} \,; \quad M \in \Omega_c \,.$$

Also we have

$$\lambda = p^c_N \cdot \left(\frac{\partial Z}{\partial \mathbf{v}} \right)^{-1} .$$

8. FINITE ELEMENTS APPROXIMATION

The approximation of the problem $(\mathscr{P})$ consists of two steps.

1° Replacing the problem $(\mathscr{P})$ by the finite dimensional problem $(\mathscr{P}_h)$.

2° Numerical solution of the problem $(\mathscr{P}_h)$.

The problem $(\mathscr{P}_h)$ means in this case the finite element approximation of the problem $(\mathscr{P})$. Let $\{\mathscr{T}_h\}$, $0 < h \leqq h_0 < \infty$, be a regular system of triangulations of the region Ω. That means that

1. $\bar{\Omega} = \bigcup_{T_i \in \mathscr{T}_h} T_i, \; i = 1, 2, \ldots, n(h)$,

where $h > 0$ is the maximal length of sides of all triangles from $\mathscr{T}_h$.

2. There exists $\vartheta_0 > 0$ such that

$$\min_{\vartheta^h \in \mathscr{T}_h} \vartheta^h \geqq \vartheta_0 \quad \text{for any} \quad h \in (0, h_0\rangle\,,$$

where ϑ^h is an arbitrary interior angle in an arbitrary triangle of the triangulation $\mathscr{T}_h$.

If $\mathscr{M}_h$ denotes the set of all nodes of the triangulation $\mathscr{T}_h$, then we assume

$$\mathscr{M}_{h_1} \subset \mathscr{M}_{h_2} \quad \text{if} \quad h_1 > h_2\,. \tag{8.1}$$

Let $T \in \mathscr{T}_h$ be the triangle with vertices a_1, a_2, a_3, mid-points $b_j = \frac{1}{2}(a_{j-1} + a_{j+1})$ and let c_j be the intersection of the sides $a_{j-1}a_{j+1}$ and their normals $\mathbf{v}_j$, $j = 1, 2, 3$. (We denote $a_0 = a_3$, $a_4 = a_1$).

It is known from the interpolation theory [2] that the following 21 values – degrees of freedom

$$\Sigma_T = \{p(a_i),\ D\,p(a_i)(a_{i-1} - a_i),\ D\,p(a_i)(a_{i+1} - a_i),\ 1 \leqq i \leqq 3\,; \tag{8.2}$$
$$D^2\,p(a_i)(a_{j+1} - a_j)^2,\ 1 \leqq i, j \leqq 3;\ D\,p(b_i)(a_i - c_i),\ 1 \leqq i \leqq 3\}$$

uniquely determine a polynomial of the fifth degree $p_5 \in P_5(T_i)$ – the Argyris element.

For every triangulation $\mathscr{T}_h$ we now introduce finite dimensional spaces

$$X_h(\Omega) = \{v_h \in C^0(\bar{\Omega}) \mid v_{h|T_i} \in P_2(T_i) \text{ for any } T_i \in \mathscr{T}_h,\ v_h = 0 \text{ on } \partial\Omega\} \tag{8.3}$$

$$Y_h(\Omega) = \left\{v_h \in C^1(\bar{\Omega}) \mid v_{h|T_i} \in P_5(T_i) \text{ for any } T_i \in \mathscr{T}_h,\ v_h = \frac{\partial v_h}{\partial n} = 0 \text{ on } \partial\Omega\right\}. \tag{8.4}$$

The space

$$V_h(\Omega) = X_h(\Omega) \times X_h(\Omega) \times Y_h(\Omega) \tag{8.5}$$

is a finite dimensional subspace of the space $V(\Omega)$. We define a finite dimensional approximation of the convex set $K(\Omega)$ by

$$K_h(\Omega) = \{\mathbf{u}_h = \langle u_h, v_h, w_h\rangle \in V_h(\Omega) \mid \Pi(a_i; \mathbf{u}_h(a_i)) \geqq 0 \tag{8.6}$$
$$\text{for any } a_i \in \mathscr{M}_h \cap \Omega_c\}\,.$$

We can now proceed to the finite dimensional approximation of the problem $(\mathscr{P})$:

Problem $(\mathscr{P}_h)$. *To find a vector-function $\mathbf{u}_h \in K_h(\Omega)$ such that*

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \geqq 0 \quad \text{for any} \quad \mathbf{v}_h \in K_h(\Omega)\,. \tag{8.7}$$

It can be verified in the same way as in the case of the set $K(\Omega) \subset V(\Omega)$ that the set $K_h(\Omega) \subset V_h(\Omega)$ is convex and closed. Then we obtain a theorem analogous to Theorem 2.

