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ON ISHLINSKII'S MODEL FOR NON-PERFECTLY ELASTIC BODIES

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Summary. The main goal of the paper is to formulate some new properties of the Ishlinskii hysteresis operator F, which characterizes e.g. the relation between the deformation and the stress in a non-perfectly elastic (elastico-plastic) material. We introduce two energy functionals and derive the energy inequalities. As an example we investigate the equation $u'' + F(u) = 0$ describing the motion of a mass point at the extremity of an elastico-plastic spring.

Keywords: Ishlinskii operator, hysteresis, non-perfect elasticity, energy inequalities, damped vibrations.

AMS Classification: 73E99, 34K15.

In 1944 Ishlinskii [1] proposed to describe the relation between the internal stress and the prolongation (Hooke's law) in non-perfectly elastic materials by means of a hysteresis scheme. A very complete mathematical theory of hysteresis phenomena has been elaborated by Krasnoselskii and Pokrovskii [2], who introduced the notion of the Ishlinskii operator and studied its basic properties.

Our aim is to derive further properties of the Ishlinskii operator. In particular, we introduce potential energies P_1 and P_2 and prove the corresponding energy inequalities. Next we investigate the asymptotic behaviour and oscillatory properties of the solution to the problem (where F is the Ishlinskii operator)

$$u'' + F(u) = 0\,, \quad u(0) = K\,, \quad u'(0) = 0\,,$$

describing the free vibrations of a mass point at the extremity of a non-perfectly elastic spring. We observe a strong dissipation of energy and study the relation between the rate of decay of the solution and the form of the corresponding hysteresis loop.

This paper is supposed to be an improved version of the first two parts of [4]. The last part of [4] concerning a nonlinear wave equation will be published later in another framework.

We first recall the definition of hysteresis operators (cf. [2], [3]). Let $C([0, T])$ denote the B-space of all continuous functions $v: [0, T] \to R^1$ with sup-norm $\|\cdot\|_{[0,T]}$, and let $v \in C([0, T])$ be piecewise monotone. For $h > 0$ we define

$$(1.1) \quad \text{(i)} \qquad l_h(v)(t) = \begin{cases} \max\{l_h(v)(t_o), v(t) - h\}, & t \in (t_o, t_1] \\ \quad \text{if } v \text{ is nondecreasing in } [t_o, t_1], \\ \min\{l_h(v)(t_o), v(t) + h\}, & t \in (t_o, t_1] \\ \quad \text{if } v \text{ is nonincreasing in } [t_o, t_1]; \end{cases}$$

$$\text{(ii)} \qquad l_h(v)(0) = \begin{cases} v(0) - h & \text{if } v(0) > h, \\ 0 & \text{if } |v(0)| \leqq h, \\ v(0) + h & \text{if } v(0) < -h; \end{cases}$$

$$\text{(iii)} \qquad f_h(v)(t) = v(t) - l_h(v)(t), \quad t \in [0, T].$$

The functions $l_h(v)$, $f_h(v)$ are continuous and piecewise monotone. Moreover, if $v, w \in C([0, T])$ are continuous and piecewise monotone, then

$$(1.2) \qquad |l_h(v)(t) - l_h(w)(t)| \leqq \|v - w\|_{[0,T]}, \quad t \in [0, T].$$

This property enables us to define the values of $l_h(v)$ $(f_h(v))$ for arbitrary $v \in C([0, T])$ as $\lim_{n\to\infty} l_h(v_n)$ $(f_h(v_n))$, where $v_n \in C([0, T])$ are piecewise monotone and $\|v_n - v\|_{[0,T]} \to 0$. Thus l_h, f_h become Lipschitz continuous operators from $C([0, T])$ into $C([0, T])$.

Further, let $\chi \in L^1(0, \infty)$ be a function such that

$$(1.3) \qquad \text{(i)} \quad \chi(x) \geqq 0 \quad \text{a.e.},$$

$$\text{(ii)} \quad \forall \varepsilon > 0, \quad \int_0^\varepsilon \chi(x)\, dx > 0.$$

We denote

$$(1.4) \qquad \varphi(z) = \int_0^z \int_x^\infty \chi(\xi)\, d\xi\, dx, \quad \Phi(z) = \int_0^z \varphi(x)\, dx.$$

The Ishlinskii operator is given by the formula

$$(1.5) \qquad F(v)(t) = \int_0^\infty f_h(v)(t)\, \chi(h)\, dh \quad \text{for} \quad v \in C([0, T]), \quad t \in [0, T].$$

Properties of hysteresis operators

(1.6) (i) l_h, f_h, F are continuous and odd operators $C([0, T]) \to C([0, T])$,

(ii) for $v, w \in C([0, T])$ we have

$$|F(v)(t) - F(w)(t)| \leqq 2\varphi(\|v - w\|_{[0,t]}) \leqq 2\varphi'(0+)\, \|v - w\|_{[0,t]},$$

(iii) for v absolutely continuous, $F(v)$ is absolutely continuous; if $v'(t) = 0$, then $(F(v))'(t) = 0$ and if both $v'(t) \neq 0$, $(F(v))'(t)$ exist, then $F(v)'(t) \cdot v'(t) > 0$. Moreover, the inequality

$$\varphi'(\|v\|_{[0,t]}) \cdot |v'(t)| \leqq |(F(v))'(t)| \leqq 2\,\varphi'(0+)|v'(t)|$$

holds almost everywhere.

