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DOMAIN OPTIMIZATION IN AXISYMMETRIC ELLIPTIC BOUNDARY VALUE PROBLEMS BY FINITE ELEMENTS

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Summary. An axisymmetric second order elliptic problem with mixed boundary conditions is considered. A part of the boundary has to be found so as to minimize one of four types of cost functionals. The existence of an optimal boundary is proven and a convergence analysis for piecewise linear approximate solutions presented, using weighted Sobolev spaces.

Keywords: domain optimization, finite elements, elliptic problems.
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INTRODUCTION

One often meets elliptic boundary value problems in three-dimensional domains Ω, which are generated by the rotation of a bounded plane domain D around an axis. Then the most suitable approach is to use the cylindrical coordinates. If the data of the problem are axially symmetric, the problem is reduced to the two-dimensional domain D (see [2] for a detailed finite element analysis).

Let a part Γ of the boundary ∂D be optimized. We arrive at a generalization of a two-dimensional domain optimization problem, which has been analyzed thoroughly by Bégis and Glowinski in [1]. In the axisymmetric problem, however, we are forced to employ weighted Sobolev spaces, where the radial coordinate plays the role of the weight.

Section 1 contains some results valid for the whole family $\{W_{2,r}^{(1)}(D(\alpha))\}$, $\alpha \in U_{\mathrm{ad}}$, of the weighted spaces under consideration. In Section 2 we formulate the state problem and prove the continuous dependence of its solution on the design variable. Section 3 contains the definitions of four domain optimization problems and the proof of existence of optimal solutions. Approximate problems are proposed in Section 4 and a convergence proof is presented for three types of the cost functionals.

1. SOME PRELIMINARY RESULTS

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$, axially symmetric with respect to the x_3-axis, if $\boldsymbol{x} = (x_1, x_2, x_3)$ denote the Cartesian coordinates. Let $H^k(\Omega)$ denote the standard Sobolev space $W^{(k),2}(\Omega)$, $k = 1, 2, \dots$, with the usual norm $\|\cdot\|_{k,\Omega}$. Passing to the cylindrical coordinates r, z, ϑ, for any function $u(x_1, x_2, x_3)$ we define

$$\hat{u}(r, z, \vartheta) = u(r\cos\vartheta, r\sin\vartheta, z)\,. \tag{1}$$

If $x \in \Omega$, then $(r, z) \in D$, $\vartheta \in [0,2\pi)$ and by (1) a mapping $Z: H^1(\Omega) \to ZH^1(D \times [0, 2\pi))$ is defined. Introducing the norm in ZH^1

$$\|\hat{u}\|_1 = \left(\int_D \int_0^{2\pi} \left[\hat{u}^2 + \left(\frac{\partial \hat{u}}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial \hat{u}}{\partial \vartheta}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\vartheta\right)^{1/2},$$

for $\hat{u} = Zu$ we obtain

$$\|\hat{u}\|_1 = \|u\|_{1,\Omega}\,. \tag{2}$$

Let W_0 be the subspace of axisymmetric functions, i.e.

$$W_0 = \left\{\hat{u} \in ZH^1(D \times [0,2\pi)) \,\middle|\, \frac{\partial \hat{u}}{\partial \vartheta} = 0 \quad \text{a.e.}\right\}.$$

It is easy to see that Z is an isomorphism and W_0 is closed in ZH^1. Moreover, we have

$$(2\pi)^{-1}\,\|\hat{u}\|_1^2 = \int_D \left[u^2 + \left(\frac{\partial \hat{u}}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] r\,\mathrm{d}r\,\mathrm{d}z \quad \forall \hat{u} \in W_0\,. \tag{3}$$

The square-root of the integral in the right-hand side represents a norm

$$\|\hat{u}\|_{1,r,D}$$

in the weighted Sobolev space $W^{(1)}_{2,r}(D)$ (see e.g. [3]). Consequently, the subspace W_0 can be identified with the space $W^{(1)}_{2,r}(D)$.

In the optimization process we shall consider a specific class of admissible domains $D(\alpha)$, where

$$D(\alpha) = \{(r, z) \mid 0 < r < \alpha(z),\ 0 < z < 1\}$$

and the function $\alpha(z)$ – the design variable – belongs to the following set of admissible functions

$$U_{\mathrm{ad}} = \left\{\alpha \in C^{(0),1}([0, 1])\,, \quad \text{(i.e. Lipschitz functions)}\,,\right.$$

$$\left.\alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max}\,, \quad |\mathrm{d}\alpha/\mathrm{d}z| \leqq C_1\,, \quad \int_0^1 \alpha^2(z)\,\mathrm{d}z = C_2\right\}$$

with given positive constants $\alpha_{\min}, \alpha_{\max}, C_1, C_2$. Assume that $\alpha_{\min} > \alpha_{\max}/2$.

Let $\Gamma(\alpha)$ denote the graph of the function α,

$$\Gamma_1 = \partial D(\alpha) \cap \{z = 0\}\,, \quad \Gamma_2 = \partial D(\alpha) \cap \{z = 1\}$$

(see Fig. 1).

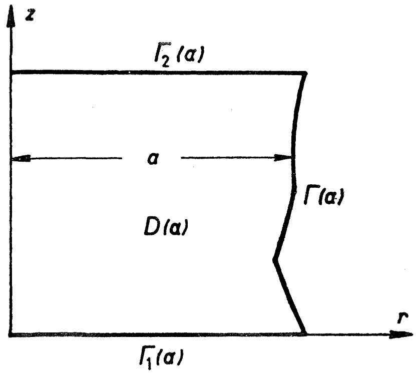

Fig. 1

Next we present several properties of the spaces $W_{2,r}^{(1)}(D(\alpha))$, $\alpha \in U_{\text{ad}}$.

For any $\alpha \in U_{\text{ad}}$ the space $W_{2,r}^{(1)}(D(\alpha))$ is a *reflexive Banach space* [4] and the *set* $C^\infty(\bar{D}(\alpha))$ *is dense in it* (see e.g. [3] – chapt. 6. § 2).

Let $S_0(\alpha) \subset \partial\Omega(\alpha)$ be generated by the rotation of a curve segment $\Gamma_0(\alpha) \subset \partial D(\alpha)$ around the z-axis $\mathcal{O}$, $\text{mes}_2\, S_0(\alpha) > 0$. Then the Trace Theorem [3] says that a linear mapping $\tilde{\gamma}$: $H^1(\Omega(\alpha)) \to L^2(S_0(\alpha))$ exists such that $\tilde{\gamma}u = u$ holds for smooth functions and

$$\|\tilde{\gamma}u\|_{L^2(S_0(\alpha))} \leqq C(\alpha)\, \|u\|_{1,\Omega(\alpha)}\,. \tag{4}$$

Passing to the cylindrical coordinates, we obtain the existence of a linear mapping

$$\gamma\colon W_{2,r}^{(1)}(D(\alpha)) \to L_{2,r}(\Gamma_0(\alpha))\,, \quad \Gamma_0(\alpha) \subset \partial D(\alpha) \dot{-} \mathcal{O}\,,$$

such that $\gamma\hat{u} = \hat{u}$ for smooth functions and

$$\|\gamma\hat{u}\|_{0,r,\Gamma_0(\alpha)} \leqq C(\alpha)\, \|\hat{u}\|_{1,r,D(\alpha)}\,, \tag{5}$$

where

$$L_{2,r}(\Gamma_0(\alpha)) = \left\{v \middle| \int_{\Gamma_0(\alpha)} v^2 r\, \mathrm{d}s \equiv \|v\|^2_{0,r,\Gamma_0(\alpha)} < +\infty\right\}.$$

The constant $C(\alpha)$ in (5) is the same as that in (4), but it depends on the domain $D(\alpha)$ and on $\Gamma_0(\alpha)$. Hence a question arises, if a "common" constant for all admissible variables exists. An answer is contained in the following

Lemma 1. *There exists a positive constant C such that*

$$\|\gamma u\|_{0,r,\Gamma_1(\alpha)\cup\Gamma(\alpha)} \leqq C\|u\|_{1,r,D(\alpha)}$$

holds for all $\alpha \in U_{ad}$ and $u \in W^{(1)}_{2,r}(D(\alpha))$.

Proof. Let $D_0 = (0, 1) \times (0, 1)$ be the unit square.