Theorem 4. *There exists a unique solution $\boldsymbol{u}_h \in K_h(\Omega)$ of the problem $(\mathscr{P}_h)$ for every $h \in (0, h_0\rangle$.*

We further show that a sequence $\boldsymbol{u}_h$ of solutions of the problems $(\mathscr{P}_h)$ converges to the solution $\boldsymbol{u}$ of the problem $(\mathscr{P})$. First we verify the weak convergence.

Lemma 8. *Let $\boldsymbol{u}_h \in K_h(\Omega)$ be a solution of the problem $(\mathscr{P}_h)$ for every $h \in (0, h_0\rangle$, let $\boldsymbol{u} \in K(\Omega)$ be a solution of the problem $(\mathscr{P})$. Then*

$$\boldsymbol{u}_h \rightharpoonup \boldsymbol{u} \quad \text{weakly in } V(\Omega) \text{ for } \quad h \to 0\,. \tag{8.8}$$

Proof. Choose a sequence $\{\boldsymbol{v}_h\} \in K_h(\Omega)$ such that $\|\boldsymbol{v}_h\|_{V(\Omega)} \leqq C_1$ for every $h \in (0, h_0\rangle$. Due to the coercivity of the form $a(\cdot\,;\,\cdot)$ we obtain the boundedness of the sequence $\{\boldsymbol{u}_h\} \in K_h(\Omega)$ of solutions of the problems $(\mathscr{P}_h)$:

$$\|\boldsymbol{u}_h\|_{V(\Omega)} \leqq C \text{ for any } h \in (0, h_0\rangle\,. \tag{8.9}$$

Then there exists a subsequence chosen from $\{\boldsymbol{u}_h\}$ (denoted again by $\{\boldsymbol{u}_h\}$) such that

$$\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}^* \quad \text{weakly in} \quad V(\Omega)\,. \tag{8.10}$$

We have to verify $\boldsymbol{u}^* = \boldsymbol{u}$ is a solution of the problem $(\mathscr{P})$. First we show that $\boldsymbol{u}^* = \langle u^*, v^*, w^* \rangle \in K(\Omega)$, i.e.

$$\Pi(M, \boldsymbol{u}^*(x, \varphi)) \geqq 0 \quad \text{for any} \quad (x, \varphi) \in \Omega\,. \tag{8.11}$$

As $\boldsymbol{u}_h = \langle u_h, v_h, w_h \rangle \in K_h(\Omega)$, we have

$$\Pi(M, \boldsymbol{u}_h(a_i)) \geqq 0 \quad \text{for any} \quad a_i \in \mathscr{M}_h \cap \Omega_c\,. \tag{8.12}$$

Let $\varepsilon > 0$. As the function $\boldsymbol{u}_h$ is uniformly continuous on $\bar{\Omega}$, there exists such a number $h_1 \in (0, h_0\rangle$ that

$$\Pi(M, \boldsymbol{u}_h(x, \varphi)) \geqq -\varepsilon \quad \text{for any} \quad (x, \varphi) \in \Omega_c\,. \tag{8.13}$$

The set

$$K_\varepsilon(\Omega) = \{\boldsymbol{u} = \langle u, v, w \rangle \in V(\Omega) \mid \Pi(M, \boldsymbol{u}(M)) \geqq -\varepsilon \text{ for any } M \in \Omega_c\}$$

is convex, closed and hence weakly closed in $V(\Omega)$. Thus we have with respect to (8.10) $\boldsymbol{u}^* \in K(\Omega)$ and hence

$$\Pi(M, \boldsymbol{u}^*(x, \varphi)) \geqq -\varepsilon \quad \text{for any} \quad (x, \varphi) \in \Omega_c\,. \tag{8.14}$$

As $\varepsilon > 0$ is an arbitrary positive number, we obtain

$$\Pi(M, \boldsymbol{u}^*(x, \varphi)) \geqq 0 \quad \text{for any} \quad (x, \varphi) \in \Omega_c\,, \tag{8.15}$$

which means that $\boldsymbol{u}^* \in K(\Omega)$.

