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INTERNAL FINITE ELEMENT APPROXIMATION IN THE DUAL VARIATIONAL METHOD FOR THE BIHARMONIC PROBLEM

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1. INTRODUCTION

The aim of this paper is to present a conforming finite element method for the dual variational formulation of the biharmonic problem with mixed boundary conditions on domains with a piecewise smooth curved boundary. We use C^0-elements while any conforming primal finite element method for the biharmonic problem requires C^1-elements, which are more complicated especially for curved boundaries [13, 21].

Note that by the dual method we calculate all second derivatives of the solution of the biharmonic problem, which are often more interesting than the solution itself. For instance we can get bending moments of an elastic plate.

In the next section we introduce a "pure" equilibrium model for the biharmonic problem. We justify the so-called static-geometric analogy ([6, 19]) by proving the existence of a vector potential $\boldsymbol{v} \in (H^1(\Omega))^2$ of an equilibrium bending moment $\boldsymbol{\mu}$, which satisfies the equilibrium condition $\operatorname{div} \operatorname{Div} \boldsymbol{\mu} = 0$ in the domain Ω and some conditions on a part of the boundary $\partial\Omega$ (Sections 3 and 4). Then we present (Section 5) a general dual finite element method for the biharmonic problem (employing polynomials of arbitrary order). We prove the convergence of this method in the L^2-norm without any regularity assumptions on the solution. The paper generalizes the results of [11], where the dual finite element analysis of a clamped plate problem was studied on polygonal domains. Let us note that piecewise linear equilibrium elements have been proposed in [7, 19]. Bending moments can also be obtained by mixed finite element methods [1, 2, 3, 4].

Let us introduce some notations. Throughout the paper, $\Omega \subset \mathbb{R}^2$ will always be a bounded domain with a Lipschitz boundary $\partial\Omega$ (see [16], p. 17). Let $\nu = (\nu_1, \nu_2)^{\mathsf{T}}$ be the outward unit normal to $\partial\Omega$ and let $\tau = (-\nu_2, \nu_1)^{\mathsf{T}}$. By $P_j(\Omega)$ we mean the space of polynomials of the order at most j defined on Ω. Notations $H^k(\Omega)$ ($k \geqq 0$, integer) are used for the Sobolev spaces of functions, the generalized derivatives of which up to the order k exist and are square integrable in Ω. The usual norm and seminorm in $H^k(\Omega)$ and also in $(H^k(\Omega))^p$ ($p \geqq 1$, integer) are denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. The scalar product in $(L^2(\Omega))^p$ is denoted by $(\cdot, \cdot)_{0,\Omega}$. The space of symmetric tensors

$$(L^2(\Omega))^4_{\text{sym}} = \{\tau \in (L^2(\Omega))^4 \mid \tau = \tau^{\mathrm{T}}\}$$

will be equipped with the scalar product

$$(\tau, \mu)_{0,\Omega} = \sum_{i,j=1}^{2} (\tau_{ij}, \mu_{ij})_{0,\Omega} \quad \text{for} \quad \tau, \mu \in (L^2(\Omega))^4_{\text{sym}} .$$

For simplicity, the subscript $_\Omega$ will sometimes be omitted. Let us introduce the operator $\varepsilon: (H^1(\Omega))^2 \to (L^2(\Omega))^4_{\text{sym}}$ defined by

$$\varepsilon(\mathbf{v}) = \begin{pmatrix} v_{1,1}, & \frac{1}{2}(v_{1,2} + v_{2,1}) \\ \text{sym.}, & v_{2,2} \end{pmatrix}, \quad \mathbf{v} = (v_1, v_2)^{\mathrm{T}} \in (H^1(\Omega))^2 ,$$

where $v_{i,k} = \partial v_i / \partial x_k$.

The space of infinitely differentiable functions with a compact support in Ω will be denoted by $\mathscr{D}(\Omega)$.

Further, let $\mathbf{g} \in (L^2(\Omega))^2$ be arbitrary. If

$$(1.1) \qquad (\tau, \varepsilon(\mathbf{v}))_0 = (\mathbf{g}, \mathbf{v})_0 \quad \forall \mathbf{v} \in (\mathscr{D}(\Omega))^2$$

holds for some $\tau \in (L^2(\Omega))^4_{\text{sym}}$, we say that the divergence of the tensor function τ exists in the sense of distributions in Ω and define

$$\operatorname{Div} \tau = -\mathbf{g} .$$

Evidently, for smooth τ we have

$$\operatorname{Div} \tau = (\tau_{11,1} + \tau_{12,2}, \tau_{12,1} + \tau_{22,2})^{\mathrm{T}} .$$

2. DUAL VARIATIONAL FORMULATION OF THE BIHARMONIC PROBLEM

Let us suppose that the boundary $\partial\Omega$ consists of four mutually disjoint parts $\mathscr{R}_1$, $\Gamma_1, \Gamma_2, \Gamma_3$ such that

$$\partial\Omega = \mathscr{R}_1 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 ,$$

where $\mathscr{R}_1$ is the union of a finite number of points and $\Gamma_1, \Gamma_2, \Gamma_3$ are open in $\partial\Omega$. Assume that $\Gamma_1 \neq \emptyset$ or Γ_2 is not contained in a single straight line, and let Γ_1 be piecewise $C^{(2)}$.

The biharmonic problem for an elastic homogeneous and isotropic plate with mixed boundary conditions can be formulated [14, 16, 18] as follows: Find $z \in C^4(\bar{\Omega})$ such that

$$(2.1) \qquad \begin{aligned} \Delta^2 z = f/D \quad &\text{in} \quad \Omega , \\ z = 0 , \quad \partial z/\partial\nu = 0 \quad &\text{on} \quad \Gamma_1 , \\ z = 0 , \quad \mathscr{M}(z) = 0 \quad &\text{on} \quad \Gamma_2 , \\ \mathscr{M}(z) = 0 , \quad \mathscr{N}(z) = 0 \quad &\text{on} \quad \Gamma_3 , \end{aligned}$$

where z is the deflection,

$$\mathscr{M}(z) = \sigma \Delta z + (1 - \sigma)(\nu_1^2 z_{,11} + 2\nu_1\nu_2 z_{,12} + \nu_2^2 z_{,22}),$$

$$\mathscr{N}(z) = -\frac{\partial \Delta z}{\partial \nu} + (1 - \sigma)\frac{\partial(\nu_1\nu_2(z_{,11} - z_{,22}) - (\nu_1^2 - \nu_2^2) z_{,12})}{\partial \tau},$$

$\partial/\partial\nu$ and $\partial/\partial\tau$ is the normal and tangential derivative, respectively, $0 < \sigma < 1/2$ is the Poisson constant, f is a given load ($f \in L^2(\Omega)$ or more generally $f \in (C(\bar{\Omega}))'$),

$$D = \frac{2Eh^3}{3(1 - \sigma^2)},$$

E is Young's modulus of elasticity and $2h$ is the (constant) thickness of the plate.