The mapping $(r, z) \to (\xi_1, \xi_2)$, defined through the relations

$$\xi_1 = r/\alpha(z)\,, \quad \xi_2 = z$$

maps the domain $D(\alpha)$ onto D_0. It is readily seen that for the functions

$$\tilde{u}(\xi_1, \xi_2) = u(\xi_1\alpha(\xi_2), \xi_2)$$

the following estimate holds

$$\int_{D_0} (|\nabla \tilde{u}|^2 + \tilde{u}^2)\, \xi_1 \,\mathrm{d}\xi \leqq C_0\|u\|^2_{1,r,D(\alpha)} \quad \forall\alpha \in U_{ad}\,, \tag{6}$$

where C_0 depends on α_{min}, α_{max} and C_1, but not on α. We may write

$$\begin{aligned}\int_{\Gamma_1\cup\Gamma(\alpha)} u^2 r \,\mathrm{d}s &= \int_{\Gamma_1} u^2 r \,\mathrm{d}r + \int_0^1 u^2\, \alpha(z)\, (1 + (\alpha')^2)^{1/2}\, \mathrm{d}z \leqq \\ &\leqq \alpha^2_{max} \int_{\tilde{\Gamma}_1} (\tilde{u}(\xi_1))^2\, \xi_1 \,\mathrm{d}\xi_1 + (1 + C_1^2)^{1/2}\, \alpha_{max} \int_0^1 \tilde{u}^2(1, \xi_2)\, \mathrm{d}\xi_2 \leqq \\ &\leqq C_4 \int_{\tilde{\Gamma}_0} \tilde{u}^2 \xi_1 \,\mathrm{d}s\,, \quad C_4 = C_4(\alpha_{max}, C_1)\,,\end{aligned} \tag{7}$$

where $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_0$ denotes the corresponding image of Γ_1 and $\Gamma_0 = \Gamma_1 \cup \Gamma(\alpha)$, respectively. Combining (7), (6) and the inequality (5), applied to the square D_0, we obtain

$$\int_{\Gamma_0} u^2 r \,\mathrm{d}s \leqq C_4 C^2 C_0 \|u\|^2_{1,r,D(\alpha)} \quad \text{Q.E.D.}$$

Lemma 2. *The set*

$$M(D(\alpha)) = \{u \in C^\infty(\mathrm{Cl}\,(D(\alpha)))|\ \mathrm{supp}\, u \cap \Gamma_2(\alpha) = \emptyset\}$$

is dense in the subspace

$$V(D(\alpha)) = \{v \in W^{(1)}_{2,r}(D(\alpha))\ |\ \gamma v = 0 \text{ on } \Gamma_2(\alpha)\}$$

for any positive $\alpha \in C^{(0),1}([0, 1])$.

Proof. Let $\hat{u} \in V(D(\alpha))$ and denote $u = Z^{-1}\hat{u}$. By the above mentioned density a sequence $\{\hat{w}_k\}$, $k \to \infty$, exists such that $\hat{w}_k \in C^\infty(\mathrm{Cl}\,(D(\alpha)))$ and

$$\|\hat{w}_k - \hat{u}\|_{1,r,D(\alpha)} \to 0\,. \tag{8}$$

From (2), (3)

$$\|Z^{-1}\hat{w}_k - u\|_{1,\Omega(\alpha)} \to 0 \tag{9}$$

follows. It is well-known (see e.g. [3]), that a sequence $\{u_n\}$, $n \to \infty$, $u_n \in C^\infty(\mathrm{Cl}\,(\Omega(\alpha)))$, exists such that

$$\operatorname{supp} u_n \cap S_2(\alpha) = \emptyset\,,$$

$$\|u_n - u\|_{1,\Omega(\alpha)} \to 0\,, \tag{10}$$

(where $S_2(\alpha)$ is generated by rotation of $\Gamma_2(\alpha)$).

Let us define $\hat{u}_n = Zu_n$ and

$$\omega_n(r, z) = (2\pi)^{-1} \int_0^{2\pi} \hat{u}_n(r, z, \vartheta)\,\mathrm{d}\vartheta\,.$$

We can see that

$$\operatorname{supp} \omega_n \cap \Gamma_2(\alpha) = \emptyset\,, \quad \omega_n \in C^\infty(\mathrm{Cl}\,(D(\alpha)))\,.$$

Moreover, we may write

$$|\omega_n(r, z) - \hat{w}_k(r, z)|^2 = (2\pi)^{-2}\left[\int_0^{2\pi} (\hat{u}_n(r, z, \vartheta) - \hat{w}_k(r, z))\,\mathrm{d}\vartheta\right]^2 \leqq$$

$$\leqq (2\pi)^{-1} \int_0^{2\pi} (\hat{u}_n - \hat{w}_n)^2\,\mathrm{d}\vartheta\,,$$

$$\left|\frac{\partial}{\partial r}(\omega_n - \hat{w}_k)\right|^2 \leqq (2\pi)^{-1} \int_0^{2\pi} \left(\frac{\partial \hat{u}_n}{\partial r} - \frac{\partial \hat{w}_k}{\partial r}\right)^2 \mathrm{d}\vartheta$$

and a similar estimate for $\partial(\omega_n - \hat{w}_n)/\partial z$.

Consequently, we have for $n \to \infty$, $k \to \infty$

$$\|\omega_n - \hat{w}_k\|^2_{1,r,D(\alpha)} = \tag{11}$$

$$= \int_0^1 \mathrm{d}z \int_0^{\alpha(z)} [(\omega_n - \hat{w}_k)^2 + |\nabla(\omega_n - \hat{w}_k)|^2]\, r\,\mathrm{d}r \leqq$$

$$\leqq \int_0^1 \mathrm{d}z \int_0^{\alpha(z)} (2\pi)^{-1} \int_0^{2\pi} [(\hat{u}_n - \hat{w}_k)^2 + |\nabla(\hat{u}_n - \hat{w}_k)|^2]\,\mathrm{d}\vartheta\, r\,\mathrm{d}r \leqq$$

$$\leqq (2\pi)^{-1}\, \|\hat{u}_n - \hat{w}_k\|^2_1 = (2\pi)^{-1}\, \|u_n - Z^{-1}w_k\|^2_{1,\Omega(\alpha)} \to 0,$$

since

$$\|u_n - Z^{-1}\hat{w}_k\|_{1,\Omega(\alpha)} \leqq \|u_n - u\|_{1,\Omega(\alpha)} + \|u - Z^{-1}\hat{w}_k\|_{1,\Omega(\alpha)} \to 0$$

on the basis of (10), (9).

Combining (11) and (8), we derive finally

$$\|\omega_n - \hat{u}\|_{1,r,D(\alpha)} \leqq \|\omega_n - \hat{w}_k\|_{1,r,D(\alpha)} + \|\hat{w}_k - \hat{u}\|_{1,r,D(\alpha)} \to 0 .$$

Since $\omega_n \in M(D(\alpha))$, the lemma is proved.

Lemma 3. *There exists a positive constant C_3 such that*

$$\int_{D(\alpha)} |\nabla u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \geqq C_3 \|u\|^2_{1,r,D(\alpha)}$$

holds for all $u \in V(D(\alpha))$ and $\alpha \in U_{\mathrm{ad}}$.

Proof. Let us introduce a constant $\delta \in (\alpha_{\max}, 2\alpha_{\min})$ and the rectangle

$$\hat{D} = (0, \delta) \times (0, 1) .$$

We shall construct an extension $Eu \in W^{(1)}_{2,r}(\hat{D})$ of the function $u \in V(D(\alpha))$ as follows:

$$Eu(r, z) = u(2\,\alpha(z) - r, z) \quad \text{on} \quad \hat{D} - D(\alpha) \tag{12}$$

and $Eu = u$ on $D(\alpha)$. Then we have obviously $\gamma(Eu) = 0$ on $z = 1$ and we can prove that

$$\int_{\hat{D}} |\nabla(Eu)|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \tilde{C} \int_{D(\alpha)} |\nabla u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z , \tag{13}$$

$$\|Eu\|_{1,r,\hat{D}} \leqq C \|u\|_{1,r,D(\alpha)} \tag{13'}$$

where the constants $\tilde{C}, C$ are independent of α. To this end, one employs the estimate

$$|\nabla Eu(r, z)|^2 \leqq (2 + 4(\alpha')^2) \, |\nabla u(2\alpha(z) - r, z)|^2$$

for $(r. z) \in \hat{D} - D(\alpha)$.

Then we prove by a standard way, that

$$\int_{\hat{D}} (Eu)^2 \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \int_{\hat{D}} \left(\frac{\partial Eu}{\partial z}\right)^2 r \, \mathrm{d}r \, \mathrm{d}z \quad \forall u \in V(D(\alpha)) .$$

(First we verify this for $u_n \in M(\hat{D})$ and then pass to the limit on the basis of Lemma 2.)

Therefore we have

$$\int_{\hat{D}} (Eu)^2 \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \int_{\hat{D}} |\nabla Eu|^2 \, r \, \mathrm{d}r \, \mathrm{d}z$$

and from (13)

(14)

$$\int_{D(\alpha)} u^2 r \, \mathrm{d}r \, \mathrm{d}z \leqq \int_{\hat{D}} (Eu)^2 \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \int_{\hat{D}} |\nabla Eu|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \leqq \tilde{C} \int_{D(\alpha)} |\nabla u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z$$

follows. The assertion of the lemma is an immediate consequence of (14), with $C_3 = (1 + \tilde{C})^{-1}$. Q.E.D.

Lemma 4. *The embedding*

$$W^{(1)}_{2,r}(D) \hookrightarrow L_{2,r}(D)$$

is compact.