(iv) Let $v \in C([0, T])$ and $v(t_m) = -\|v\|_{[0,T]}$ (or $v(t_M) = \|v\|_{[0,T]}$). Then $f_h(v)(t_m) = -\|f_h(v)\|_{[0,T]} = -\min\{h, -v(t_m)\}$ (or $f_h(v)(t_M) = \|f_h(v)\|_{[0,T]} = \min\{h, v(t_M)\}$, respectively) for all $h > 0$, and $F(v)(t_m) = -\varphi(-v(t_m))$ $(F(v)(t_M) = \varphi(v(t_M)))$. In particular, $\|F(v)(t)\|_{[0,T]} = \varphi(\|v\|_{[0,T]})$. Further, let $t_m < T$ and let v be nondecreasing in $[t_m, t_1]$ (or nonincreasing in in $[t_M, t_1]$). Then $f_h(v)(t) = f_h(v)(t_m) + \min\{2h, v(t) - v(t_m)\}$, $h > 0$, $t \in [t_m, t_1]$ $(f_h(v)(t) = f_h(v)(t_M) - \min\{2h, v(t_M) - v(t)\}$, $h > 0$, $t \in [t_M, t_1]$, respectively).

(v) Let $0 \leqq t_o < t_1 \leqq T$, $v_i(t_o) \leqq v_i(t) \leqq v_i(t_1)$ (or $v_i(t_0) \geqq v_i(t) \geqq v_i(t_1)$) for $t \in [t_o, t_1]$, $i = 1, 2$, $v_1(t_k) = v_2(t_k)$, $k = 0, 1$, $f_h(v_1)(t_o) = f_h(v_2)(t_o)$. Then $f_h(v_1)(t_1) = f_h(v_2)(t_1)$.

(vi) Let $0 \leqq t_o < t_1 < t_2 \leqq T$, v nonincreasing in $[t_1, t_2]$, $v(t_2) \geqq v(t_o)$, $f_h(v)(t_1) = f_h(v)(t_0) + \min\{2h, v(t_1) - v(t_0)\}$. Then for $t \in [t_1, t_2]$ we have $f_h(v)(t) = f_h(v)(t_1) - \min\{2h, v(t_1) - v(t)\}$ (analogously, if v is nondecreasing in $[t_1, t_2]$, $v(t_2) \leqq v(t_o)$, $f_h(v)(t_1) = f_h(v)(t_o) - \min\{2h, v(t_o) - v(t_1)\}$, then $f_h(v)(t) = f_h(v)(t_1) + \min\{2h, v(t) - v(t_1)\}$ for $t \in [t_1, t_2]$).

(vii) Let $f_h(v)(t) = f_h(v)(t_o) \pm \min\{2h, |v(t) - v(t_0)|\}$ for $t \in [t_1, t_2]$ and for all $h > 0$. Then $F(v)(t) = F(v)(t_o) \pm 2\varphi(\frac{1}{2}|v(t) - v(t_0)|)$, $t \in [t_1, t_2]$.

(viii) Let u, v be absolutely continuous in $[0, T]$. Then $f_h(u), f_h(v)$ are absolutely continuous and $([(f_h(u))'(t) - u'(t)] - [(f_h(v))'(t) - v'(t)])(f_h(u)(t) - f_h(v)(t)) \leqq 0$ is satisfied for almost every $h > 0$ and almost every $t \in (0, T)$. In particular,

$$\int_s^t (F(u)(\sigma) - F(v)(\sigma))(u'(\sigma) - v'(\sigma))\,d\sigma \geqq$$

$$\geqq \tfrac{1}{2}\int_0^\infty [(f_h(u)(t) - f_h(v)(t))^2 - (f_h(u)(s) - f_h(v)(s))^2]\,\chi(h)\,dh$$

holds for every s, t, $0 \leqq s < t \leqq T$.

(ix) Let u, v be absolutely continuous and ω-periodic functions. Then $f_h(u)$, $f_h(v)$, $F(u)$, $F(v)$ are ω-periodic for $t \geqq \omega$. Let us assume $\chi(h) > 0$ a.e. in $(0, \infty)$ and

$$\int_\omega^{2\omega} (F(u)(t) - F(v)(t))(u'(t) - v'(t))\,dt = 0\,.$$

Then $u' = v'$ almost everywhere.

Proof of (1.6). Parts (i)–(vii) in this form are proved in [3]. The assertion of (viii) corresponds to Sec. 36.3 of [2]. The ω-periodicity of $f_h(u)$, $F(u)$ for an ω-periodic function u is also proved in [2].

Let us assume that the relation in (ix) is satisfied, i.e.

$$\int_0^\infty \int_\omega^{2\omega} (f_h(u)(t) - f_h(v)(t))(u'(t) - v'(t))\,dt\,\chi(h)\,dh = 0\,.$$

By (viii) we have for almost all $h > 0$ and a.e. $t > 0$

$$(f_h(u)(t) - f_h(v)(t))(u'(t) - v'(t) \geqq \tfrac{1}{2}\frac{d}{dt}[f_h(u)(t) - f_h(v)(t)]^2\,.$$

We conclude that

$$\text{(1.7)} \qquad ([(f_h(u))'(t) - u'(t)] - [(f_h(v))'(t) - v'(t)])(f_h(u)(t) - f_h(v)(t)) = 0$$

is fulfilled for a.e. $h > 0$ and $t \geqq \omega$.