For the primal variational formulation of (2.1) let us introduce the space

$$Z = \left\{ z \in H^2(\Omega) \mid z = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\}, \tag{2.2}$$

and the operator hes: $H^2(\Omega) \to (L^2(\Omega))^4_{\text{sym}}$,

$$\text{hes } z = \begin{pmatrix} z_{,11}, & z_{,12} \\ z_{,12}, & z_{,22} \end{pmatrix}.$$

Further, let $\mathbb{A} = (A_{ijkl})^2_{i,j,k,l=1}$, where $A_{1111} = A_{2222} = D$, $A_{1122} = D\sigma$, $A_{1212} = D(1 - \sigma)/2$, $A_{1112} = A_{2221} = 0$, and let

$$A_{jikl} = A_{ijkl} = A_{klij}. \tag{2.3}$$

Then we have

$$(\mathbb{A} . \boldsymbol{\mu}, \boldsymbol{\mu})_0 \geqq C\|\boldsymbol{\mu}\|_0^2 \quad \forall \boldsymbol{\mu} \in (L^2(\Omega))^4_{\text{sym}}. \tag{2.4}$$

Here we write $\boldsymbol{\tau} = \mathbb{A} . \boldsymbol{\mu}$, when

$$\tau_{ij} = \sum_{k,l=1}^{2} A_{ijkl}\mu_{kl}$$

for $\boldsymbol{\tau}, \boldsymbol{\mu} \in (L^2(\Omega))^4_{\text{sym}}$.

Let us recall [10, 16] that *the primal problem* consists in minimizing the functional (of the potential energy)

$$I(z) = \tfrac{1}{2}(\mathbb{A} . \text{hes } z, \text{hes } z)_0 - \langle f, z \rangle$$

over Z ($\langle \cdot, \cdot \rangle$ denotes the dual pairing between $(H^2(\Omega))'$ and $H^2(\Omega)$).

Henceforth, we introduce the set of statically admissible bending moments

$$M(f) = \{ \boldsymbol{\mu} \in (L^2(\Omega))^4_{\text{sym}} \mid (\boldsymbol{\mu}, \text{hes } z)_0 = \langle f, z \rangle \ \forall z \in Z \}.$$

The dual problem consists (see [1, 16]) in minimizing the functional (of the complementary energy)

$$J(\boldsymbol{m}) = \tfrac{1}{2}(\mathbb{A}^{-1} . \boldsymbol{m}, \boldsymbol{m})_0$$

over $M(f)$, where $\mathbb{A}^{-1}$ is the inverse to $\mathbb{A}$.

For $\mathbb{A}^{-1}$ the relations analogous to (2.3) and (2.4) hold, and we have $A^{-1}_{1111} = A^{-1}_{2222} = D^{-1}(1 - \sigma^2)^{-1}$, $A^{-1}_{1122} = -\sigma D^{-1}(1 - \sigma^2)^{-1}$, $A^{-1}_{1212} = D^{-1}(1 - \sigma)^{-1}/2$, $A^{-1}_{1112} = A^{-1}_{2221} = 0$.

We define the space of *equilibrium bending moments* in the following way

$$M = M(0).$$

Note that for smooth $\boldsymbol{\mu} \in M$ it holds that

$$\operatorname{div} \operatorname{Div} \boldsymbol{\mu} = \mu_{11,11} + 2\mu_{12,12} + \mu_{22,22} = 0.$$

Clearly, the dual problem can be formulated in an equivalent way: Given $\bar{\boldsymbol{\lambda}} \in M(f)$ (see Remark 2.1 below), find $\boldsymbol{\lambda} \in M$ which minimizes the functional

$$\bar{J}(\boldsymbol{\mu}) = \tfrac{1}{2}(\mathbb{A}^{-1} . \boldsymbol{\mu}, \boldsymbol{\mu})_0 + (\mathbb{A}^{-1} . \boldsymbol{\mu}, \bar{\boldsymbol{\lambda}})_0 \tag{2.5}$$

over the space M. The tensor $\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}}$ is considered to be the solution of the dual problem and to any $\bar{\boldsymbol{\lambda}} \in M(f)$ there exists exactly one $\boldsymbol{\lambda}$. Moreover, we have the following equality (see [16], p. 250)

$$\boldsymbol{\lambda} + \bar{\boldsymbol{\lambda}} = \mathbb{A} . \operatorname{hes} \bar{z},$$

where $\bar{z}$ is the solution of the foregoing primal problem.

Remark 2.1. We shall describe a way of finding some $\bar{\boldsymbol{\lambda}} \in M(f)$ in practical cases. Let all the functions occurring below be sufficiently smooth so that the corresponding symbols have the correct sense, and let us look for $\bar{\boldsymbol{\lambda}}$ in the form

$$\bar{\boldsymbol{\lambda}} = \boldsymbol{\varphi} + \boldsymbol{\psi}.$$

We define $\boldsymbol{\varphi} = (\varphi_{ij}) \in (L^2(\Omega))^4_{\text{sym}}$ by

$$\varphi_{11} = \varphi_{12} = 0,$$

$$\varphi_{22}(x_1, x_2) = \int_0^{x_2} \int_0^{\xi} \bar{f}(x_1, \eta) \, d\eta \, d\xi, \quad (x_1, x_2) \in \Omega,$$

where $\bar{f} = f$ in Ω and $\bar{f} = 0$ in $\mathbb{R}^2 - \Omega$.

We introduce the operator $\omega: (H^1(\Omega))^2 \to (L^2(\Omega))^4_{\text{sym}}$

$$\omega(\boldsymbol{v}) = \begin{pmatrix} v_{2,2}, & -\tfrac{1}{2}(v_{1,2} + v_{2,1}) \\ \text{sym.}, & v_{1,1} \end{pmatrix}.$$

We put

$$\boldsymbol{\psi} = \omega(\boldsymbol{v}),$$

where $\boldsymbol{v} \in (H^1(\Omega))^2$ is an arbitrary function satisfying

$$\omega(\boldsymbol{v}) \,.\, \nu = -\boldsymbol{\varphi} \,.\, \nu \quad \text{on} \quad \Gamma_2 \cup \Gamma_3 \,,$$

$$(\operatorname{Div} \omega(\boldsymbol{v})) \,.\, \nu = -(\operatorname{Div} \boldsymbol{\varphi}) \,.\, \nu \quad \text{on} \quad \Gamma_3 \,.$$

Then by the Green formulae we get for $z \in Z$

$$(\boldsymbol{\varphi} + \boldsymbol{\psi}, \operatorname{hes} z)_0 = (-\operatorname{Div}(\boldsymbol{\varphi} + \boldsymbol{\psi}), \operatorname{grad} z)_0 + \int_{\partial\Omega} (\operatorname{grad} z)^{\mathrm{T}} (\boldsymbol{\varphi} + \boldsymbol{\psi}) \,.\, \nu \, \mathrm{d}s =$$

$$= (\operatorname{div} \operatorname{Div}(\boldsymbol{\varphi} + \omega(\boldsymbol{v})), z)_0 - \int_{\Gamma_3} z(\operatorname{Div}(\boldsymbol{\varphi} + \omega(\boldsymbol{v}))) \,.\, \nu \, \mathrm{d}s +$$

$$+ \int_{\Gamma_2 \cup \Gamma_3} (\operatorname{grad} z)^{\mathrm{T}} (\boldsymbol{\varphi} + \omega(\boldsymbol{v})) \,.\, \nu \, \mathrm{d}s = (\operatorname{div} \operatorname{Div} \boldsymbol{\varphi}, z)_0 = (f, z)_0 = \langle f, z \rangle \,,$$

that is $\bar{\boldsymbol{\lambda}} = \boldsymbol{\varphi} + \boldsymbol{\psi} \in M(f)$.