Proof. Let M be a bounded subset of $W^{(1)}_{2,r}(D)$. Since

$$\|Z^{-1}\hat{u}\|^2_{1,\Omega} = \|\hat{u}\|^2_1 = 2\pi\|\hat{u}\|^2_{1,r,D},$$

the set $Z^{-1}M$ is bounded in $H^1(\Omega)$.

Rellich's Theorem ([3]) implies that $Z^{-1}M$ is precompact in $L^2(\Omega)$.

Let $Z^{-1}\hat{u}_n \to \omega$ in $L^2(\Omega)$. We also have

$$\|Z^{-1}\hat{u}\|^2_{0,\Omega} = 2\pi\|\hat{u}\|^2_{0,r,D}$$

and therefore

$$\|Z^{-1}\hat{u}_n - Z^{-1}\hat{u}_m\|_{0,\Omega} \to 0 \Rightarrow \|\hat{u}_n - \hat{u}_m\|_{0,r,D} \to 0$$

(for $n \to \infty$, $m \to \infty$). Since the space $L_{2,r}(D)$ is complete, the sequence $\{\hat{u}_n\}$ is convergent in $L_{2,r}(D)$. Consequently, the set M is precompact in $L_{2,r}(D)$.

2. THE STATE PROBLEM AND THE CONTINUOUS DEPENDENCE OF ITS SOLUTION ON THE DESIGN VARIABLE

We shall consider the following boundary value problem

$$-\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(A_i(x)\frac{\partial u}{\partial x_i}\right) = F \quad \text{in} \quad \Omega(\alpha), \tag{15}$$

$$\sum_{i=1}^{3} \nu_i A_i \frac{\partial u}{\partial x_i} = G \quad \text{on} \quad S_1(\alpha) \cup S(\alpha), \tag{16}$$

$$u = 0 \quad \text{on} \quad S_2(\alpha), \tag{17}$$

where $\Omega(\alpha)$ is generated by rotation of $D(\alpha)$ around the x_3-axis, $S_i(\alpha)$ by rotation of $\Gamma_i(\alpha)$, $i = 1, 2$ and $S(\alpha)$ by rotation of $\Gamma(\alpha)$, (see Fig. 1), ν_i are components of the unit outward normal with respect to $\partial\Omega(\alpha)$.

Let $\hat{\Omega}$ be the cylindrical domain generated by rotation of the rectangle $\hat{D} = (0, \delta) \times (0, 1)$, $\delta > \alpha_{\max}$.

Assume that the function F in (15) is determined as the restriction to $\Omega(\alpha)$ of an axisymmetric function $F \in L^2(\hat{\Omega})$,

$$G = \begin{cases} 0 & \text{on} \quad S(\alpha), \\ G_1 & \text{on} \quad S_1(\alpha), \end{cases}$$

where G_1 is determined as the restriction to $S_1(\alpha)$ of an axisymmetric function $G_1 \in L^2(\hat{S}_1)$, $\hat{S}_1 = \partial\hat{\Omega} \cap \{x \mid x_3 = 0\}$.

Assume that the coefficients A_i are restrictions to $\Omega(\alpha)$ of axisymmetric functions $A_i \in L^\infty(\hat{\Omega})$, $A_1 = A_2$ a.e. and a positive constant a_0 exists such that

$$A_i(x) \geqq a_0 \quad \text{a.e.} \quad \text{in} \quad \hat{\Omega}\,.$$

Let us denote $A_1 = A_2 = a_r$, $A_3 = a_z$.

Passing to the cylindrical coordinate system, we transform the standard variational formulation of the problem (15), (16), (17) to the following *state problem*:

$$\text{find} \quad y \in V(D(\alpha) \quad \text{such that}$$

$$a(\alpha; y, v) = L(\alpha; v) \quad \forall v \in V(D(\alpha))\,, \tag{18}$$

where

$$a(\alpha; y, v) = \int_{D(\alpha)} \left(a_r \frac{\partial y}{\partial r}\frac{\partial v}{\partial r} + a_z \frac{\partial y}{\partial z}\frac{\partial v}{\partial z} \right) r \, \mathrm{d}r \, \mathrm{d}z\,,$$

$$L(\alpha; v) = \int_{D(\alpha)} fvr \, \mathrm{d}r \, \mathrm{d}z + \int_{\Gamma_1(\alpha)} g\gamma vr \, \mathrm{d}s\,,$$

where the function $f \in L_{2,r}(\hat{D})$ and $g \in L_{2,r}(\tilde{\Gamma}_1)$, $\tilde{\Gamma}_1 = \partial\hat{D} \cap \{z = 0\}$, are given. Finally, $a_r, a_z \in L^\infty(\hat{D})$,

$$a_r(r, z) \geqq a_0\,, \quad a_z(r, z) \geqq a_0 \quad \text{a.e.} \quad \text{in} \quad \hat{D}\,. \tag{19}$$

Remark 1. The variational formulation (18) corresponds with the following "classical" one:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(ra_r\frac{\partial y}{\partial r}\right) - \frac{\partial}{\partial z}\left(a_z\frac{\partial y}{\partial z}\right) = f \quad \text{in} \quad D(\alpha)\,,$$

$$v_r a_r \frac{\partial y}{\partial r} + v_z a_z \frac{\partial y}{\partial z} = \begin{cases} 0 & \text{on} \quad \Gamma(\alpha) \\ g & \text{on} \quad \Gamma_1(\alpha)\,, \end{cases}$$

$$y = 0 \quad \text{on} \quad \Gamma_2(\alpha)\,.$$

Lemma 5. *Positive constants C_4, C_5 exist such that the inequalities*

$$a(\alpha; u, u) \geqq C_4 \|u\|^2_{1,r,D(\alpha)} \quad \forall u \in V(D(\alpha))\,, \tag{20}$$

$$|a(\alpha; u, v)| \leqq C_5 \|u\|_{1,r,D(\alpha)} \|v\|_{1,r,D(\alpha)} \quad \forall u, v \in W^{(1)}_{2,r}(D(\alpha)) \tag{21}$$

hold for any $\alpha \in U_{\text{ad}}$.

Proof. Using (19) and Lemma 3, we obtain

$$a(\alpha; u, u) \geqq a_0 \int_{D(\alpha)} |\nabla u|^2 \, r \, \mathrm{d}r \, \mathrm{d}z \geqq C_3 a_0 \|u\|^2_{1,r,D(\alpha)}\,.$$

By virtue of the boundedness of a_r, a_z, we may write

$$a(\alpha; u, v) \leqq C_5 \int_{D(\alpha)} |\nabla u| \, |\nabla v| \, r \, \mathrm{d}r \, \mathrm{d}z \leqq C_5 \|u\|_{1,r,D(\alpha)} \, \|v\|_{1,r,D(\alpha)} \,,$$

where $C_5 = \max \{ \|a_r\|_{\infty,\hat{D}}, \|a_z\|_{\infty,\hat{D}} \}$.

Lemma 6. *A positive constant C_6 exists such that*

$$|L(\alpha; u)| \leqq C_6 \|u\|_{1,r,D(\alpha)} \quad \forall u \in W_{2,r}^{(1)}(D(\alpha))$$

holds for any $\alpha \in U_{\mathrm{ad}}$.

Proof. Using Lemma 1, we may write

$$|L(\alpha; u)| = \left| \int_{D(\alpha)} fur \, \mathrm{d}r \, \mathrm{d}z + \int_{\Gamma_1(\alpha)} g \gamma u r \, \mathrm{d}r \right| \leqq$$

$$\leqq \|f\|_{0,r,D} \, \|u\|_{0,r,D(\alpha)} + \|g\|_{0,r,\Gamma_1(\alpha)} \, \|\gamma u\|_{0,r,\Gamma_1(\alpha)} \leqq$$

$$\leqq (\|f\|_{0,r,\hat{D}} + C \|g\|_{0,r,\hat{\Gamma}_1}) \, \|u\|_{1,r,D(\alpha)} \,.$$

Lemma 7. *The state problem (18) has a unique solution $y = y(\alpha)$ for any $\alpha \in U_{\mathrm{ad}}$.*

Proof – follows from the Riesz Theorem, since the space $V(D(\alpha))$ can be equipped with the inner product $a(\alpha; u, v)$, on the basis of Lemma 5. Moreover, we employ Lemma 6 to show the continuity of the right-hand side in (18). Q.E.D.

Proposition 1. *Assume that a sequence $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \in U_{\mathrm{ad}}$, converges to a function α in $C([0, 1])$. Let us construct extensions $E\, y(\alpha_n) \in W_{2,r}^{(1)}(\hat{D})$ of the solutions $y(\alpha_n)$ of the state problem (18) by means of the formula (12).*
Then

$$(22) \qquad Ey(\alpha_n)|_{D(\alpha)} \rightharpoonup y(\alpha) \quad (\textit{weakly}) \quad \textit{in} \quad W_{2,r}^{(1)}(D(\alpha)) \,,$$

where $y(\alpha)$ is the solution of the state problem (18) on $D(\alpha)$.