Let us choose $t \geqq 2\omega$ such that $u'(t)$, $v'(t)$ exist. Let e.g. $u'(t) > 0$ and put $\bar{t} = \max\{\tau \in [0, \omega],\ |u(\tau)| = \|u\|_{[0,\omega]}\}$. For $u(\bar{t}) > 0$ we put $t_1 = \bar{t}$, $t_2 = \max\{\tau \in [t_1, t_1 + \omega],\ u(\tau) = \min\{u(\eta),\ \eta \in [t_1, t_1 + \omega]\}\}$. For $u(\bar{t}) < 0$ we put $t_0 = \bar{t}$, $t_1 = \max\{\tau \in [t_0, t_0 + \omega], u(\tau) = \max\{u(\eta),\ \eta \in [t_0, t_0 + \omega]\}\}$, $t_2 = t_0 + \omega$. By induction we define

$$\text{(1.8)} \qquad \begin{aligned} t_{2j+1} &= \max\{\tau \in [t_{2j}, t],\ u(\tau) = \max\{u(\eta), \eta \in [t_{2j}, t]\}\}\,, \\ t_{2j+2} &= \max\{\tau \in [t_{2j+1}, t],\ u(\tau) = \min\{u(\eta), \eta \in [t_{2j+1}, t]\}\}\,, \end{aligned}$$

until $t_k = t$.

The assumption $u'(t) > 0$ ensures that there exists $n \geqq 1$ such that $t = t_{2n+1}$. Following (1.6) (iv)–(vi) we have for all $h > 0$ and $j = 1, \ldots, n$

$$\text{(1.9)} \qquad \begin{aligned} f_h(u)(t_{2j+1}) &= f_h(u)(t_{2j}) + \min\{2h, u(t_{2j+1}) - u(t_{2j})\}\,, \\ f_h(u)(t_{2j}) &= f_h(u)(t_{2j-1}) - \min\{2h, u(t_{2j-1}) - u(t_{2j})\}\,. \end{aligned}$$

The following lemma is obvious.

(1.10) **Lemma.** *Let u: $[a, b] \to R^1$ be absolutely continuous, $u(a) \leqq u(\tau) \leqq u(b)$ ($u(a) \geqq u(\tau) \geqq u(b)$) for $\tau \in [a, b]$. Put $\bar{u}(\tau) = \max\{u(\eta),\ a \leqq \eta \leqq \tau\}$ ($\bar{u}(\tau) = \min\{u(\eta),\ a \leqq \eta \leqq \tau\}$, respectively). Then $\bar{u}$ is absolutely continuous and if we denote $M = \{\tau \in (a, b), \bar{u}'(\tau) \neq 0\}$, then $u(\tau) = \bar{u}(\tau)$ for every $\tau \in M$ and $u'(\tau) = \bar{u}'(\tau)$ for a.e. $\tau \in M$.*

(1.11) **Lemma.** *Let u be absolutely continuous, let $u'(t) > 0$, $(F(u))'(t)$ exist, and let $\{t_k, k = 1, \ldots, 2n\}$ be the sequence (1.8). If $2h > u(t) - u(t_{2n})$, then $(f_h(u))'(t) = u'(t)$, if $2h < u(t) - u(t_{2n})$, then $(f_h(u))'(t) = 0$.*

Proof of (1.11). The first part is obvious. For $2h < u(t) - u(t_{2n})$ we introduce the function $\bar{u}$ from (1.10) in $[t_{2n}, t]$. For every $\tau < t$ we have $\bar{u}(\tau) < \bar{u}(t) = \bar{u}(\tau) + \int_\tau^t \bar{u}'(\eta)\,d\eta$, hence meas $([\tau, t] \cap M) > 0$. Thus we can find $\tau_0 < t$ such that $u(\tau_0) = \bar{u}(\tau_0)$ and for $\tau \in (\tau_0, t)$ we have $u(\tau) - u(t_{2n}) > 2h$, $u(t) - u(\tau) < 2h$. For $\tau > \tau_0$ we have $f_h(\bar{u})(\tau) = h$ and $f_h(\bar{u})(\tau) - f_h(u)(\tau) = \bar{u}(\tau) - u(\tau)$. Similarly we find $\tau_1 > t$ such that $\bar{u}(\tau_1) = u(\tau_1) < u(t_{2n-1})$ and $u(\tau_1) - u(\tau) < 2h$ for $\tau \in [t, \tau_1)$. The assertion of (1.11) follows from the fact that $\bar{u}'(t) = u'(t)$. □

The function v in (1.7) is nonconstant. Indeed, if $v(t) \equiv v_0 \leqq 0$, then for sufficiently small $h > 0$ we have $f_h(u)(t) = h$, $(f_h(u))'(t) = 0$, $f_h(v)(t) \leqq 0$ and this contradicts (1.7). For $v_0 > 0$ we find t' such that $u'(t') < 0$ and we proceed as above.

Let us denote by $\{s_k\}$ the sequence (1.8) corresponding to v. If $\{s_k\}$ is infinite, we have for sufficiently small $h > 0$: $f_h(u)(t) = h$, $(f_h(u))'(t) = 0$, and for sufficiently large j: $-h < f_h(v)(s_{2j}) < f_h(v)(t) < f_h(v)(s_{2j+1}) < h$, $v'(t) = (f_h(v))'(t) = 0$, which contradicts (1.7).

Let us assume $t = s_{2m}$ for some $m > 1$. Then $v'(t) \leqq 0$ and for sufficiently small $h > 0$ we again obtain a contradiction to (1.7).