3. EXISTENCE OF A VECTOR POTENTIAL OF EQUILIBRIUM BENDING MOMENTS

In this section we shall rectrict ourselves to the case $\Gamma_2 = \emptyset$ (the case $\partial\Omega = \Gamma_2$ of the simply supported plate will be discussed in Section 4). Let us define the space

(3.1) $$V = \{\boldsymbol{v} \in (H^1(\Omega))^2 \mid \boldsymbol{v} = 0 \text{ on } \Gamma_3\} \,.$$

In the following theorem we show (under certain assumptions) that for any equilibrium bending moment $\boldsymbol{\mu} \in M$ there exists a vector potential $\boldsymbol{v} \in V$ such that $\boldsymbol{\mu} = \omega(\boldsymbol{v})$.

Theorem 3.1. *Let Γ_1 and Γ_3 be connected. Then*

$$M = \omega(V) \,.$$

Proof. For an arbitrary $\boldsymbol{\mu} = (\mu_{ij}) \in M$ let us set

$$\boldsymbol{\mu}^* = \begin{pmatrix} \mu_{22}, & -\mu_{21} \\ -\mu_{21}, & \mu_{11} \end{pmatrix}.$$

Then we have

(3.2) $$(\boldsymbol{\mu}^*, \varrho(z))_0 = (\boldsymbol{\mu}, \operatorname{hes} z)_0 = 0 \quad \forall z \in Z \,,$$

where Z is defined by (2.2) and

$$\varrho(z) = \begin{pmatrix} z_{,22}, & -z_{,12} \\ -z_{,12}, & z_{,11} \end{pmatrix}.$$

As Γ_1 and Γ_3 are connected, it holds that (see [11], p. 51)

$$(L^2(\Omega))^4_{\text{sym}} = \varepsilon(V) \oplus \varrho(Z) . \tag{3.3}$$

Hence, by (3.2) there exists $\mathbf{v} \in V$ such that $\boldsymbol{\mu}^* = \varepsilon(\mathbf{v})$. Thus $\boldsymbol{\mu} = \omega(\mathbf{v})$ and $\boldsymbol{\mu} \in \omega(V)$.

Conversely, let $\mathbf{v} \in V$ be given. Then by (3.3) we obtain

$$(\omega(\mathbf{v}), \text{hes } z))_0 = (\varepsilon(\mathbf{v}), \varrho(z))_0 = 0 \quad \forall z \in Z .$$

Therefore, $\omega(\mathbf{v}) \in M$. ∎

Remark 3.1. We can easily ascertain as in [16], p. 78, that $\{(1, 0)^{\mathrm{T}}, (0, 1)^{\mathrm{T}}, (x_2, -x_1)^{\mathrm{T}}\}$ is a basis of the space

$$V^0 = \{v \in (H^1(\Omega))^2 |\ \omega(v) = 0\} . \tag{3.4}$$

Thus for $\Gamma_3 \neq \emptyset$ (Γ_3 is open in $\partial\Omega$), the vector potential is unique, while for $\Gamma_1 = \partial\Omega$ (a clamped plate) it is unique apart from a function of V^0.

Remark 3.2. As a consequence of Theorem 3.1 we get

$$(L^2(\Omega))^4_{\text{sym}} = \omega(V) \oplus \text{hes } Z ,$$

when Γ_1 and Γ_3 are connected.

4. SIMPLY SUPPORTED RECTANGULAR PLATE

Throughout this section we shall assume that $\Gamma_2 = \partial\Omega$. Consequently, the space Z (used in the definition of M) will have the form

$$Z = \{z \in H^2(\Omega) |\ z = 0 \text{ on } \partial\Omega\} . \tag{4.1}$$

First we prove the following lemma.

Lemma 4.1. *Let Ω be an arbitrary domain and let an open part $\Gamma \subset \partial\Omega$ be from $C^{(2)}$. Assume that $z \in C^2(\bar{\Omega})$. Then*

$$t_\nu(z) \equiv \nu \,.\, (\varrho(z) \,.\, \nu) = k \frac{\partial z}{\partial \nu} + \frac{\partial^2 z}{\partial s^2} \quad on \quad \Gamma , \tag{4.2}$$

$$t_\tau(z) \equiv \tau \,.\, (\varrho(z) \,.\, \nu) = k \frac{\partial z}{\partial s} - \frac{\partial^2 z}{\partial s \, \partial \nu} \quad on \quad \Gamma , \tag{4.3}$$

where k is the curvature of Γ and $\partial z/\partial s = \partial z/\partial \tau = -\nu_2 z_{,1} + \nu_1 z_{,2}$.

Proof. As $\tau \equiv (\tau_1, \tau_2)^{\mathrm{T}} = (-\nu_2, \nu_1)^{\mathrm{T}}$, we have

$$(\varrho(z) \,.\, \nu)_1 = \varrho_{11}(z)\, \nu_1 + \varrho_{12}(z)\, \nu_2 = z_{,22}\tau_2 + z_{,12}\tau_1 = \tau \,.\, \text{grad}\,(z_{,2}) = \frac{\partial z_{,2}}{\partial s} , \tag{4.4}$$

(4.5) $$(\varrho(z) \cdot \nu)_2 = \varrho_{21}(z)\,\nu_1 + \varrho_{22}(z)\,\nu_2 = -z_{,12}\tau_2 - z_{,11}\tau_1 = \\ = -\tau \cdot \operatorname{grad}(z_{,1}) = -\frac{\partial z_{,1}}{\partial s}\,.$$

Furthermore, it holds that

$$t_\nu(z) = \nu \cdot (\varrho(z) \cdot \nu) = \nu_1(\varrho(z) \cdot \nu)_1 + \nu_2(\varrho(z) \cdot \nu)_2 = \\ = \tau_2(z_{,22}\tau_2 + z_{,12}\tau_1) + \tau_1(z_{,12}\tau_2 + z_{,11}\tau_1) = \sum_{i,j=1}^{2} \tau_i \tau_j z_{,ij}\,.$$

Using Frenet's formulae ([17], p. 308)

$$\frac{\partial \tau_i}{\partial s} = -k\nu_i\,, \quad i = 1, 2\,,$$

and $\partial x_i/\partial s = \tau_i$, we arrive at the equation

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial s}(z_{,1}\tau_1 + z_{,2}\tau_2) = \\ = z_{,11}\tau_1^2 + z_{,12}\tau_1\tau_2 + z_{,21}\tau_1\tau_2 + z_{,22}\tau_2^2 - z_{,1}k\nu_1 - z_{,2}k\nu_2 = t_\nu(z) - k\frac{\partial z}{\partial \nu}\,,$$

which proves (4.2).

From (4.4) and (4.5) we have

$$t_\tau(z) = \tau_1(\varrho(z) \cdot \nu)_1 + \tau_2(\varrho(z) \cdot \nu)_2 = \tau_1 \frac{\partial z_{,2}}{\partial s} - \tau_2 \frac{\partial z_{,1}}{\partial s}\,,$$

$$-\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial \nu}\right) = -\frac{\partial}{\partial s}(z_{,1}\tau_2 - z_{,2}\tau_1) = -\tau_2\frac{z_{,1}}{\partial s} + \tau_1\frac{\partial z_{,2}}{\partial s} + z_{,1}k\nu_2 - z_{,2}k\nu_1 = \\ = t_\tau(z) - k\frac{\partial z}{\partial s}\,,$$

which proves (4.3). ∎

Henceforth, we shall investigate only a *rectangular plate*, which is one of the most important cases.