Proof. Let us denote $y(\alpha_n) = y_n$, $D(\alpha_n) = D_n$, $D(\alpha) = D$. Using Lemmas 5 and 6, we may write

$$C_4 \|y_n\|_{1,r,D_n}^2 \leqq a(\alpha_n; y_n, y_n) = L(\alpha_n; y_n) \leqq C_6 \|y_n\|_{1,r,D_n} \,.$$

Consequently, we have

$$(23) \qquad \|y_n\|_{1,r,D_n} \leqq C_6/C_4 \quad \forall n \,.$$

On the basis of the relation (13')

$$(24) \qquad \|Ey_n\|_{1,r,\hat{D}} \leqq C \|y_n\|_{1,r,D_n} \leqq C_7$$

holds. Since $W_{2,r}^{(1)}(\hat{D})$ is reflexive Banach space, there exist a subsequence $\{y_k\} \subset \{y_n\}$

and $y \in W_{2,r}^{(1)}(\hat{D})$ such that

$$Ey_k \rightharpoonup y \quad \text{(weakly)} \quad \text{in} \quad W_{2,r}^{(1)}(\hat{D}) . \tag{25}$$

Let us show that $y|_D$ solves the state problem (18). It is easy to verify that the limit function $\alpha \in U_{\text{ad}}$. Let $v \in V(D)$ be arbitrary and let us construct $Ev \in W_{2,r}^{(1)}(\hat{D})$. Obviously, $Ev \in V(\hat{D})$, so that the restriction

$$Ev|_{D_n} \in V(D_n) \quad \forall n ,$$

since the trace $\gamma(Ev)$ vanishes on $\Gamma_2(\alpha_n)$. Consequently, we may insert Ev into the definition (18) to obtain

$$a(\alpha_k; y_k, Ev) = L(\alpha_k; Ev) . \tag{26}$$

Let us pass to the limit with $k \to \infty$ on both sides of (26). We have

$$|a(\alpha; y_k, Ev) - a(\alpha; y, v)| \leqq |a(\alpha_k; y_k, Ev) - a(\alpha; Ey_k, Ev)| +$$

$$+ |a(\alpha; Ey_k, Ev) - a(\alpha; y, v)| \equiv I_k + J_k ,$$

$$I_k \leqq \int_{\triangle(D_k, D)} \left| a_r \frac{\partial Ey_k}{\partial r} \frac{\partial Ev}{\partial r} + a_z \frac{\partial Ey_k}{\partial z} \frac{\partial Ev}{\partial z} \right| r \, dr \, dz \leqq$$

$$\leqq C \|\nabla Ey_k\|_{0,r,\hat{D}} \|\nabla Ev\|_{0,r,\triangle(D_k,D)} ,$$

where

$$\triangle(D_k, D) = (D_k \dot{-} D) \cup (D \dot{-} D_k) .$$

Since

$$\lim_{k \to \infty} \text{mes} \left(\triangle(D_k, D)\right) = 0$$

and

$$\|\nabla Ey_k\|_{0,r,\hat{D}} \leqq \|Ey_k\|_{1,r,\hat{D}} \leqq C_7 ,$$

we conclude that $I_k \to 0$ for $k \to \infty$.

By virtue of Lemma 5 the functional

$$u \to a(\alpha; u, v)$$

is linear and continuous on $W_{2,r}^{(1)}(\hat{D})$. Consequently,

$$J_k \to 0$$

follows from the weak convergence (25). Thus we arrive at the conclusion

$$\lim_{k \to \infty} a(\alpha_k; y_k, Ev) = a(\alpha; y, v) . \tag{27}$$

We may write

$$|L(\alpha_k; Ev) - L(\alpha; v)| \leqq \left|\int_{\Gamma_1(\alpha_k)} gEvr \, \mathrm{d}s - \int_{\Gamma_1(\alpha)} gEvr \, \mathrm{d}s\right| +$$

$$+ \left|\int_{D_k} fEvr \, \mathrm{d}r \, \mathrm{d}z - \int_D fEvr \, \mathrm{d}r \, \mathrm{d}z\right| = M_k + N_k ,$$

$$M_k \leqq \int_{\triangle(\Gamma_1(\alpha_k), \Gamma_1(\alpha))} |gEv| \, r \, \mathrm{d}r \leqq \|g\|_{0,r,\Gamma_1} \|Ev\|_{0,r,\triangle} \to 0 ,$$

since

(28′) $$\lim_{k\to\infty} \operatorname{mes} \left(\triangle(\Gamma_1(\alpha_k), \Gamma_1(\alpha))\right) = 0 .$$

Moreover, it holds

$$N_k \leqq \int_{\triangle(D_k, D)} |fEv| \, r \, \mathrm{d}r \, \mathrm{d}z \to 0 .$$

Altogether, we have

(28) $$\lim_{k\to\infty} L(\alpha_k; Ev) = L(\alpha; v) .$$

Passing to the limit in (26) and using (27), (28), we arrive at the condition (18).

It is not difficult to verify that the subspace $V(\hat{D})$ is weakly closed in $W_{2,r}^{(1)}(\hat{D})$. Since every function $Ey_n \in V(\hat{D})$, the weak limit $y \in V(\hat{D})$ and its restriction $y|_D \in V(D)$. Hence $y|_D$ is a solution of (18).

Since the solution of (18) is unique and (25) implies that

$$Ey_k|_D \to y|_D \quad \text{(weakly)} \quad \text{in} \quad W_{2,r}^{(1)}(D) ,$$

the latter convergence holds for the *whole* sequence $\{Ey_n|_D\}$. Q.E.D.

3. DOMAIN OPTIMIZATION PROBLEMS AND THE EXISTENCE OF OPTIMAL SOLUTION

We shall consider the following four types of the cost functional:

$$j_1(\alpha, y) = \int_{D(\alpha)} (y - y_{\mathrm{d}})^2 \, r \, \mathrm{d}r \, \mathrm{d}z ,$$

$$j_2(\alpha, y) = \int_0^1 (y(\alpha(z), z) - y_\gamma)^2 \, \mathrm{d}z ,$$

where $y(\alpha(z), z) \equiv \gamma y$ denotes the trace of y on the curve $\Gamma(\alpha)$, $y_{\mathrm{d}} \in L_{2,r}(\hat{D})$ and $y_\gamma \in L^2([0, 1])$ are given functions,

$$j_3(\alpha, y) = a(\alpha; y, y) ,$$

$$j_4(\alpha, y) = \int_{D(\alpha)} \left[\left(a_r \frac{\partial y}{\partial r} - K_1\right)^2 + \left(a_z \frac{\partial y}{\partial z} - K_2\right)^2\right] r \, dr \, dz \, ,$$

where $K_i \in L_{2,r}(\hat{D})$ are given functions, $i = 1, 2$.

We define the *Domain Optimization Problems*:

$$\text{find} \quad \alpha^0 \in U_{ad} \quad \text{such that}$$

(29i)
$$j_i(\alpha^0, y(\alpha^0)) \leqq j_i(\alpha, y(\alpha)) \quad \forall \alpha \in U_{ad}\,, \quad i \in \{1, 2, 3, 4\}\,,$$

where $y(\alpha)$ denotes the solution of the state problem (18).

To the proof of existence of an optimal α^0 we shall need the following

Proposition 2. *Let the assumptions of Proposition* 1 *be satisfied. Then*

$$\lim_{n\to\infty} j_i(\alpha_n, y(\alpha_n)) = j_i(\alpha, y(\alpha))\,, \quad i = 1, 2, 3\,,$$

$$\liminf_{n\to\infty} j_4(\alpha_n, y(\alpha_n)) \geqq j_4(\alpha, y(\alpha))\,.$$

Proof. *Case* $i = 1$. Denoting again $y(\alpha_n) = y_n$, $y(\alpha) = y$, $D(\alpha_n) = D_n$, $D(\alpha) = D$, we conclude on the basis of Proposition 1 and Lemma 4 that

(30)
$$Ey_n|_D \to y \quad \text{in} \quad L_{2,r}(D)\,.$$

We may write

$$\int_D [(Ey_n - y_d)^2 - (y - y_d)^2]\, r \, dr \, dz = \int_D (Ey_n - y)(Ey_n + y - 2y_d)\, r \, dr \, dz \leqq$$
$$\leqq \|Ey_n - y\|_{0,r,D} \| Ey_n + y - 2y_d\|_{0,r,D}\,.$$

By virtue of (30), however, we obtain for $n \to \infty$

$$\|Ey_n - y\|_{0,r,D} \to 0\,, \quad \|Ey_n + y - 2y_d\|_{0,r,D} \leqq C\,.$$

Consequently,

(31)
$$\int_D (Ey_n - y_d)^2\, r \, dr \, dz \to \int_D (y - y_d)^2\, r \, dr \, dz\,.$$

It is easy to see that

(32)
$$\int_{D_n} (y_n - y_d)^2\, r \, dr \, dz = \int_D (Ey_n - y_d)^2\, r \, dr \, dz +$$
$$+ \int_{D_n - D} (Ey_n - y_d)^2\, r \, dr \, dz - \int_{D - D_n} (Ey_n - y_d)^2\, r \, dr \, dz\,.$$

Let us estimate the last two terms as follows:

$$(33)\qquad \left|\int_{D_n-D} - \int_{D-D_n}\right| \leqq \int_{\triangle(D_n,D)} (Ey_n - y_d)^2\, r\, dr\, dz \leqq$$

$$\leqq \left(\int_{\triangle(D_n,D)} r^2\, dr\, dz\right)^{1/2} \left(\int_{\hat{D}_0} (Ey_n - y_d)^4\, dr\, dz\right)^{1/2},$$

where $\hat{D}_0 = \hat{D} - (0, \alpha_{\min}/2) \times (0, 1)$.