We conclude that $t = s_{2m+1}$ for some $m \geqq 1$ and relations analogous to (1.9) hold for v and $\{s_k\}$. Moreover, for the same reasons as above the assumption $v(t) - v(s_{2m}) < 2h < u(t) - u(t_{2n})$ leads to a contradiction.

Thus we have proved (notice that the case $u'(t) < 0$ is analogous: the mapping f_h is odd):

(1.12) **Lemma.** *Let the assumptions of* (1.6) (ix) *be satisfied and let* $u'(t)$, $v'(t)$, $(F(u))'(t)$, $(F(v))'(t)$ *exist*, $t \geqq 2\omega$.

If $u'(t) > 0$, *then* $t = t_{2n+1} = s_{2m+1}$ *and*

(i) $u(t) - u(t_{2n}) \leqq v(t) - v(s_{2m})$,

if $u'(t) < 0$, *then* $t = t_{2n} = s_{2m}$ *and*

(ii) $u(t_{2n-1}) - u(t) \leqq v(s_{2m-1}) - v(t)$.

Let us prove

(1.13) **Lemma.** *Let the assumptions of* (1.12) *be satisfied*, $u'(t) > 0$. *Then* $t_{2n} = s_{2m}$, $u(t) - u(t_{2n}) = v(t) - v(s_{2m})$.

Proof of (1.13). *A. Assume* $s_{2m} \leqq t_{2n-1}$. Following (1.10) we construct the function $\bar{u}$ in $[t_{2n-1}, t_{2n}]$. Put $r_0 = \min\{\tau \in [t_{2n-1}, t_{2n}],\ \bar{u}(\tau) = u(t_{2n})\}$. For every $\tau < r_0$ we have meas $[\tau, r_0] \cap M > 0$, hence there exists a sequence $\tau_j \nearrow r_0$, $\bar{u}'(\tau_j) = u'(\tau_j) < 0$, $v'(\tau_j)$ exist. Following (1.10), (1.12) (we have $\tau_j > t_1$, which is sufficient for (1.12)) we find $\sigma_j \in (s_{2m}, \tau_j)$ such that $u(t_{2n-1}) - u(\tau_j) \leqq v(\sigma_j) - v(\tau_j)$. We can assume $\sigma_j \to \sigma_0$ and passing to the limit as $j \to \infty$ we obtain

$$u(t_{2n-1}) - u(t_{2n}) \leqq v(\sigma_0) - v(r_0)\,, \tag{1.14}$$

where $v(t) \geqq v(\sigma_0) \geqq v(\sigma) \geqq v(r_0)$ for $\sigma \in [\sigma_0, r_0]$. Next we put

$$r_1 = \max\{\sigma \in [\sigma_0, t];\ v(\sigma) = \min\{v(\eta),\ \eta \in [\sigma_0, t]\}\}\,,$$

$$r_2 = \min\{\sigma \in [r_1, t],\ v(\sigma) = v(\sigma_0)\}\,.$$

We have $r_2 > r_1 \geqq r_0$, $v(r_1) \leqq v(r_0)$. Using the function $\bar{v}$ in $[r_1, r_2]$ we find sequences $\sigma_j \nearrow r_2$, $\tau_j \in (t_{2n-1}, \sigma_j)$ such that $v'(\sigma_j) > 0$, $u'(\sigma_j)$ exist and

$$v(\sigma_j) - v(r_1) \leqq u(\sigma_j) - u(\tau_j) \leqq u(\sigma_n) - u(t_{2n}),$$

$$v(r_2) - v(r_1) \leqq u(r_2) - u(t_{2n}).$$

From (1.14) we conclude $u(r_2) \geqq u(t_{2n-1})$, which contradicts (1.8).

B. Assume $t_{2n-1} < s_{2m} < t_{2n}$. Following (1.10) we construct $\bar{v}$ in $[s_{2m}, t]$. Obviously $\bar{v}(t_{2n}) > v(s_{2m})$. Put $r_0 = \min\{\sigma \in [s_{2m}, t_{2n}], v(\sigma) = \bar{v}(t_{2n})\}$. We find again the sequences $\sigma_j \nearrow r_0$, $\tau_j \in (t_{2n-1}, \sigma_j)$, $\tau_j \to \tau_0$ such that $v'(\sigma_j) > 0$, $u'(\sigma_j)$ exist and $v(\sigma_j) - v(s_{2m}) \leqq u(\sigma_j) - u(\tau_j)$,

$$v(r_0) - v(s_{2m}) \leqq u(r_0) - u(\tau_0). \tag{1.15}$$

We see that $r_0 < t_{2n}$, $u(t_{2n}) \leqq u(\tau_0) \leqq u(\tau) \leqq u(r_0)$ for $\tau \in [\tau_0, r_0]$. Put $r_1 = \max\{\tau \in [\tau_0, t_{2n}], \quad u(\tau) = \max\{u(\eta), \eta \in [\tau_0, t_{2n}]\}\}, \quad r_2 = \min\{\tau \in [r_1, t_{2n}], u(\tau) = u(\tau_0)\}$. We define the function $\bar{u}$ in $[r_1, r_2]$ and find the sequences $\tau_j \nearrow r_2$, $\sigma_j \in (r_0, \tau_j)$ such that $u'(\tau_j) < 0$, $v'(\tau_j)$ exist and

$$u(r_1) - u(\tau_j) \leqq v(\sigma_j) - v(\tau_j) \leqq v(r_0) - v(\tau_j),$$

$$u(r_1) - u(r_2) \leqq v(r_0) - v(r_2).$$

Now (1.15) implies $v(r_2) \leqq v(s_{2m})$, which contradicts (1.8).