Let us introduce the space (cf. (1.1))

$$Q_0 = \{\boldsymbol{\tau} \in (L^2(\Omega))^4_{\text{sym}} \mid \operatorname{Div} \boldsymbol{\tau} = 0 \text{ in } \Omega\}\,,$$

and the following subspace of Q_0

(4.6) $$T = T(\Omega) = \{\boldsymbol{\tau} \in (L^2(\Omega))^4_{\text{sym}} \mid (\boldsymbol{\tau}, \varepsilon(\mathbf{v}))_0 = 0 \ \forall \mathbf{v} \in V\}\,,$$

where

$$V = \{\mathbf{v} \in (H^1(\Omega))^2 \mid v_{\tau|S_i} = c_i,\ i = 1, 2, 3, 4,\ c_i \in R^1\}\,,$$

$v_\tau = \tau \cdot \mathbf{v}$ is the tangential component of $\mathbf{v}$ and S_i are sides of the rectangular domain Ω (see Fig. 1).

Definition 4.1. *Let Ω be a rectangle with the sides S_1, S_2, S_3, S_4. Introduce the following spaces*

$$\mathscr{V}^1 = \mathscr{V}^1(\Omega) = \{v_1 \in H^1(\Omega) |\ v_1 = 0 \ \text{on}\ S_1 \cup S_3\},$$
$$\mathscr{V}^2 = \mathscr{V}^2(\Omega) = \{v_2 \in H^1(\Omega) |\ v_2 = 0 \ \text{on}\ S_2 \cup S_4\}.$$

For $\tau \in Q_0$ we define the funotional $t_\nu(\tau) \in (\gamma\mathscr{V}^1 \times \gamma\mathscr{V}^2)'$ by the relation

$$\langle t_\nu(\tau), \gamma \mathbf{v}\rangle = (\tau, \varepsilon(\mathbf{v}))_0, \quad \mathbf{v} = (v_1, v_2)^\mathsf{T},$$

where γ: $(H^1(\Omega))^p \to (H^{1/2}(\partial\Omega))^p$, $p = 1, 2$, denotes the trace operator.

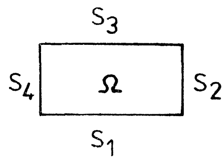

Fig. 1.

Remark 4.1. From the definition of Q_0 it follows that $t_\nu(\tau)$ does not depend on extensions of γv_1 and γv_2 into the interior of Ω. The notation t_ν is in agreement with the fact that it represents an extension of the mapping

$$\tau \to \nu \,.\, (\tau \,.\, \nu) \equiv t_\nu(\tau),$$

which is defined for all symmetric $\tau \in (C(\bar{\Omega}))^4$.

Theorem 4.1. *Let Ω be a rectangle. Then*

$$T = \varrho(Z),$$

where T and Z are given by (4.6) and (4.1), respectively.

The proof is based on three auxiliary lemmas.

Lemma 4.2. *Let Ω be a rectangle. Then $\tau \in T$ if and only if it satisfies the following three conditions:*

(a) $\operatorname{Div} \tau = 0$ *in* Ω,
(b) $(\tau_{12}, 1)_0 = 0$,
(c) $t_\nu(\tau) = 0$ (*in the sense of Definition* 4.1).

Proof. Let us write $\Omega = (0, a) \times (0, b)$ and choose $\mathbf{w}^j \in (P_1(\Omega))^2$, $j = 1, 2, 3, 4$, such that

$$w^j_\tau = 1 \ \text{on}\ S_j, \quad w^j_\tau = 0 \ \text{on}\ \partial\Omega - S_j.$$

We have

(4.7)
$$\mathbf{w}^1(x) = \begin{pmatrix} 1 - x_2/b \\ 0 \end{pmatrix}, \quad \mathbf{w}^2(x) = \begin{pmatrix} 0 \\ x_1/a \end{pmatrix}, \quad \mathbf{w}^3(x) = \begin{pmatrix} x_2/b \\ 0 \end{pmatrix}, \quad \mathbf{w}^4(x) = \begin{pmatrix} 0 \\ 1 - x_1/a \end{pmatrix}.$$

Then we may write

$$(4.8) \qquad V = \sum_{j=1}^{4} c_j \mathbf{w}^j + \mathscr{V}^1 \times \mathscr{V}^2 , \quad c_j \in \mathbb{R}^1 .$$

For $\tau \in T$ we get

$$0 = (\tau, \varepsilon(\mathbf{w}^1))_0 = -\frac{1}{b} \int_\Omega \tau_{12} \, \mathrm{d}x ,$$

since

$$(4.9) \qquad \varepsilon(\mathbf{w}^1) = \begin{pmatrix} 0 & -\dfrac{1}{2b} \\ -\dfrac{1}{2b} & 0 \end{pmatrix},$$

i.e. the condition (b) holds. The condition (a) follows from (1.1) and (4.6).

Let us choose $\mathbf{v} \in \mathscr{V}^1 \times \mathscr{V}^2$, that is $v_\tau = 0$ on $\partial\Omega$. Then

$$0 = (\tau, \varepsilon(\mathbf{v}))_0 = \langle t_\nu(\tau), \gamma\mathbf{v} \rangle ,$$

which is the condition (c).

Conversely, let $\tau \in (L^2(\Omega))^4_{\mathrm{sym}}$ fulfil (a), (b), (c). As $\varepsilon(\mathbf{w}^j)$ is of the form (4.9) for every $j = 1, 2, 3, 4$, we find by (b) that

$$(4.10) \qquad (\tau, \varepsilon(\mathbf{w}^j))_0 = \text{const.} \int_\Omega \tau_{12} \, \mathrm{d}x = 0 .$$

For any $\mathbf{v} \in \mathscr{V}^1 \times \mathscr{V}^2$ the conditions (a) and (c) imply

$$(4.11) \qquad (\tau, \varepsilon(\mathbf{v}))_0 = \langle t_\nu(\tau), \gamma\mathbf{v} \rangle = 0 .$$

The combination of (4.10) and (4.11) with the use of (4.8) yields $\tau \in T$. ∎

Lemma 4.3. *Let Ω_1 and Ω_2 be identical rectangles having one common side $S = \bar{\Omega}_1 \cap \bar{\Omega}_2$, which is parallel with the axis x_1 or x_2. For $\tau \in T(\Omega_1)$ let $E\tau$ be defined on $\Omega_3 = \operatorname{int}(\bar{\Omega}_1 \cup \bar{\Omega}_2)$ by the extension of τ $(E\tau|_{\Omega_1} = \tau)$ such that $E\tau_{ii}$ $(i = 1, 2)$ is an antisymmetric function and $E\tau_{12}$ is a symmetric function with respect to S. Then $E\tau \in T(\Omega_3)$.*

Proof. Let S lie on the axis x_2 and let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows

$$(y_1, y_2) = F(x_1, x_2) = (-x_1, x_2) .$$

By assumptions we have for $y \in \bar{\Omega}_2$

$$E\,\tau_{ii}(y) = -\tau_{ii}(F^{-1}(y)) ,$$
$$E\,\tau_{12}(y) = \tau_{12}(F^{-1}(y)) .$$