It remains to show that $\boldsymbol{u}^*$ is a solutions of the problem $(\mathscr{P})$. Let $\boldsymbol{v} \in K(\Omega) \cap \cap [C^\infty(\bar{\Omega})]^3$. Denote by $\boldsymbol{v}_h^I$ the $V_h(\Omega)$ – interpolation polynomial belonging to the function $\boldsymbol{v} \in K(\Omega)$. Obviously $\boldsymbol{v}_h^I \in K_h(\Omega)$, because $\Pi(M, \boldsymbol{v}_h^I(a_i)) = \Pi(M, \boldsymbol{v}(a_i)) \geqq 0$ for every $a_i \in \mathscr{M}_h \cap \Omega_c$. This means

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h^I - \boldsymbol{u}_h) \geqq 0 . \tag{8.16}$$

Using the estimate [2]

$$\|\boldsymbol{v} - \boldsymbol{v}_h^I\|_{V(\Omega)} \leqq c\, h \|\boldsymbol{v}\|_{[H^2(\Omega) \times H^2(\Omega) \times H^4(\Omega)]} \text{ for any } \boldsymbol{v} \in V(\Omega) \cap [C^\infty(\bar{\Omega})]^3 , \tag{8.17}$$

we can write

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h^I - \boldsymbol{u}_h) = -a(\boldsymbol{u}_h, \boldsymbol{v} - \boldsymbol{v}_h^I) + a(\boldsymbol{u}_h, \boldsymbol{v}) - a(\boldsymbol{u}_h, \boldsymbol{u}_h) .$$

The function $a(\cdot, \cdot)$ is lower semicontinuous on $V(\Omega)$ and by (8.10), (8.17) we obtain the inequality

$$a(\boldsymbol{u}^*, \boldsymbol{v} - \boldsymbol{u}^*) \geqq 0 \quad \text{for any} \quad \boldsymbol{v} \in K(\Omega) \cap [C^\infty(\bar{\Omega})]^3 \tag{8.18}$$

by letting $h \to 0+$.

It is verified in [6] that the set $K(\Omega) \cap [C^\infty(\bar{\Omega})]^3$ is dense in $K(\Omega)$ and the inequality (8.18) holds also for arbitrary $\boldsymbol{v} \in K(\Omega)$. Hence $\boldsymbol{u}^*$ is a solution of the problem $(\mathscr{P})$. We have then $\boldsymbol{u}^* = \boldsymbol{u}$ due to the unicity of the solution of $(\mathscr{P})$ and the proof is completed.

The following theorem asserts the strong convergence of the sequence $\{\boldsymbol{u}_h\}$.

Theorem 5. *Let $\boldsymbol{u}_h \in K_h(\Omega)$, $h \in (0, h_0\rangle$ and $\boldsymbol{u} \in K(\Omega)$ be solutions of the problem $(\mathscr{P}_h)$ and $(\mathscr{P})$, respectively. Then*

$$\lim_{h \to 0+} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{V(\Omega)} = 0 . \tag{8.19}$$

Proof. Using the inequalities (6.5), (8.7) we obtain the estimates

$$c_1 \|\boldsymbol{u}_h - \boldsymbol{u}\|^2_{V(\Omega)} \leqq a(\boldsymbol{u}_h - \boldsymbol{u},\ \boldsymbol{u}_h - \boldsymbol{u}) \leqq a(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_h) +$$
$$+ a(\boldsymbol{u}_h, \boldsymbol{v}_h^I - \boldsymbol{u}) , \quad c_1 > 0 , \quad \text{for any} \quad \boldsymbol{v} \in K(\Omega) \cap [C^\infty(\bar{\Omega})]^3 .$$

Passing to the limit we arrive at

$$0 \leqq c_1 \limsup_{h \to 0+} \|\boldsymbol{u}_h - \boldsymbol{u}\|^2_{V(\Omega)} \leqq a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \quad \text{for any} \quad \boldsymbol{v} \in K(\Omega) \cap [C^\infty(\bar{\Omega})]^3 . \tag{8.20}$$