C. Assume $s_{2m} \geqq t_{2n}$. We define $\bar{v}$ in $[s_{2m}, t]$ and similarly as above we obtain

$$v(t) - v(s_{2m}) \leqq u(t) - u(t_{2n}).$$

For $s_{2m} > t_{2n}$ we use A, B, where u is replaced by v and vice versa. Lemma (1.13) is proved. □

We can complete the proof of (1.6) (ix). For a fixed $t \geqq 2\omega$ such that $u'(t)$, $v'(t)$ exist, $u'(t) > 0$ there are two possibilities:

(a) $\forall s \in (t_{2n}, t), u(s) < u(t)$. In $[t_{2n}, t]$ we define $\bar{u}$ and find $\tau_j \nearrow t$ as above. Following (1.13) we have $u(\tau_j) - u(t_{2n}) = v(\tau_j) - v(t_{2n})$, hence

$$\frac{u(t) - u(\tau_j)}{t - \tau_j} = \frac{v(t) - v(\tau_j)}{t - \tau_j},$$

which implies $v'(t) = u'(t)$.

(b) $\exists s \in (t_{2n}, t)$, $u(s) = u(t)$. Then the derivative $(F(u))'(t)$ does not exist. Since the set of such points is of measure zero, the proof of (1.6) (ix) is complete.

Remark. In general, in the nonperiodic case the assumption that (1.7) holds for all $h > 0$ and $t > 0$ does not imply $u'(t) = v'(t)$: it suffices to choose u bounded and non-decreasing, v constant, $v(t) \equiv v_0 > \|u\|$.

Potential energies

Let us denote by $W^{k,p}(0, T)$, $k = 1, 2, \ldots, 1 \leqq p \leqq \infty$ the usual Sobolev space. We define the expressions

$$(1.16) \qquad \text{(i)} \quad P_1(u)(t) = \tfrac{1}{2}\int_0^\infty f_h^2(u)(t)\,\chi(h)\,\mathrm{d}h \quad \text{for} \quad u \in C([0, T]),$$

$$\text{(ii)} \quad P_2(u)(t) = \tfrac{1}{2}(F(u))'(t)\,u'(t) \quad \text{for} \quad u \in W^{1,\infty}(0, T).$$

(1.17) Energy inequalities.

(i) *Let* $u \in W^{1,1}(0, T)$. *Then* $P_1(u) \in W^{1,1}(0, T)$ *and*

$$(P_1(u))'(t) - F(u)(t)\,u'(t) \leqq 0 \quad \textit{a.e. in} \quad [0, T].$$

(ii) *Let us assume that there exists a positive decreasing function* γ *such that for all* $K > 0$ *and for almost all* $x \in (0, K]$ *we have* $\chi(x) \geqq \gamma(K)$. *Then for every* $u \in W^{2,1}(0, T)$ *the inequality*

$$P_2(u)(s) - P_2(u)(t) - \int_t^s (F(u))'(\sigma)\,u''(\sigma)\,\mathrm{d}\sigma \leqq$$

$$\leqq -\tfrac{1}{4}\gamma(\|u\|_{[0,s]})\int_t^s |u'(\sigma)|^3\,\mathrm{d}\sigma$$

holds for a.e. t, s, $0 \leqq t < s \leqq T$.

Proof. Part (i) follows immediately from (1.6) (viii).

(ii) Let $0 \leqq t < s \leqq T$ be given. The set $\{\sigma \in (t, s), u'(\sigma) \neq 0\}$ is a countable union of open intervals. Hence it suffices to assume that u is strictly monotone in $[t, s]$. Let u be increasing in $[t, s]$ (the other case is symmetric). Put $\bar{t} = \max\{\tau \in [0, t], |u(\tau)| = \|u\|_{[0,t]}\}$, and assume

A. $\bar{t} < t$. We put $\bar{t} = t_0$ for $u(\bar{t}) < 0$ and $\bar{t} = t_1$ for $u(\bar{t}) > 0$. Let $\{t_k\}$ be the sequence (1.8), $k = 0, 1, \ldots$, For j such that $u(t_{2j+1}) \in [u(t), u(s)]$ we denote by $\tau_{2j+1} \in [t, s]$ the point where $u(\tau_{2j+1}) = u(t_{2j+1})$. In the case $\bar{t} = t_0$, $u(s) > |u(t_0)|$ we find $\tau_0 \in (t, s)$, $u(\tau_0) = -u(t_0)$. The singular points τ_{2j+1} are those where $(F(u))'$ and consequently $P_2(u)$ are not defined. Thus we deal only with the case $t, s \neq \tau_{2j+1}$. Moreover, we can assume that t, s are separated by at most one point τ_{2j+1}. In the opposite case we replace the interval $[t, s]$ by a countable union of intervals having the required property (notice that the only possible limit point of τ_{2j+1} is t, and in that case we have $u'(t) = 0$, $|F(u))'(\sigma)| \leqq c|u'(\sigma)| \to 0$ for $\sigma \searrow t$, $\sigma \neq \tau_{2j+1}$). By (1.6) (iv)–(vii) the following cases are possible:

$$(1.18) \qquad \text{(i)} \quad F(u)(\sigma) = F(u)(t_{2n}) + 2\varphi(\tfrac{1}{2}(u(\sigma) - u(t_{2n}))), \quad \sigma \in [t, s];$$

$$\text{(ii)} \quad F(u)(\sigma) = \begin{cases} F(u)(t_0) + 2\varphi(\tfrac{1}{2}(u(\sigma) - u(t_0))), & \sigma \in [t, \tau), \\ \varphi(u(\sigma)), & \sigma \in [\tau, s]; \end{cases}$$

(iii) $F(u)(\sigma) = F(u)(t_{2n+2}) + 2\varphi(\frac{1}{2}(u(\sigma) - u(t_{2n+2})))\,, \quad \sigma \in [t, \tau)\,,$
$F(u)(t_{2n}) + 2\varphi(\frac{1}{2}(u(\sigma) - u(t_{2n})))\,, \quad \sigma \in [\tau, s]\,,$

where $u(t_{2n}) < u(t_{2n+2})$.