Given $\mathbf{v} \in \mathscr{V}^1(\Omega_3) \times \mathscr{V}^2(\Omega_3)$, we set for $x \in \bar{\Omega}_1$

$$\tilde{v}_1(x) = -v_1(F(x)),$$
$$\tilde{v}_2(x) = \quad v_2(F(x)).$$

Then

$$\int_{\Omega_2} [E\,\tau_{11}(y)\,v_{1,1}(y) + E\,\tau_{12}(y)\,(v_{1,2}(y) + v_{2,1}(y)) + E\,\tau_{22}(y)\,v_{2,2}(y)]\,\mathrm{d}y =$$
$$= \int_{\Omega_1} (-\tau_{11}(x)\,\tilde{v}_{1,1}(x) + \tau_{12}(x)\,(-\tilde{v}_{1,2}(x) - \tilde{v}_{2,1}(x)) - \tau_{22}(x)\,\tilde{v}_{2,2}(x))\,\mathrm{d}x$$

and we get

$$(E\tau, \varepsilon(\mathbf{v}))_{0,\Omega_3} = (\tau, \varepsilon(\mathbf{v}))_{0,\Omega_1} + (E\tau, \varepsilon(\mathbf{v}))_{0,\Omega_2} =$$
$$= (\tau, \varepsilon(\mathbf{v}))_{0,\Omega_1} - (\tau, \varepsilon(\tilde{\mathbf{v}}))_{0,\Omega_1} = (\tau, \varepsilon(\mathbf{v} - \tilde{\mathbf{v}}))_{0,\Omega_1}.$$

The last term, however, vanishes, since $\tau \in T(\Omega_1)$ and $\mathbf{v} - \tilde{\mathbf{v}}|_{\Omega_1} \in \mathscr{V}^1(\Omega_1) \times \mathscr{V}^2(\Omega_1)$. Hence $\operatorname{Div} E\tau = 0$ in Ω_3 and $t_\nu(E\tau) = 0$ on $\partial\Omega_3$ follows. Moreover,

$$(E\tau_{12}, 1)_{0,\Omega_3} = (\tau_{12}, 1)_{0,\Omega_1} + (E\tau_{12}, 1)_{0,\Omega_2} = 2(\tau_{12}, 1)_{0,\Omega_1} = 0,$$

by virtue of Lemma 4.2. Consequently, we have $E\tau \in T(\Omega_3)$.

Evidently, the lemma remains true when S is only parallel with the axis x_2. The case when S is parallel with x_1 can be handled in an analogous way.

Lemma 4.4. *Let Ω be a rectangle. Then the set*

$$T \cap (C^\infty(\bar{\Omega}))^4$$

is dense in T with respect to the $\|\cdot\|_0$-norm.

Proof. Let $\tau \in T$ be given. Using Lemma 4.3 four times, we can get an extension $E\tau$ defined on some domain $\Omega^* \supset \bar{\Omega}$, such that

$$\operatorname{Div} E\tau = 0 \quad \text{in} \quad \Omega^*.$$

The domain Ω^* can be chosen for instance in the way shown in Fig. 2.

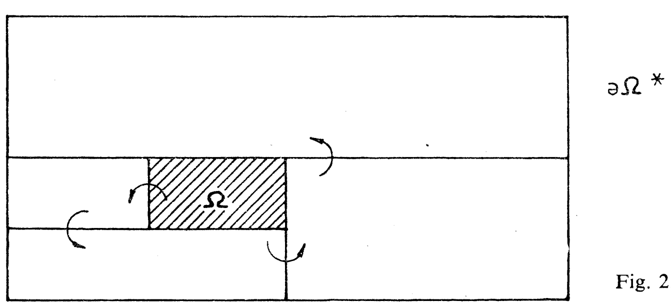

Fig. 2.

We make a regularization of the function $E\tau$ using the kernel

$$A^{-1}\varkappa^2 \mathscr{K}_\varkappa(y) = \begin{cases} \exp\left(|y|^2/(|y|^2 - \varkappa^2)\right) & \text{for} \quad |y| < \varkappa\,, \\ 0 & \text{for} \quad |y| \geqq \varkappa\,, \end{cases}$$

where $A = \text{const} > 0$ and $\varkappa < \operatorname{dist}(\partial\Omega, \partial\Omega^*)$. So let us put

$$R_\varkappa E\,\tau(x) = \int_{\Omega^*} \mathscr{K}_\varkappa(x - \xi)\, E\,\tau(\xi)\,\mathrm{d}\xi\,.$$

As the functions $E\tau_{11}$ and $E\tau_{22}$ are antisymmetric functions with regard to the lines $x_1 = 0$, $x_1 = a$ and $x_2 = 0$, $x_2 = b$, respectively, we have

$$\begin{aligned} &R_\varkappa E\tau_{11}(0, x_2) = R_\varkappa E\tau_{11}(a, x_2) = 0\,, \quad x_2 \in [0, b]\,, \\ &R_\varkappa E\tau_{22}(x_1, 0) = R_\varkappa E\tau_{22}(x_1, b) = 0\,, \quad x_1 \in [0, a]\,. \end{aligned} \tag{4.12}$$

Setting

$$c_\varkappa = \int_\Omega R_\varkappa E\,\tau_{12}(x)\,\mathrm{d}x\,, \quad \boldsymbol{\alpha}^\varkappa = c_\varkappa(\operatorname{mes}\Omega)^{-1}\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix},$$

we define

$$\boldsymbol{\tau}^\varkappa = R_\varkappa E\boldsymbol{\tau} - \boldsymbol{\alpha}^\varkappa\,.$$

Then obviously $\boldsymbol{\tau}^\varkappa \in (C^\infty(\bar{\Omega}))^4$ and (see [8], p. 450)

$$\operatorname{Div}\boldsymbol{\tau}^\varkappa = \operatorname{Div} R_\varkappa E\boldsymbol{\tau} = 0 \quad \text{in} \quad \bar{\Omega}\,. \tag{4.13}$$

We have also

$$(\tau_{12}^\varkappa, 1)_0 = \int_\Omega R_\varkappa E\tau_{12}\,\mathrm{d}x - c_\varkappa = 0\,. \tag{4.14}$$

Let $\mathbf{v} \in \mathscr{V}^1 \times \mathscr{V}^2$. Then we may write

$$\langle t_\nu(\boldsymbol{\tau}^\varkappa), \gamma v\rangle = (\boldsymbol{\tau}^\varkappa, \varepsilon(\mathbf{v}))_0 = \sum_{i,j=1}^{2} \int_{\partial\Omega} \tau_{ij}^\varkappa \nu_j v_i\,\mathrm{d}s = \tag{4.15}$$

$$= -\int_{S_1} \tau_{22}^\varkappa v_2\,\mathrm{d}x_1 + \int_{S_3} \tau_{22}^\varkappa v_2\,\mathrm{d}x_1 + \int_{S_2} \tau_{11}^\varkappa v_1\,\mathrm{d}x_2 - \int_{S_4} \tau_{11}^\varkappa v_1\,\mathrm{d}x_2 = 0\,,$$

using e.g. $\tau_{22}^\varkappa = R_\varkappa E\tau_{22} = 0$ on S_1 and S_3 by (4.12). Since $\boldsymbol{\tau}^\varkappa$ satisfies (4.13), (4.14), and (4.15), we have $\boldsymbol{\tau}^\varkappa \in T$ on the basis of Lemma 4.2.