Since the restriction $Ey_{n|\hat{D}_0} \in H^1(\hat{D}_0)$ and the embedding of $H^1(\hat{D}_0)$ into $L^4(\hat{D}_0)$ is continuous, we may write

$$\|Ey_n - y_d\|_{L^4(\hat{D}_0)} \leqq C\|Ey_n - y_d\|_{1,\hat{D}_0} \leqq$$

$$\leqq C(2/\alpha_{\min})^{1/2}\, \|Ey_n - y_d\|_{1,r,\hat{D}} \leqq \tilde{C},$$

where also (24) has been used.

Since

$$(34)\qquad \operatorname{mes}(\triangle(D_n, D)) \to 0,$$

the right-hand side of (33) tends to zero. Combining this result with (31) and (32), the assertion of the Proposition follows.

Case $i = 2$. Denoting $\alpha_m(z) = \alpha(z) - 1/m$, $m = 2, 3, \ldots$, we may write for n sufficiently great with respect to m:

$$(35)\qquad j_2(\alpha_n, y_n) - j_2(\alpha, y) = M_1 + M_2 + M_3,$$

$$M_1 = \int_0^1 (y_n(\alpha_n(z)) - y_\gamma)^2\, dz - \int_0^1 (y_n(\alpha_m(z)) - y_\gamma)^2\, dz,$$

$$M_2 = \int_0^1 (y_n(\alpha_m(z)) - y_\gamma)^2\, dz - \int_0^1 (y(\alpha_m(z)) - y_\gamma)^2\, dz,$$

$$M_3 = \int_0^1 (y(\alpha_m(z)) - y_\gamma)^2\, dz - \int_0^1 (y(\alpha(z)) - y_\gamma)^2\, dz$$

and estimate the individual terms.

It is readily seen that

$$|M_1| \leqq \int_0^1 |y_n(\alpha_n) - y_n(\alpha_m)|\, |y_n(\alpha_n) + y_n(\alpha_m) - 2y_\gamma|\, dz \leqq$$

$$\leqq \|y_n(\alpha_n) - y_n(\alpha_m)\|_0\, \|y_n(\alpha_n) + y_n(\alpha_m) - 2y_\gamma\|_0,$$

where $\|\cdot\|_0$ denotes the norm in $L^2([0, 1])$. Furthermore,

$$(36)\qquad \|y_n(\alpha_n) - y_n(\alpha_m)\|_0^2 = \int_0^1 dz \left(\int_{\alpha_m(z)}^{\alpha_n(z)} \frac{\partial y_n}{\partial r}\, dr\right)^2 \leqq$$

$$\leqq \int_0^1 \mathrm{d}z(\beta_n + 1/m) \int_{\alpha_m}^{\alpha_n} \left(\frac{\partial y_n}{\partial r}\right)^2 \mathrm{d}r \leqq (\beta_n + 1/m) \, \|y_n\|_{1,D_n^0}^2 ,$$

where

$$\beta_n = \max_{z \in [0,1]} |\alpha_n(z) - \alpha(z)| ,$$

$$D_n^0 = D_n - (0, \alpha_{\min}/2) \times (0, 1) ,$$

(considering only m such that $\alpha(z) - 1/m \geqq \alpha_{\min}/2$).

By virtue of (23)

(37) $$\|y_n\|_{1,D_n^0}^2 \leqq (2/\alpha_{\min}) \|y_n\|_{1,r,D_n}^2 \leqq C .$$

For sufficiently great $n, m, n > n_0(m)$, we derive

(38) $$\|y_n(\alpha_n) + y_n(\alpha_m) - 2y_\gamma\|_0^2 \leqq 3 \int_0^1 (y_n^2(\alpha_n) + y_n^2(\alpha_m) + 4y_\gamma^2) \, \mathrm{d}z \leqq C .$$

In fact, Lemma 1 and (23) yield that

(39) $$\int_0^1 y_n^2(\alpha_n) \, \mathrm{d}z \leqq \int_{\Gamma(\alpha_n)} (\gamma y_n)^2 \, \mathrm{d}s \leqq$$

$$\leqq \|y_n\|_{0,r,\Gamma(\alpha_n)}^2 \cdot (2/\alpha_{\min}) \leqq C(2/\alpha_{\min}) \|y_n\|_{1,r,D_n}^2 \leqq \tilde{C} .$$

Using (39), (36) and (37), we obtain

$$\int_0^1 y_n^2(\alpha_m) \, \mathrm{d}z \leqq 2 \int_0^1 y_n^2(\alpha_n) \, \mathrm{d}z + 2 \int_0^1 (y_n(\alpha_m) - y_n(\alpha_n))^2 \, \mathrm{d}z < C .$$

Combining (36), (37) and (38), we arrive at the estimate

(40) $$|M_1| \leqq C(\beta_n + 1/m)^{1/2} .$$

In a similar way we derive

(41) $$|M_2| \leqq \|y_n(\alpha_m) - y(\alpha_m)\|_0 \, \|y_n(\alpha_m) + y(\alpha_m) - 2y_\gamma\|_0 .$$

It holds, however, that

(42) $$\|y_n - y\|_{0,\Gamma(\alpha_m)}^2 = \int_{\Gamma(\alpha_m)} (y_n(\alpha_m) - y(\alpha_m))^2 \, \mathrm{d}s \to 0 \quad \text{for} \quad n \to \infty .$$

In fact, the mapping

$$\gamma \colon H^1(G_m^0) \to L^2(\partial G_m^0) ,$$

where

$$G_m^0 = G_m \dot{-} (0, \alpha_{\min}/2) \times (0, 1)$$

and

$$G_m = \{(r, z) \mid 0 < r < \alpha_m(z), \ 0 < z < 1\} ,$$

is completely continuous ([3] — chapt. 2, § 6.2). The weak convergence (22) of $\{Ey_n\}$ implies that

$$y_n \to y \quad \text{(weakly)} \quad \text{in} \quad H^1(G_m^0),$$

since $H^1(G_m^0) = W_{2,r}^{(1)}(G_m^0)$ and the norms are equivalent. Consequently, $y_n \to y$ in $L^2(\Gamma(\alpha_m))$ follows, i.e. (42) holds.

Thus we obtain that

$$\|y_n(\alpha_m) - y(\alpha_m)\|_0^2 \leqq \int_{\Gamma(\alpha_m)} (y_n(\alpha_m) - y(\alpha_m))^2 \, \mathrm{d}s = \|y_n - y\|_{0,\Gamma(\alpha_m)}^2 \to 0 \tag{43}$$

for $n \to \infty$.

Using (43), it is easy to see that

$$\begin{aligned} &\|y_n(\alpha_m) + y(\alpha_m) - 2y_\gamma\|_0 \leqq \\ &\leqq \|y_n(\alpha_m) - y(\alpha_m)\|_0 + 2\|y(\alpha_m) - y_\gamma\|_0 \leqq C \end{aligned} \tag{44}$$

for any fixed m and sufficiently great n.

Substituting (43), (44) into (41), we are led to the conclusion that

$$M_2 \to 0 \quad \text{for} \quad n \to \infty \quad \text{and any fixed} \quad m\,. \tag{45}$$

Finally, we may write

$$|M_3| \leqq \|y(\alpha_m) - y(\alpha)\|_0 \, \|y(\alpha_m) + y(\alpha) - 2y_\gamma\|_0\,, \tag{46}$$

$$\begin{aligned} \|y(\alpha_m) - y(\alpha)\|_0^2 &= \int_0^1 (y(\alpha_m) - y(\alpha))^2 \, \mathrm{d}z = \\ = \int_0^1 \mathrm{d}z \left(\int_{\alpha_m(z)}^{\alpha(z)} \frac{\partial y}{\partial r} \, \mathrm{d}r \right)^2 &\leqq \frac{1}{m} \int_0^1 \mathrm{d}z \int_{\alpha_m}^{\alpha} \left(\frac{\partial y}{\partial r} \right)^2 \mathrm{d}r \leqq \\ \leqq m^{-1} \|y\|_{1,D^0}^2 &\leqq C m^{-1} \|y\|_{1,r,D}^2 \to 0 \end{aligned}$$

for $m \to \infty$ (here $D^0 = D \dot{-} (0, \alpha_{\min}/2) \times (0, 1)$).