B. $\bar{t} = t$. For $u(t) < 0$ we have (1.18) (i) or (ii), for $u(t) > 0$ we obtain

(1.18) (iv) $F(u)(\sigma) = \varphi(u(\sigma))\,, \quad \sigma \in [t, s]\,.$

The proof follows from an easy integration by parts.

(1.19) Remark. Integrating directly in (1.16) (i) we obtain another expression for $P_1(u)$:

$$P_1(u)(t) = u(t)\,\varphi(u(t)) - \Phi(u(t)) \quad \text{if} \quad F(u)(t) = \varphi(u(t))\,,$$

$$P_1(u)(t) = P_1(u)(t_0) + \tfrac{1}{2}(u(t) - u(t_0))(F(u)(t_0) + F(u)(t))$$

if

$$F(u)(t) = F(u)(t_0) + 2\varphi(\tfrac{1}{2}(u(t) - u(t_0)))\,,$$

and similarly for u nonincreasing.

2. AN ORDINARY DIFFERENTIAL EQUATION

Let us consider the problem

(2.1) (i) $u'' + F(u) = 0\,,$

(ii) $u(0) = K > 0\,, \quad u'(0) = 0\,,$

where F is given by (1.3), (1.5).

(2.2) **Theorem.** *There exists a unique classical solution $u\colon [0, \infty) \to R^1$ to the problem* (2.1) (i), (ii) *and this solution has the following properties:*

(i) *There exists a sequence $0 = t_0 < t_1 < t_2 < \dots$ such that $(-1)^n u$ is decreasing in $[t_n, t_{n+1}]$ and*

$$\lim_{n\to\infty} (t_{n+1} - t_n) = \frac{\pi}{\sqrt{\varphi'(0+)}}\,.$$

(ii) *There exist positive decreasing functions $R(t)$, $\varrho(t)$ such that*

$$\lim_{t\to\infty} R(t) = 0\,, \quad \lim_{t\to\infty} \frac{\log(\varrho(t))}{t} = 0$$

and

$$\varrho(t) \leqq |u'(t)| + |F(u)(t)| \leqq R(t)\,.$$

Proof. The existence and uniqueness of the solution follows from the Lipschitz continuity (1.6) (ii) of F. The total energy of the system is given by

(2.3) $$E_1(t) = \tfrac{1}{2}u'^2(t) + P_1(u)(t)\,.$$

From (1.17) (i) we immediately have $E_1'(t) \leqq 0$ a.e. Let us denote $F_0 = F(u)(0) = \varphi(K)$. There exists $t_1 > 0$ such that u is decreasing in $[0, t_1]$. In the case $u(t_1) < -K$ we have

$$E_1(t_1) = P_1(u)(t_1) > P_1(u)(0) = E_1(0),$$

which contradicts the energy inequality. Hence $u(t_1) \geqq -K$, $u'(t_1) = 0$ and (cf. (1.6) (iv)) $F(u)(t) = F_0 - 2\varphi(\frac{1}{2}(K - u(t)))$ for $t \in [0, t_1]$. Consequently, $u'^2(t) = 2F_0(K - u(t)) - 8\Phi(\frac{1}{2}(K - u(t)))$. Putting $z(t) = \frac{1}{2}(K - u(t))$ we obtain $z'(t) = \sqrt{(F_0 z - 2\,\Phi(z))}$, $z(0) = 0$.

Let $0 < z_1 < K$ be the unique positive root of the equation $F_0 z = 2\,\Phi(z)$. We have $z(t_1) = z_1$ and $\varphi'(z_1) > 0$. Put $K_1 = K - 2z_1$ and $F_1 = -F(u)(t_1) = -F(u)(t_1) = -F_0 + 2\,\varphi(z_1) = -(2\,\Phi(z_1)/z_1) + 2\,\varphi(z_1) > 0$.

We continue by induction, finding the sequences z_n, F_n, t_n, K_n such that $0 = t_0 < t_1 < \ldots$, $u(t_n) = K_n$, $u'(t_n) = 0$, $F_n = (-1)^n F(u)(t_n) > 0$, $z_{n+1} = \frac{1}{2}(-1)^n \cdot (K_n - K_{n+1})$, $F_n z_{n+1} = 2\,\Phi(z_{n+1})$, and for $t \in [t_n, t_{n+1}]$ the function $(-1)^n u$ is decreasing and $F(u)(t) = (-1)^n [F_n - 2\,\varphi(\frac{1}{2}(-1)^n (K_n - u(t)))]$. In particular, $F_{n+1} = -F_n + 2\,\varphi(z_{n+1})$, hence

$$F_n - F_{n+1} = 2\left(2\frac{\Phi(z_{n+1})}{z_{n+1}} - \varphi(z_{n+1})\right) = \frac{2}{z_{n+1}} \int_0^{z_{n+1}} \int_0^{\xi} \eta\,\chi(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi > 0\,.$$