Defining $E\boldsymbol{\tau} = 0$ in $\mathbb{R}^2 - \Omega^*$, we obtain for $\varkappa \to 0$

$$\begin{aligned} \|R_\varkappa E\boldsymbol{\tau} - \boldsymbol{\tau}\|_{0,\Omega} &\leqq \|R_\varkappa E\boldsymbol{\tau} - E\boldsymbol{\tau}\|_{0,\Omega^*} \to 0\,, \\ |c_\varkappa| = |(R_\varkappa E\tau_{12} - \tau_{12}, 1)_0| &\leqq C\|R_\varkappa E\tau_{12} - \tau_{12}\|_0 \to 0\,, \end{aligned}$$

and therefore

$$\|\boldsymbol{\tau}^\varkappa - \boldsymbol{\tau}\|_0 = \|R_\varkappa E\boldsymbol{\tau} - \boldsymbol{\alpha}^\varkappa - \boldsymbol{\tau}\|_0 \leqq \|R_\varkappa E\boldsymbol{\tau} - \boldsymbol{\tau}\|_0 + \|\boldsymbol{\alpha}^\varkappa\|_0 \to 0\,. \quad \blacksquare$$

Proof of Theorem 4.1. 1°. For given $\tau \in T$ we find by Lemma 4.4 a sequence

$$\tau^n \in T \cap (C^\infty(\bar{\Omega}))^4 , \quad \tau^n \to \tau \quad \text{in} \quad (L^2(\Omega))^4 . \tag{4.16}$$

As $\operatorname{Div} \tau^n = 0$, there exists an Airy function $z^n \in C^\infty(\bar{\Omega})$ such that (see [11], p. 39)

$$\tau^n = \varrho(z^n) .$$

For each $\mathbf{v} \in V$ it holds that

$$0 = (\tau^n, \varepsilon(\mathbf{v}))_0 = \int_{\partial\Omega} (v_\nu \, t_\nu(z^n) + v_\tau \, t_\tau(z^n)) \, ds ,$$

where (see Lemma 4.1)

$$t_\nu(z^n) = \nu \, . \, (\tau^n \, . \, \nu) = \frac{\partial^2 z^n}{\partial s^2} , \quad t_\tau(z^n) = - \frac{\partial^2 z^n}{\partial s \, \partial \nu} .$$

Hence

$$0 = \sum_{i=1}^4 \int_{S_i} \left(v_\nu \frac{\partial^2 z^n}{\partial s^2} - v_\tau \frac{\partial^2 z^n}{\partial s \, \partial \nu} \right) ds . \tag{4.17}$$

Let us choose $\mathbf{v}$ such that $v_\tau = 0$ and the support of the trace of v_ν is in S_1 (i.e. we choose $v_1 = 0$, $v_2 \in \mathscr{V}^2$). Thus we get

$$\int_{S_1} \frac{\partial^2 z^n}{\partial s^2} \varphi \, ds = 0 \quad \forall \varphi \in \gamma \mathscr{V}^2 .$$

Consequently, we have

$$\frac{\partial^2 z^n}{\partial s^2} = 0 \quad \text{on} \quad S_i , \quad i = 1, 2, 3, 4 .$$

Hence $z^n|_{S_i}$ is a linear function. We shall prove that there exists a linear function $p \in P_1(\bar{\Omega})$ such that $p|_{S_i} = z^n|_{S_i}$, $i = 1, 2, 3, 4$. Obviously, there exist q, r linear, i.e.

$$q(x_1, x_2) = a_0 + a_1 x_1 + a_2 x_2 , \quad r(x_1, x_2) = b_0 + b_1 x_1 + b_2 x_2 ,$$

such that

$$q|_{S_i} = z^n|_{S_i} , \quad i = 1, 2 ,$$
$$r|_{S_i} = z^n|_{S_i} , \quad i = 3, 4 .$$

Choosing $\mathbf{v} = \mathbf{w}^i \in V$ in (4.17), where $\mathbf{w}^i$ are given by (4.7), we find

$$0 = \int_{S_i} \frac{\partial^2 z^n}{\partial s \, \partial \nu} \, ds = \left[\frac{\partial z^n}{\partial \nu} \right]_{s_{i-1}}^{s_i}$$

(s_i and s_{i-1} are the end-points of S_i). Thus we have for example

$$q_{,2} = \frac{d}{dx_2} (z^n|_{S_2}) = \frac{d}{dx_2} (z^n|_{S_4}) = r_{,2} ,$$

that is $a_2 = b_2$. Analogously we show that $a_1 = b_1$ and from the continuity of z^n we obtain $a_0 = b_0$. Altogether, $q = r = p$. As the Airy function z^n is determined by τ^n uniquely apart from a linear function, we may set $z^n = 0$ on $\partial\Omega$.

By (4.16) we know that

$$\tau^n = \varrho(z^n) \to \tau \quad \text{in} \quad (L^2(\Omega))^4 \quad \text{for} \quad n \to \infty .$$

Therefore, the second derivatives of z^n are bounded in $L^2(\Omega)$ and the well-known estimate holds

$$C_1 \|z^n\|_2 \leqq |z^n|_2 \leqq C_2 \quad \forall n .$$

Consequently, there exist a subsequence $\{z^m\}$ and $z \in H^2(\Omega)$ such that for $m \to \infty$

$$z^m \rightharpoonup z \quad \text{(weakly)} \quad \text{in} \quad H^2(\Omega) , \tag{4.18}$$

$$\varrho(z^m) \to \tau \quad \text{in} \quad (L^2(\Omega))^4_{\text{sym}} . \tag{4.19}$$

Passing to the limit with $m \to \infty$ in the definition of the second derivatives

$$(z^m_{,ij}, \varphi)_0 = (z^m, \varphi_{,ij})_0 \quad \forall \varphi \in \mathscr{D}(\Omega) ,$$

and making use of (4.18) and (4.19), we arrive at

$$(-1)^{i+j} (\tau_{kl}, \varphi)_0 = (z, \varphi_{,ij})_0 \quad \forall \varphi \in \mathscr{D}(\Omega) ,$$

where $k = 3 - i$ and $l = 3 - j$, i.e. $\tau = \varrho(z)$.

It remains to show that $z \in Z$. As the imbedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ is completely continuous (see [15], p. 107), by (4.18) z^m converges to z in $C(\bar{\Omega})$ (strongly — see [12], p. 178). Then $z = 0$ on $\partial\Omega$ follows from $z^m = 0$ on $\partial\Omega$ for every m.

2°. Let $z \in Z$ be given. First we show that the set $V \cap (C^\infty(\bar{\Omega}))^2$ is dense in V. Using (4.8), any $\boldsymbol{u} \in V$ can be written in the form

$$\boldsymbol{u} = \sum_{j=1}^{4} c_j \boldsymbol{w}^j + (v_1, 0)^{\mathrm{T}} + (0, v_2)^{\mathrm{T}} , \quad v_i \in \mathscr{V}^i .$$

According to [5], p. 618, there are sequences $\{v_i^n\}_{n=1}^{\infty}$ such that

$$v_i^n \in \mathscr{V}^i \cap C^\infty(\bar{\Omega}) , \quad \|v_i^n - v_i\|_1 \to 0 , \quad i = 1, 2 .$$

Setting

$$\boldsymbol{v}^n = \sum_{j=1}^{4} c_j \boldsymbol{w}^j + (v_1^n, 0)^{\mathrm{T}} + (0, v_2^n)^{\mathrm{T}} ,$$

we find that

$$\boldsymbol{v}^n \in V \cap (C^\infty(\bar{\Omega}))^2 , \quad \boldsymbol{v}^n \to \boldsymbol{u} \quad \text{in} \quad (H^1(\Omega))^2 . \tag{4.20}$$

Let us consider the product

$$I_n = (\varrho(z), \varepsilon(\boldsymbol{v}^n))_0 = \int_\Omega (z_{,22} v^n_{1,1} - z_{,12}(v^n_{1,2} + v^n_{2,1}) + z_{,11} v^n_{2,2}) \, \mathrm{d}x .$$