Moreover,

$$\|y(\alpha_m) + y(\alpha) - 2y_\gamma\|_0 \leqq \|y(\alpha_m) - y(\alpha)\|_0 + 2\|y(\alpha) - y_\gamma\|_0 \leqq C$$

follows easily. Substituting into (46), we obtain

$$M_3 \to 0 \quad \text{for} \quad m \to \infty\,. \tag{47}$$

It suffices to combine (34), (40), (45) and (47) to arrive at

$$\lim_{n \to \infty} j_2(\alpha_n, y_n) = j_2(\alpha, y)\,.$$

Case $i = 3$. By virtue of the definition (18) we have

$$j_3(\alpha, y) = a(\alpha; y, y) = L(\alpha; y),$$
$$j_3(\alpha_n, y_n) = a(\alpha_n; y_n, y_n) = L(\alpha_n; y_n).$$

Consequently, we may write

$$j_3(\alpha_n, y_n) - j_3(\alpha, y) = K_1 + K_2,$$

where

$$K_1 = L(\alpha_n; y_n) - L(\alpha; Ey_n),$$
$$K_2 = L(\alpha; Ey_n - y).$$

Let us estimate

$$|K_1| \leqq \left|\int_{D_n} f y_n r \,\mathrm{d}r\,\mathrm{d}z - \int_D fEy_n r \,\mathrm{d}r\,\mathrm{d}z\right| + \left|\int_{\Gamma_1(\alpha_n)} g y_n r \,\mathrm{d}r - \int_{\Gamma_1(\alpha)} gEy_n r \,\mathrm{d}r\right| \leqq$$
$$= \int_{\triangle(D_n, D)} |fEy_n| \, r \,\mathrm{d}r\,\mathrm{d}z + \int_{\triangle(\Gamma_1(\alpha_n), \Gamma_1(\alpha))} |gEy_n| \, r \,\mathrm{d}r = K_{11} + K_{12}.$$

We obviously have

$$K_{11} \leqq \|f\|_{0,r,\triangle(D_n,D)} \, \|Ey_n\|_{0,r,\hat{D}} \to 0$$

on the basis of (24), (34) and

$$K_{12} \leqq \|g\|_{0,r,\triangle(\Gamma_1(\alpha_n),\Gamma_1(\alpha))} \, \|Ey_n\|_{0,r,\hat{\Gamma}_1} \to 0$$

on the basis of (28′), (24) and (5) (which can be applied to $\alpha(z) \equiv \delta$, as well), since

$$\|Ey_n\|_{0,r,\hat{\Gamma}_1} \leqq C\|Ey_n\|_{1,r,\hat{D}} \leqq \tilde{C}.$$

Altogether,

$$\lim_{n\to\infty} K_1 = 0.$$

By virtue of Lemma 6 the functional $u \to L(\alpha; u)$ is continuous on $W_{2,r}^{(1)}(D)$. Consequently, the weak convergence (22) implies that

$$\lim_{n\to\infty} K_2 = 0.$$

Combining these two results we are led to the assertion to be proved.

Case $i = 4$. We have

$$j_4(\alpha_n, y_n) \geqq \int_{G_m} \left[\left(a_r \frac{\partial y_n}{\partial r} - K_1\right)^2 + \left(a_z \frac{\partial y_n}{\partial z} - K_2\right)^2\right] r \,\mathrm{d}r\,\mathrm{d}z$$

for any n, m such that $G_m \subset D_n$. The functional on the right-hand side is weakly lower semi-continuous on $W_{2,r}^{(1)}(G_m)$ (being convex and Gâteaux-differentiable).

The restrictions $y_n|_{G_m}$ converge to $y|_G$ weakly in the space $W_{2,r}^{(1)}(G_m)$, as follows from (22). Consequently, we have for any m

$$\liminf_{n\to\infty} j_4(\alpha_n, y_n) \geqq \int_{G_m} \left[\left(a_r \frac{\partial y}{\partial r} - K_1\right)^2 + \left(a_z \frac{\partial y}{\partial z} - K_2\right)^2\right] r \, dr \, dz \, .$$

Passing to the limit with $m \to \infty$, we obtain the assertion

$$\liminf_{n\to\infty} j_4(\alpha_n, y_n) \geqq j_4(\alpha, y) \, . \qquad \text{Q.E.D.}$$

Now we are able to prove the main result of the Section 3.

Theorem 1. *There exists at least one solution of the Domain Optimization Problem* (29i), $i \in \{1, 2, 3, 4\}$.

Proof. Let $\{\alpha_n\}$, $\alpha_n \in U_{\text{ad}}$, be a minimizing sequence of $j_i(\alpha, y(\alpha))$, where $i \in \in \{1, 2, 3, 4\}$, i.e.

$$\lim_{n\to\infty} j_i(\alpha_n, y(\alpha_n)) = \inf_{\alpha \in U_{\text{ad}}} j_i(\alpha, y(\alpha)) \, . \tag{48}$$

By means of Arzelà-Ascoli Theorem we show that the set U_{ad} is compact in $C([0, 1])$. Hence there exist a subsequence $\{\alpha_k\}$ and $\alpha^0 \in U_{\text{ad}}$ such that

$$\alpha_k \to \alpha^0 \quad \text{in} \quad C([0, 1]) \, .$$

Then Proposition 2 and (48) imply that

$$j_i(\alpha^0, y(\alpha^0)) \leqq \liminf_{k\to\infty} j_i(\alpha_k, y(\alpha_k)) = \inf_{\alpha \in U_{\text{ad}}} j_i(\alpha, y(\alpha)) \, .$$

Consequently, at α^0 a minimum is attained. Q.E.D.

4. APPROXIMATIONS BY FINITE ELEMENT METHOD

In the present Section we propose an approximate solution of the domain optimization problem, making use of piecewise linear design variable and linear triangular finite elements for solving the state problem.

Let N be a positive integer and $h = 1/N$. We denote by $\triangle_j$, $j = 1, 2, \ldots, N$, the subintervals $[(j - 1)\, h, jh]$ and introduce the set

$$U_{\text{ad}}^h = \{\alpha_h \in U_{\text{ad}}: \alpha_h|_{\triangle j} \in P_1(\triangle_j) \ \forall j\} \, .$$

where $P_1(\triangle_j)$ is the set of linear functions defined on $\triangle_j$. Let $D_h = D(\alpha_h)$ denote the domain bounded by the graph $\Gamma_h = \Gamma(\alpha_h)$ of the function $\alpha_h \in U_{\text{ad}}^h$. The polygonal domain D_h will be carved into triangles by the following way. We choose $\alpha_0 \in (0, \alpha)$

and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, independent of α_h, if h is fixed.

In the remaining part $D_h \dot{-} \mathscr{R}$ let the nodal points divide the segments $[\alpha_0, \alpha_h(jh)]$, $j = 1, 2, \ldots, N$, into M equal segments, where

$$M = 1 + [(\beta - \alpha_0) N]$$

and the square brackets denote the integer part of the number inside.

One can verify that the segments parallel with the r-axis are not longer than h and shorter than $h(\alpha - \alpha_0)/(\beta - \alpha_0)$. One also deduces the following estimate for the interior angles ω of the triangles T:

$$\operatorname{tg} \omega \geqq \frac{\alpha - \alpha_0}{\beta - \alpha_0} (1 + C_1 + C_1^2)^{-1} .$$

Consequently, one obtains a *regular family* $\{\mathscr{T}_h(\alpha_h)\}$, $h \to 0$, $\alpha_h \in U_{\text{ad}}^h$, *of triangulations*, with

$$\max_{T \in \mathscr{T}_h(\alpha_h)} (\operatorname{diam} T) \leqq h / \sin \omega_0 ,$$

$$\omega_0 = \operatorname{arctg} \left[\frac{\alpha - \alpha_0}{\beta - \alpha_0} (1 + C_1 + C_1^2)^{-1} \right] .$$

The family is even *strongly regular*, i.e. the ratio of the maximal and minimal side in $\mathscr{T}_h$ is not greater than a constant, independent of h and α_h.

Let us consider the standard space V_h of linear finite elements

$$V_h = \{ v_h \in C(\bar{D}_h) \cap V(D_h) \mid v_h|_T \in P_1(T) \ \forall T \in \mathscr{T}_h(\alpha_h) \} .$$

We define the approximate state problem:

$$\text{find} \quad y_h \equiv y_h(\alpha_h) \in V_h \quad \text{such that}$$

$$a(\alpha_h; y_h, v_h) = L_h(\alpha_h; v_h) \quad \forall v_h \in V_h . \tag{49}$$

Here $L_h(\alpha_h; v_h)$ denotes a suitable approximation of the functional $L(\alpha_h; v_h)$, which satisfies the following conditions:

positive constants C_8, C_9 and ϑ exist such that

$$|L_h(\alpha_h; v_h) - L(\alpha_h; v_h)| \leqq C_8 h^{\vartheta} \|v_h\|_{1,r,D_h} , \tag{50}$$

$$|L_h(\alpha_h; v_h)| \leqq C_9 \|v_h\|_{1,r,D_h} \tag{51}$$

hold for any $\alpha_h \in U_{\text{ad}}^h$ and any $v_h \in V_h$.