As the set $K(\Omega) \cap [C^\infty(\bar{\Omega})]^3$ is dense in $K(\Omega)$, the inequality (8.20) holds for every $\boldsymbol{v} \in K(\Omega)$. We can now put $\boldsymbol{v} = \boldsymbol{u}$ and obtain

$$0 \leqq \liminf_{h \to 0+} \|\boldsymbol{u}_h - \boldsymbol{u}\|^2_{V(\Omega)} \leqq \limsup_{h \to 0+} \|\boldsymbol{u}_h - \boldsymbol{u}\|^2_{V(\Omega)} \leqq 0 ,$$

and the relation (8.19) follows.

References

[1] *J. Céa:* Optimisation, Théorie et Algorithmes. Dunod Paris, 1971.

[2] *P. G. Ciarlet:* The finite element method for elliptic problems. North-Holland 1978.

[3] *G. Duvaut, J. L. Lions:* Inequalities in mechanics and physics. Berlin, Springer Verlag 1975.

[4] *R. Glowinski, J. L. Lions, R. Trémolieres:* Numerical analysis of variational inequalities. North-Holland, Amsterdam, New York 1981.

[5] *I. Ekeland, R. Temam:* Analyse convexe et problémes variationnels. Dunod, Paris 1974.

[6] *J. Haslinger, I. Hlaváček:* Contact between elastic bodies II. Finite element analysis. Apl. mat. 26, 1981, p. 263—290.

[7] *I. Hlaváček, J. Haslinger, J. Nečas, J. Lovíšek:* Solution of variational inequalities in mechanics (Slovak). ALFA-SNTL, 1982.

[8] *I. Hlaváček, J. Nečas:* On inequalities of Korn's type. Arch. Rat. Mech. Anal., 36, 1970, 305—334.

[9] *N. D. Hung, G. de Saxcé:* Finite element analysis of contact problems based on the unilateral constraints formulation. Structural Control. H.H.E. Leipholz (ed) IUTAM, 1980, p. 341—373.

[10] *N. Kikuchi, T. Oden:* Contact problems in elasticity. SIAM, Philadelphia, 1981.

[11] *N. Kikuchi, Y. Joon Song:* Penalty, finite-element approximations of a class of unilateral problems in linear elasticity. Quarterly of Appl. Math. Vol. XXXVIV No 1. April 1981.

[12] *D. Kinderlehrer, G. Stampacchia:* An introduction to variational inequalities and their applications. Academic Press, 1980.

[13] *A. C. Kravčuk:* On Hertz's problem for linearly and nonlinearly elastic bodies of finite dimensions (Russian). Prikladnaja matematika i mechanika, 1977, t. 41, No 2, s. 28—30.

[14] *J. L. Lions:* Quelques méthodes de résolution des problémes aux limites non linéaires. Dunod, Paris, 1969.

[15] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia Praha, 1967.

[16] *J. Nečas, I. Hlaváček:* Mathematical theory of elastic and elasto-plastic bodies: An introduction. Elsevier 1981.

[17] *B. L. Pelech:* Generalized theory of shells (Russian). Lvov 1978.

Súhrn

ANALÝZA KONTAKTNEJ ÚLOHY PRE VALCOVÚ ŠKRUPINU: PRIMÁRNA A DUÁLNA FORMULÁCIA

IGOR BOCK, JÁN LOVÍŠEK

V tejto práci je študovaná kontaktná úloha pre valcovú škrupinu a tuhý razník (bez uvažovania trenia). Pre primárnu a duálnu formuláciu úlohy je dokázaná existencia a jednoznačnosť riešenia na množine prípustných funkcií. Pre primárnu úlohu je dokázaná konvergencia numerického riešenia metódou konečných prvkov k riešeniu východzej úlohy.

Author's address: RNDr. *Igor Bock,* CSc., Elektrotechnická fakulta SVŠT, Gottwaldovo nám. 19, 812 19 Bratislava; Doc. RNDr. Ing. *Ján Lovíšek,* CSc., Stavebná fakulta SVŠT, Radlinského nám. 11, 813 68 Bratislava.