Let us introduce the function α: $[0, 2\,\varphi(+\infty)) \to R^1$, $\alpha(x) = 2(x - \varphi(z))$, where $x = 2\,\Phi(z)/z$. We have $\alpha(0) = 0$,

$$\alpha'(x) = \left(2 - \frac{z^2\,\varphi'(z)}{z\,\varphi(z) - \Phi(z)}\right) = \left(\int_0^z \xi^2\,\chi(\xi)\,\mathrm{d}\xi\right)\left(\int_0^z \xi\,\varphi'(\xi)\,\mathrm{d}\xi\right)^{-1} > 0$$

and $\lim_{x \to 0+} \alpha'(x) = 0$. Moreover, $\alpha(x)/x < 1$ for every $x > 0$. Put

$$G(x) = -\int_x^{F_0} \frac{\mathrm{d}\xi}{\alpha(\xi)}\,.$$

We see that G is increasing, $\lim_{x \to 0+} G(x) = -\infty$ and $\lim_{x \to 0+} \log x/G(x) = \lim_{x \to 0+} \alpha(x)/x = 0$.

We have

$$G(F_n) - G(F_{n+1}) = \frac{F_n - F_{n+1}}{\alpha(F_{n+1} + \theta_n(F_n - F_{n+1}))} = \frac{\alpha(F_n)}{\alpha(F_{n+1} + \theta_n(F_n - F_{n+1}))}$$

for some $\theta_n \in [0, 1]$, hence

$$1 \leqq G(F_n) - G(F_{n+1}) \leqq \frac{\alpha(F_n)}{\alpha(F_{n+1})} = \frac{\alpha(F_n)}{\alpha(F_n - \alpha(F_n))}\,.$$

We immediately obtain $G(F_n) \leqq -n$, hence $F_n \leqq G^{-1}(-n) \to 0$ (and therefore $z_n \to 0$) as $n \to +\infty$. Further,

$$\lim_{x \to 0+} \frac{\alpha(x - \alpha(x))}{\alpha(x)} = 1 - \lim_{x \to 0+} \frac{1}{\alpha(x)} \int_{x - \alpha(x)}^{x} \alpha'(\xi)\,\mathrm{d}\xi = 1\,,$$

and consequently

$$(2.4)\qquad F_n \geqq G^{-1}(-\alpha_0 n) \quad \text{for some} \quad \alpha_0 > 1\,.$$

Let us estimate the difference $t_{n+1} - t_n$. We have

$$t_{n+1} - t_n = \int_0^{z_{n+1}} \frac{\mathrm{d}z}{\sqrt{(F_n z - 2\,\Phi(z))}} = \frac{z_{n+1}}{\sqrt{2}} \int_0^1 \frac{\mathrm{d}\sigma}{\sqrt{(\sigma\,\Phi(z_{n+1}) - \Phi(\sigma z_{n+1}))}}\,.$$

On the other hand, the relation

$$\sigma\,\Phi(z) - \Phi(\sigma z) = \sigma \int_0^z \int_{\sigma\xi}^{\xi} \varphi'(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi$$

implies

$$\tfrac{1}{2}\,\varphi'(z)\,z^2\sigma(1 - \sigma) \leqq \sigma\,\Phi(z) - \Phi(\sigma z) \leqq \tfrac{1}{2}\,\varphi'(0+)\,z^2\sigma(1 - \sigma)\,.$$

This yields

$$(2.5)\qquad \frac{\pi}{\sqrt{(\varphi'(0+))}} \leqq t_{n+1} - t_n \leqq \frac{\pi}{\sqrt{(\varphi'(z_{n+1}))}}\,,$$

and (2.2) (i) is proved.

Let us now choose $t \in (t_n, t_{n+1})$, $n \geqq 1$. We have $u'^2(t) = 4(F_n\,z(t) - 2\,\Phi(z(t)))$ and $(F(u)\,(t))^2 = (F_n - 2\,\varphi(z(t)))^2$, where $z(t) = \frac{1}{2}(-1)^n\,(K_n - u(t))$. We obtain $|u'(t)| + |F(u)\,(t)| \leqq c_1 F_n \leqq c_2 F_{n+1}$ (notice that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 \quad \text{and} \quad t \leqq t_{n+1} \leqq \frac{\pi(n+1)}{\sqrt{\varphi'(z_1)}}\Big)\,,$$

hence

$$|u'(t)| + |F(u)\,(t)| \leqq c_2\,G^{-1}\left(-\frac{\sqrt{\varphi'(z_1)}}{\pi}\,t\right).$$

On the other hand,

$$(F(u)\,(t))^2 + \frac{\varepsilon}{2}\,\frac{\varphi(z(t))}{z(t)}\,.\,(u'(t))^2 =$$

$$= F_n^2 - 2(2 - \varepsilon)\,F_n\,\varphi(z) + 4\,\varphi^2(z) - 4\varepsilon\,\frac{\varphi(z)\,\Phi(z)}{z}\,.$$

Substituting $\Phi(z) \leqq \varkappa z\,\varphi(z)$ for $z \in [0, z_1]$, where $\varkappa = (1 - \varphi'(z_1)/2\,\varphi'(0+)) < 1$, we obtain

$$(F(u)\,(t))^2 + \tfrac{1}{2}\varepsilon\,\varphi'(0+)\,(u'(t))^2 \geqq F_n^2 - 2(2 - \varepsilon)\,F_n\,\varphi(z) + 4(1 - \varkappa\varepsilon)\,\varphi^2(z) \geqq$$
$$\geqq c_3 F_n^2$$

provided $\varepsilon < 4(1 - \varkappa)$. Therefore (cf. (2.4), (2.5)).