Let us define

$$L_h(\alpha_h; v_h) = \sum_{T \in \mathscr{T}_h(\alpha_h)} [f r v_h]_{G(T)} \operatorname{mes}(T) + \sum_{I \in \mathscr{T}_h(\alpha_h) \cap \Gamma_1(\alpha_h)} [g r v_h]_{G(I)} \operatorname{mes}(I) , \tag{52}$$

where $G(T)$ denotes the centre of gravity of the triangle T and $G(I)$ the midpoint of the interval $I = T \cap \Gamma_1(\alpha_h)$.

Lemma 8. *Let* $L_h(\alpha_h; v_h)$ *be defined by the formula* (52). *Assume that* $f \in H^1(\hat{D}) \cap \cap C(\hat{D})$, $r^2 D^\alpha f \in L^2(\hat{D})$ *for* $|\alpha| = 2$ *and* g *is piecewise from* C^2.
Then (50), (51) *hold, with* $\vartheta = 1$.

Proof. If we denote the local error by

$$E_T(w) = \int_T w \, dx - w(G) \operatorname{mes} T, \quad T \in \mathcal{T}_h(\alpha_h)$$

and $fr \equiv F$, we may write

$$\left| \int_{D_h} f v_h \, r \, dr \, dz - \sum_T [f r v_h]_{G(T)} \operatorname{mes} T \right| = \tag{53}$$

$$= \left| \sum_T E_T(F v_h) \right| \leqq \sum_T (|E_T(F \, v_h(G))| + |E_T(F(v_h - v_h(G)))|) \, .$$

Applying the affine mapping

$$(r, z) = B_T \hat{\boldsymbol{x}} + \boldsymbol{b}_T \, , \quad \hat{\boldsymbol{x}} = (\hat{r}, \hat{z}) \, ,$$

which transforms the reference unit triangle τ with the vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ onto T, we easily deduce

$$|E_T(w)| \leqq Ch^2 |\hat{E}(\hat{w})| \, , \tag{54}$$

where $\hat{w}(\hat{\boldsymbol{x}}) = w(B_T \hat{\boldsymbol{x}} + \boldsymbol{b}_T)$,

$$\hat{E}(\hat{w}) = \int_\tau (\hat{w} - \hat{w}(\gamma)) \, d\hat{\boldsymbol{x}}$$

and γ is the centroid of the triangle τ. Since

$$|\hat{E}(\hat{w})| \leqq C \|\hat{w}\|_{2,\tau} \, ,$$

$$\hat{E}(p) = 0 \quad \forall p \in P_1(\tau) \, ,$$

the Bramble-Hilbert Lemma yields [5]

$$|\hat{E}(\hat{w})| \leqq C |\hat{w}|_{2,\tau} \, .$$

Using the estimate ([5] – pp. 118, 122)

$$|\hat{w}|_{2,\tau} \leqq Ch |w|_{2,T} \, ,$$

we obtain

$$|E_T(w)| \leqq Ch^3 |w|_{2,T} \, . \tag{55}$$

Moreover, by the strong regularity of the family $\{\mathscr{T}_h(\alpha_h)\}$, we have ([5], p. 142)

$$|v_h(G)| \leqq \|v_h\|_{C(T)} \leqq Ch^{-1}\|v_h\|_{0,T} \quad \forall T \in \mathscr{T}_h(\alpha_h)\,, \tag{56}$$

so that

$$|E_T(Fv_h(G))| = |E_T(F)|\,|v_h(G)| \leqq Ch^2|F|_{2,T}\|v_h\|_{0,T} \tag{57}$$

follows from (55), (56).

Using again (54), we derive

$$|E_T(F(v_h - v_h(G)))| \leqq Ch^2|\hat{E}(\hat{F}(\hat{v}_h - \hat{v}_h(\gamma)))| = Ch^2\left|\int_\tau \hat{F}(\hat{v}_h - \hat{v}_h(\gamma))\,\mathrm{d}\hat{x}\right|. \tag{58}$$

If we define

$$\mathscr{F}(\hat{F}) = \int_\tau \hat{F}(\hat{v}_h - \hat{v}_h(\gamma))\,\mathrm{d}\hat{x}\,,$$

then

$$\mathscr{F}(p) = 0 \quad \forall p \in P_0(\tau)\,,$$

since $\hat{v}_h$ is linear and

$$|\mathscr{F}(\hat{F})| \leqq C\|\hat{v}_h - \hat{v}_h(\gamma)\|_{0,\tau}\,\|\hat{F}\|_{1,\tau}\,.$$

From the Bramble-Hilbert Lemma

$$|\mathscr{F}(\hat{F})| \leqq C\|\hat{v}_h - \hat{v}_h(\gamma)\|_{0,\tau}\,|\hat{F}|_{1,\tau}$$

follows. Since

$$|\hat{v}_h - \hat{v}_h(\gamma)| \leqq |\nabla \hat{v}_h . (\hat{x} - \gamma)| \leqq |\nabla \hat{v}_h|\,,$$

we have

$$\|\hat{v}_h - \hat{v}_h(\gamma)\|_{0,\tau} \leqq |\hat{v}_h|_{1,\tau}$$

and

$$|\mathscr{F}(\hat{F})| \leqq C|\hat{v}_h|_{1,\tau}\,|\hat{F}|_{1,\tau} \leqq \tilde{C}\|F\|_{1,T}\,\|v_h\|_{1,T}\,.$$

Substituting into (58), we arrive at

$$|E_T(F(v_h - v_h(G)))| \leqq Ch^2\|F\|_{1,T}\,\|v_h\|_{1,T}\,. \tag{59}$$

Combining (57), (59), we see that the upper bound in (53) is

$$E_f \equiv Ch^2\|F\|_{2,D_h}\,\|v_h\|_{1,D_h}\,. \tag{60}$$

For any $v_h \in V_h$ the following inequality holds

$$\|v_h\|_{1,D_h} \leqq Ch^{-1/2}\|v_h\|_{1,r,D_h}\,. \tag{61}$$

To prove this, we consider the subset $\mathscr{T}_h^{(1)}$ of triangles $T \in \mathscr{T}_h(\alpha_h)$, such that

$$T \cap \mathcal{O} \neq \emptyset$$

(where $\mathcal{O}$ is the z-axis). Let us use the mapping $r = h_1\hat{r}$, $z = z_T + h\hat{z}$, which transforms the "unit" triangle τ onto $T \in \mathcal{T}_h^{(1)}$. On T we have

$$v_h^2 + |\nabla v_h|^2 = p^2 + c^2\,, \quad p \in P_1(T)\,, \quad c \in P_0(T)\,,$$

(62) $$\|v_h\|_{1,T}^2 = \int_T (p^2 + c^2)\,\mathrm{d}r\,\mathrm{d}z = h_1 h \int_\tau (\hat{p}^2 + c^2)\,\mathrm{d}\hat{r}\,\mathrm{d}\hat{z}\,,$$

where $\hat{p}(\hat{r}, \hat{z}) = p(h_1\hat{r}, z_T + h\hat{z})$. Since in the finite-dimensional space $P_1(\tau) \times \times P_0(\tau)$ all norms are equivalent, we have

$$\int_\tau (\hat{p}^2 + c^2)\,\mathrm{d}\hat{r}\,\mathrm{d}\hat{z} \leqq C \int_\tau (\hat{p}^2 + c^2)\,\hat{r}\,\mathrm{d}\hat{r}\,\mathrm{d}\hat{z}\,.$$

Substituting into (62) leads us to the following estimate

(63) $$\|v_h\|_{1,T}^2 \leqq Ch_1 h \int_\tau (\hat{p}^2 + c^2)\,\hat{r}\,\mathrm{d}\hat{r}\,\mathrm{d}\hat{z} \leqq$$
$$\leqq Ch^{-1} \int_T (p^2 + c^2)\,r\,\mathrm{d}r\,\mathrm{d}z = Ch^{-1}\|v_h\|_{1,r,T}^2\,.$$

If $T \in \mathcal{T}_h(\alpha_h) \dot{-} \mathcal{T}_h^{(1)}$, then $h_1 \leqq r$ so that $1 \leqq Ch^{-1}r$ and

(64) $$\|v_h\|_{1,T}^2 = \int_T (v_h^2 + |\nabla v_h|^2)\,\mathrm{d}r\,\mathrm{d}z \leqq Ch^{-1}\|v_h\|_{1,r,T}^2\,.$$

Adding the results (63) and (64), we arrive at (61).