Integrating by parts, we obtain

$$(4.21) \qquad I_n = \int_{\partial\Omega} [z_{,2}(v^n_{1,1}\nu_2 - v^n_{1,2}\nu_1) + z_{,1}(v^n_{2,2}\nu_1 - v^n_{2,1}\nu_2)]\, ds =$$

$$= \int_{\partial\Omega} \left(-z_{,2}\frac{\partial v^n_1}{\partial s} + z_{,2}\frac{\partial v^n_2}{\partial s} \right) ds =$$

$$= -\int_{S_1} z_{,2}\frac{\partial v^n_1}{\partial x_1}\, dx_1 + \int_{S_2} z_{,1}\frac{\partial v^n_2}{\partial x_2}\, dx_2 + \int_{S_3} z_{,2}\frac{\partial v^n_1}{\partial x_1}\, dx_1 - \int_{S_4} z_{,1}\frac{\partial v^n_2}{\partial x_2}\, dx_2 ,$$

where $\partial z/\partial s = 0$ on $\partial\Omega$ has been used. As $\mathbf{v}^n \in V$, the functions v^n_τ are constant on every S_i, $i = 1, 2, 3, 4$. Thus all the integrals in (4.21) vanish. Making use of (4.21), we find that

$$0 = \lim_{n\to\infty} I_n = (\varrho(z), \varepsilon(\mathbf{u}))_0 .$$

As $\boldsymbol{u}$ was an arbitrary element of V, $\varrho(z) \in T$ holds. ■

Theorem 4.2. *Let Ω be a rectangle and let $\Gamma_2 = \partial\Omega$. Then*

$$M = \omega(V) .$$

Proof. An immediate consequence of Theorem 4.1 is

$$(L^2(\Omega))^4_{\mathrm{sym}} = \varepsilon(V) \oplus \varrho(Z) ,$$

where Z and V are defined by (4.1) and (4.8), respectively. Thus the proof is identical with that of Theorem 3.1. ■

5. INTERNAL FINITE ELEMENT APPROXIMATION OF THE DUAL PROBLEM

Let us recall that the dual problem consists in minimizing the functional (2.5) over the space of equilibrium bending moments M. In both previous sections we have proved that

$$M = \omega(V) ,$$

under the assumptions of Theorem 3.1 or 4.2, which will be kept in the sequel. Let us consider an arbitrary finite element space V_h such that

$$(5.1) \qquad V_h \subset V ,$$

(h is the usual mesh parameter). Introducing the space of equilibrium finite elements

$$(5.2) \qquad M_h = \omega(V_h) ,$$

we see that $M_h \subset M$. We may therefore define internal finite element approximations of the dual problem as follows. Find $\lambda_h \in M_h$, which minimizes the functional (2.5) over the space M_h. Then the sum $\lambda_h + \bar{\lambda}$ will be called the approximate solution of the dual problem.

Theorem 5.1. *Let $\{V_h\}$ be a system of finite element subspaces of V such that the union $\bigcup_h V_h$ is dense in V (with the topology of $(H^1(\Omega))^2$). Then*

$$\|\lambda - \lambda_h\|_0 \to 0 \quad for \quad h \to 0\,,$$

where λ minimizes the functional (2.5).

Proof. By Theorems 3.1 or 4.2 there exists $\mathbf{v} \in V$ such that $\lambda = \omega(\mathbf{v})$. Using now Céa's Lemma ([3], p. 104) and (5.2), we obtain

$$C\|\lambda - \lambda_h\|_0 \leqq \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_0 = \inf_{v_h \in V_h} \|\omega(\mathbf{v}) - \omega(\mathbf{v}_h)\|_0 \leqq$$
$$\leqq \inf_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_1 \to 0 \quad \text{for} \quad h \to 0\,,$$

where $C > 0$ is a constant independent of h. ∎

From [5], p. 618, it follows that

$$\overline{V \cap (C^\infty(\bar{\Omega}))^2} = V\,, \tag{5.3}$$

(the bar denotes the closure in V). Thus it is not difficult to verify the density assumption of Theorem 5.1 for polygonal domains and some C^0-elements (see e.g. [3], Chap. 3.2).

Let us consider the case $\Gamma_2 = \emptyset$ and *assume that Γ_1 and Γ_3 are connected.* We shall describe a construction of the dense subset $\bigcup_h V_h \subset V$ for a *curved boundary $\partial\Omega$* in case of linear finite elements. Consequently, M_h will consist of piecewise constant fields.

Definition 5.1. *A couple (Ω, Γ_3) is said to be from the class $\mathscr{C}^{(2)}$, if*

(i) *$\Omega \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary, which consists of a finite number of arcs from the class $C^{(2)}$. The set of the end-points of these arcs will be denoted by $\mathscr{R}_2$.*

(ii) *the part Γ_3 of the boundary $\partial\Omega$ consists of a finite number of convex and concave arcs. The set of the end-points of these arcs will be denoted by $\mathscr{R}_3$.*

An arc $\Gamma \subset \partial\Omega$ is said to be convex (concave), if there exists a convex domain $\Omega_0 \subset \Omega$ ($\Omega_0 \subset \mathbb{R}^2 - \bar{\Omega}$) such that $\Gamma \subset \partial\Omega_0$.

Let us describe now the way of triangulation of a domain from the class $\mathscr{C}^{(2)}$. The part Γ_3 of the boundary $\partial\Omega$ will be approximated by a "polygonal" curve $\Gamma_{3h} \subset \bar{\Omega}$ consisting of a finite number of straight-line segments, the length of which does not exceed h. Each of those segments is a chord or a tangent of a convex or of a concave arc, respectively (see Fig. 3).

If Γ_3 is a closed curve, we require Γ_{3h} to be also a closed curve. Moreover, we demand $\mathscr{R}_1 \cup \mathscr{R}_3 \subset \Gamma_{3h} \cap \Gamma_3$.

The subdomain of Ω, bounded by Γ_1 and Γ_{3h}, will be denoted by Ω_h, and we define

$$D_h = \Omega - \bar{\Omega}_h .$$

Now $\mathscr{T}_h$ will denote the triangulation of the domain Ω_h generated in a standard way, assuming that the "triangles" adjacent to Γ_1 may have at most one curved side. The inner triangles are "straight" only.

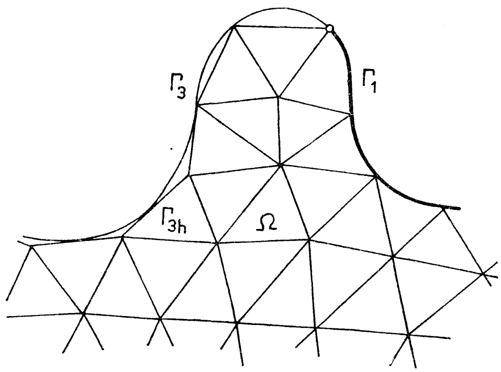

Fig. 3.

Furthermore, we shall always assume the validity of the so-called conformity condition of a triangulation, i.e. the interior of any side of any triangle $K \in \mathscr{T}_h$ is disjoint with the set $\mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{R}_3$. Each segment from $\Gamma_{3h} - \Gamma_3$ coincides with a side of one triangle $K \in \mathscr{T}_h$.