$$|F(u)\,(t)| + |u'(t)| \geqq c_4 G^{-1}\left(-\frac{\alpha_0\sqrt{\varphi'(0+)}}{\pi}\,t\right).$$

We put

$$\varrho(t) = c_4 G^{-1}\left(-\frac{\alpha_0\sqrt{\varphi'(0+)}}{\pi}\,t\right), \quad R(t) = c_2 G^{-1}\left(-\frac{\sqrt{\varphi'(z_1)}\,t}{\pi}\right).$$

Indeed,

$$\lim_{t\to\infty}\frac{\log\varrho(t)}{t} = -\frac{\pi}{\alpha_0\sqrt{\varphi'(0+)}}\lim_{x\to 0+}\frac{\log(c_4 x)}{G(x)} = 0$$

and the proof is complete.

(2.6) Examples. (i) Let us assume that for every $K > 0$ and for almost all $x \in (0, K]$ we have

$$\Gamma(K)\,x^q \geqq \chi(x) \geqq \gamma(K)\,x^p$$

for some $p \geqq q > -1$, where $\gamma \geqq 0$ does not vanish identically. We obtain

$$c_1 t^{-1/1+q} \leqq |u'(t)| + |F(u)(t)| \leqq c_2 t^{-1/1+p}$$

(cf. [4] for $p = q = 0$).

(ii) Theorem (2.2) does not imply $u(t) \to 0$ for $t \to +\infty$. Let, for example, $\Phi(x) = cx^{1+\alpha}$ for $x \geqq x_0$, $\alpha \in (0, 1)$. For sufficiently large K we have $F_0 = c(1+\alpha)K^\alpha$, $z_1 = (\alpha + 1/2)^{1/\alpha} K$. By induction we obtain $z_{n+1} = \alpha^{1/\alpha} z_n$ until $z_{n+1} < x_0$. We have $U_\infty = \lim_{t\to\infty} u(t) = K + 2\sum_{n=1}^{2N}(-1)^n z_n$. The sequence $\{z_n\}$ is decreasing, hence the inequalities

$$0 \leqq K + 2\sum_{n=1}^{2N}(-1)^n z_n - U_\infty \leqq 2z_{2N}$$

hold for all $N > 0$.

Put

$$A = 1 - \left(\frac{\alpha+1}{2}\right)^{1/\alpha}\frac{2}{1+\alpha^{1/\alpha}} > 0\,.$$

For $\varepsilon \in (0, 1)$ we find $N > 0$ such that

$$\left(\frac{\alpha+1}{2}\right)^{1/\alpha}\left(\frac{2\alpha^{1/\alpha}}{1+\alpha^{1/\alpha}} + 1\right)\alpha^{2N/\alpha} < \varepsilon A\,.$$

Choose K such that $z_{2N} \geqq x_0$. Then we have $A(1+\varepsilon) > (U_\infty/K) > A(1-\varepsilon)$. Therefore $\lim_{K\to\infty}(U_\infty/K) = A$. Notice that in general cases the problem of determination of U_∞ remains open.

(iii) In the last example the energy (2.3) of the solution of (2.1) does not tend to zero as $t \to +\infty$. By (1.19) we have $E_\infty = \lim_{t\to\infty} E_1(t) = K\,\varphi(K) - \Phi(K) - \sum_{n=0}^{\infty} z_{n+1}(F_n - F_{n+1})$. A computation analogous to (ii) yields

$$\lim_{K\to\infty}\frac{E_\infty}{cK^{1+\alpha}} = \alpha - 2\left(\frac{\alpha+1}{2}\right)^{1+1/\alpha}\frac{1-\alpha}{1-\alpha^{1+1/\alpha}} > 0$$

(notice that for $f(x) = x^{1+1/\alpha}$ we have $[f(x) - f(y)]/(x - y) > f'[(x + y)/2]$ provided $x \neq y$). This shows that the initial mechanical energy is not completely dissipated. The quantity E_∞ corresponds to the inaccessible remainder of the potential energy.

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Souhrn

O IŠLINSKÉHO MODELU PRO NE ZCELA PRUŽNÁ TĚLESA

PAVEL KREJČÍ

Hlavním obsahem práce je formulace některých nových vlastností hysterezního operátoru Išlinského, který vyjadřuje např. vztah mezi deformací a napětím v ne zcela pružném (pružně plastickém) materiálu. Jsou zde definovány dva funkcionály energie a odvozeny energetické nerovnosti. Jako příklad je zkoumána rovnice $u'' + F(u) = 0$, která popisuje volné kmity hmotného bodu na pružně plastické pružině.

Резюме

О МОДЕЛИ ИШЛИНСКОГО ДЛЯ НЕ ВПОЛНЕ УПРУГИХ ТЕЛ

PAVEL KREJČÍ

Главной целью работы является формулировка некоторых новых свойств гистерезисного оператора Ишлинского F, который выражает напр. соотношение между деформацией и напряжением в не вполне упругом (упруго-пластическом) материале. Вводятся два функционала энергии и доказываются энергетические неравенства. В качестве примера исследуется уравнение $u'' + F(u) = 0$, описывающее движение массовой точки на упруго-пластической пружине.

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