Let us define

$$V_h = \{\mathbf{v} \in V \mid \mathbf{v}|_{D_h} = 0,\ \mathbf{v}|_K \in (P_1(K))^2\ \forall K \in \mathscr{T}_h\} . \tag{5.4}$$

Lemma 5.1. *Let $(\Omega, \Gamma_3) \in \mathscr{C}^{(2)}$ and let V_h (defined by (5.4)) correspond to a regular family of triangulations of Ω_h. Then $\bigcup_h V_h$ is dense in V.*

Proof. Let $\mathbf{w} \in V$ and $\delta > 0$ be given. Using (5.3), we find $\mathbf{v} \in V \cap (C^\infty(\bar{\Omega}))^2$ such that

$$\|\mathbf{w} - \mathbf{v}\|_1 < \delta/2 . \tag{5.5}$$

By [9], p. 58, we can find an approximation $\mathbf{v}_h$ of $\mathbf{v}$ such that $\mathbf{v}_h \in V_h$ and

$$\|\mathbf{v} - \mathbf{v}_h\|_1 < C(v)\, h .$$

The righ-hand side is less than $\delta/2$ for sufficiently small h. Combining this estimate with (5.5), we arrive at the assertion of the lemma. ∎

For an approximation of V by finite element spaces of higher order curved elements, we refer to [20].

Next we shall describe a way of finding λ_h. Let $V_h \subset V$ be an arbitrary finite element space with the basis $\{\mathbf{v}^i\}_{i=1}^n$. Obviously, $\dim M_h \leqq \dim V_h$ follows from (5.2). However, for $\Gamma_3 \neq \emptyset$ ($\Gamma_2 = \emptyset$) we have

$$\dim M_h = \dim V_h = n ,$$

since by Remark 3.1 and (5.1)

$$V^0 \cap V_h \subset V^0 \cap V = \{0\} ,$$

i.e. if $\mathbf{v}_h \in V_h$, then $\omega(\mathbf{v}_h) = 0$ implies $\mathbf{v}_h = 0$. In this case $\{\omega(\mathbf{v}^i)\}_{i=1}^n$ is a basis of M_h and

$$\lambda_h = \sum_{i=1}^{u} c^i \, \omega(\mathbf{v}^i) ,$$

where $c^1, \ldots, c^n$ is the solution of the following system of algebraic equations with a symmetric and positive definite matrix

$$\sum_{j=1}^{n} c^j(\mathbb{A}^{-1} \,.\, \omega(\mathbf{v}^i), \omega(\mathbf{v}^j))_0 = (\mathbb{A}^{-1} \,.\, \omega(\mathbf{v}^i), \bar{\lambda})_0 , \quad i = 1, \ldots, n .$$

In the case $\Gamma_1 = \partial\Omega$, it is easy to see that

$$V^0 \subset V . \tag{5.6}$$

We show that (5.6) holds in the case $\Gamma_2 = \partial\Omega$ as well. According to (4.7), we get

$$\mathbf{w}^1 + \mathbf{w}^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}^2 + \mathbf{w}^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad b\mathbf{w}^3 - a\mathbf{w}^2 = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.$$

Hence, any $\mathbf{v} \in V^0$ can be expressed in the form

$$\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{w}^i ,$$

where $\mathbf{w}^i \in V$.

Therefore, in both cases $\Gamma_1 = \partial\Omega$ or $\Gamma_2 = \partial\Omega$, the set $\{\omega(\mathbf{v}^i)\}_{i=1}^n$ is not a basis of M_h in general, since $V^0 \cap V_h$ may contain some non-zero element. Let us suppose that $V^0 \subset V_h$. Then

$$\dim M_h + 3 = \dim V_h ,$$

and three convenient functions have to be omitted from the set $\{\boldsymbol{\mu}^i\}_{i=1}^n = \{\omega(\mathbf{v}^i)\}_{i=1}^n$ to obtain a basis of M_h. This can be done for instance in the following way.

Let us assume that there exists a nodal point $y = (y_1, y_2)^{\mathrm{T}}$ such that

$$\mathbf{v}^p(y) = (1, 0)^{\mathrm{T}} , \quad \mathbf{v}^q(y) = (0, 1)^{\mathrm{T}} , \quad \mathbf{v}^j(y) = (0, 0)^{\mathrm{T}} \quad j \notin \{p, q\}. \tag{5.7}$$

As $\{\mathbf{v}^i\}$ is the basis in $V_h (\supset V^0)$, there exist $\{\alpha^i\}, \{\beta^i\}, \{\gamma^i\}$ such that

$$\sum_{i=1}^{n} \alpha^i \mathbf{v}^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sum_{i=1}^{n} \beta^i \mathbf{v}^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sum_{i=1}^{n} \gamma^i \mathbf{v}^i = \begin{pmatrix} x_2 - y_2 \\ -x_1 + y_1 \end{pmatrix}. \tag{5.8}$$

With the help of (5.7), we obtain

$$\alpha^p = 1\,, \quad \alpha^q = 0\,, \quad \beta^p = 0\,, \quad \beta^q = 1\,, \quad \gamma^p = \gamma^q = 0 \tag{5.9}$$

and

$$\gamma^r \neq 0 \quad \text{for some} \quad r \in \{1, \ldots, n\}\,. \tag{5.10}$$

Applying the operator ω to (5.8), a simple calculation leads to the result that

$$\boldsymbol{\mu}^p = - \sum_{i \neq p,q} \alpha^i \boldsymbol{\mu}^i\,, \quad \boldsymbol{\mu}^q = - \sum_{i \neq p,q} \beta^i \boldsymbol{\mu}^i\,, \tag{5.11}$$

$$\gamma^r \boldsymbol{\mu}^r = - \sum_{i \neq p,q,r} \gamma^i \boldsymbol{\mu}^i\,. \tag{5.12}$$

Let $\boldsymbol{\mu}_h \in M_h$ be arbitrary. Because $\{\boldsymbol{\mu}^i\}_{i=1}^n$ generates the space M_h, we may write by (5.11)

$$\boldsymbol{\mu}_h = \sum_{i=1}^n \xi^i \boldsymbol{\mu}^i = \sum_{i \neq p,q} \eta^i \boldsymbol{\mu}^i\,,$$

for some $\{\xi^i\}, \{\eta^i\}$. Finally, from (5.12) and (5.10) we come to

$$\boldsymbol{\mu}_h = \eta^r \boldsymbol{\mu}^r + \sum_{i \neq p,q,r} \eta^i \boldsymbol{\mu}^i = \sum_{i \neq p,q,r} \zeta^i \boldsymbol{\mu}^i$$

for convenient $\{\zeta^i\}$. From this expression we conclude that $\{\boldsymbol{\mu}^i\}_{i=1}^n - \{\boldsymbol{\mu}^p, \boldsymbol{\mu}^q, \boldsymbol{\mu}^r\}$ is a basis of M_h.

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Souhrn

VNITŘNÍ APROXIMACE BIHARMONICKÉ ÚLOHY KONEČNÝMI PRVKY DUÁLNÍ VARIAČNÍ METODOU

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Na oblastech s po částech hladkou zakřivenou hranicí je vyšetřována konformní metoda konečných prvků pro duální variační formulaci biharmonického problému s kombinovanými okrajovými podmínkami. Tak jsou v okrajové úloze pružné desky přímo počítány ohybové momenty. Pro konstrukci konečných prvků se používá vektorový potenciál a prvky třídy C^0. Je dokázána konvergence této metody (bez předpokladu regularity řešení) a popsána její algoritmizace.

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