Consequently, instead of (60) we may write

(65) $$E_f \leqq Ch^{3/2}\|F\|_{2,D_h}\,\|v_h\|_{1,r,D_h}\,.$$

Next we have

(66) $$\left|\int_{\Gamma_1(\alpha_h)} g v_h r\,\mathrm{d}r - \sum_I [g v_h r]_{G(I)}\,\mathrm{mes}\,I\right| \leqq \sum_{I \in \Gamma_1} |E_I(g v_h r)|\,,$$

where the local error is defined by the formula

$$E_I(w) = \int_I w\,\mathrm{d}r - w(G)\,h_I\,, \quad h_I = \mathrm{mes}\,I\,.$$

It is easy to see that by means of the mapping

$$r = r_I + h_I\hat{r}$$

which maps the unit interval σ onto I, we obtain

$$E_I(w) = \int_0^1 (\hat{w} - \hat{w}(1/2))\,h_I\,\mathrm{d}\hat{r} = h_I\,\hat{E}(\hat{w})\,,$$
$$\hat{w}(\hat{r}) = w(r_I + h_I\hat{r})\,.$$

Since

$$|\hat{E}(\hat{w})| \leqq C\|\hat{w}\|_{2,\sigma}, \quad \hat{E}(p) = 0 \quad \forall p \in P_1(\sigma),$$

the Bramble-Hilbert Lemma yields that

$$|\hat{E}(\hat{w})| \leqq C|\hat{w}|_{2,\sigma}.$$

Making use of the relation

$$|\hat{w}|_{2,\sigma}^2 = h_I^3 |w|_{2,I}^2,$$

we obtain

$$|E_I(w)| \leqq Ch|\hat{E}(\hat{w})| \leqq Ch^{5/2}|w|_{2,I} \quad \forall w \in H^2(I).$$

If we insert $w = grv_h$, then

$$|w|_{2,I}^2 = |grv_h|_{2,I}^2 \leqq C\|g\|_{C^2}^2 \, \|v_h\|_{1,I}^2,$$

where

$$\|g\|_{C^2} = \max_i \|g\|_{C^2(S_i)}$$

and $S_i \subset \Gamma_1$ are subintervals, where $g|_{S_i} \in C^2(S_i)$.

Consequently, we obtain the following upper bound for the right-hand side in (66)

$$E_g \leqq \sum_{I \in \Gamma_1} h^{5/2} \|g\|_{C^2} \, \|v_h\|_{1,I} \leqq Ch^2 \|g\|_{C^2} \, \|v_h\|_{1,\Gamma_1(\alpha_h)}. \tag{67}$$

Here we have also used the estimate Ch^{-1} for the number of subintervals $I \in \Gamma_1$.

One can show that

$$\|v_h\|_{0,\Gamma_1(\alpha_h)}^2 \leqq Ch^{-1} \|v_h\|_{0,r,\Gamma_1(\alpha_h)}^2$$

by an argument similar to that of (61). Furthermore, we have

$$|v_h|_{1,I}^2 = \left|\frac{\mathrm{d}v_h}{\mathrm{d}r}\right|^2 h_I \leqq h_I |v_h|_{1,T}^2 (\operatorname{mes} T)^{-1} \leqq Ch^{-1} |v_h|_{1,T}^2$$

since

$$\operatorname{mes} T \geqq Ch^2$$

holds by virtue of the strong regularity of $\{\mathscr{T}_h(\alpha_h)\}$. Consequently, we may write

$$\|v_h\|_{1,\Gamma_1}^2 = \|v_h\|_{0,\Gamma_1}^2 + \sum_{I \in \Gamma_1} |v_h|_{1,I}^2 \leqq Ch^{-1} \|v_h\|_{0,r,\Gamma_1}^2 + \\ + Ch^{-1} \|v_h\|_{1,D_h}^2 \leqq Ch^{-2} \|v_h\|_{1,r,D_h}^2,$$

using again (61). Substituting into (67), we obtain

$$E_g \leqq Ch \|g\|_{C^2} \, \|v_h\|_{1,r,D_h}. \tag{68}$$

Finally, it is easy to combine (65) and (68) to get

$$\left|L_h(\alpha_h; v_h) - L(\alpha_h; v_h)\right| \leqq E_f + E_g \leqq$$
$$\leqq C(h^{3/2}\|fr\|_{2,D_h} + h\|g\|_{C^2}) \|v_h\|_{1,r,D_h} .$$

Consequently, the condition (50) is satisfied with $\vartheta = 1$.

Note that if $f \in H^1(\hat{D})$ and $r^2 D^\alpha f \in L^2(\hat{D})$ for $|\alpha| = 2$, then $fr \in H^2(\hat{D})$.

To verify the condition (51), we employ Lemma 6 in the following estimate

$$\left|L_h(\alpha_h; v_h)\right| \leqq \left|L(\alpha_h; v_h)\right| + \left|L_h(\alpha_h; v_h) - L(\alpha_h; v_h)\right| \leqq$$
$$\leqq C_6\|v_h\|_{1,r,D_h} + C_8 h^\vartheta \|v_h\|_{1,r,D_h} \leqq C_9\|v_h\|_{1,r,D_h} .$$

Lemma 9. *The approximate state problem* (49) *has a unique solution for any* $\alpha_h \in U^h_{\text{ad}}$.

Proof. Lemmas 5 and 8 guarantee that was can apply Riesz-Theorem in the space V_h with the inner product $(u, v) \equiv a(\alpha_h; u, v)$.

Proposition 3. *Let the assumptions of Lemma 8 be satisfied. Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of* $\alpha_h \in U^h_{\text{ad}}$, *converging to* α *in* $C([0, 1])$. *Let us construct extensions* Ey_h *of the solutions* $y_h(\alpha_h)$ *of the approximate state problem* (49) *by means of the formula* (12).

Then

$$Ey_h|_{D(\alpha)} \rightharpoonup y(\alpha) \quad (\textit{weakly}) \quad \textit{in} \quad W^{(1)}_{2,r}(D(\alpha)) , \tag{69}$$

where $y(\alpha)$ *is the solution of the state problem* (18) *on* $D(\alpha)$.

Proof. Let us define $y_h^* \in V_h$ to be the solution of the following problem

$$a(\alpha_h; y_h^*, v_h) = L(\alpha_h; v_h) \quad \forall v_h \in V_h . \tag{70}$$

Recall that $y_h \in V_h$ satisfies the condition

$$a(\alpha_h; y_h, v_h) = L_h(\alpha_h; v_h) \quad \forall v_h \in V_h .$$

Subtracting, we obtain

$$a(\alpha_h; y_h^* - y_h, v_h) = L(\alpha_h; v_h) - L_h(\alpha_h; v_h)$$

and inserting $v_h := y_h^* - y_h$, we arrive at

$$C_4\|y_h^* - y_h\|^2_{1,r,D_h} \leqq a(\alpha_h; y_h^* - y_h, y_h^* - y_h) = \tag{71}$$
$$= L(\alpha_h; y_h^* - y_h) - L_h(\alpha_h; y_h^* - y_h) \leqq C_8 h\|y_h^* - y_h\|_{1,r,D_h} ,$$

using Lemma 5 and Lemma 8.

For the extensions we make use of (13′) and (71) to obtain

$$\|Ey_h^* - Ey_h\|_{1,r,\hat{D}} \leqq C\|y_h^* - y_h\|_{1,r,D_h} \leqq \tilde{C}h\,. \tag{72}$$

From (70) it follows that

$$C_4\|y_h^*\|_{1,r,D_h}^2 \leqq L(\alpha_h;\, y_h^*) \leqq C_9\|y_h^*\|_{1,r,D_h}\,,$$

by virtue of Lemma 5 and Lemma 8.

Consequently,

$$\|Ey_h^*\|_{1,r,\hat{D}} \leqq C\|y_h^*\|_{1,r,D_h} \leqq \tilde{C} \tag{73}$$

follows and there exists a subsequence (we shall denote it by the same symbol) and $y \in W_{2,r}^{(1)}(\hat{D})$ such that

$$Ey_h^* \rightharpoonup y \quad \text{(weakly)} \quad \text{in} \quad W_{2,r}^{(1)}(\hat{D})\,. \tag{74}$$

Let $v \in V(D(\alpha))$ be given. Let us construct the extension $Ev \in V(\hat{D})$ by the formula (12). By virtue of Lemma 2, there exists a sequence $\{v_\varkappa\}$, $\kappa \to 0$, such that $v_\varkappa \in \in C^\infty(\mathrm{Cl}\,(\hat{D}))$ $\operatorname{supp} v_\varkappa \cap \hat{\Gamma}_2 = \emptyset$ and

$$\|v_\varkappa - Ev\|_{1,r,\hat{D}} \to 0 \quad \text{for} \quad \kappa \to 0\,. \tag{75}$$

Consider the Lagrange linear interpolate $\pi_h v_\varkappa$ of $v_\varkappa|_{D_h}$ over the triangulation $\mathscr{T}_h(\alpha_h)$. Obviously, we have $\pi_h v_\varkappa \in V_h$. Let κ be fixed, for a time being. We can insert $\pi_h v_\varkappa$ into (70) to obtain

$$a(\alpha_h;\, y_h^*,\, \pi_h v_\varkappa) = L(\alpha_h;\, \pi_h v_\varkappa)\,. \tag{76}$$

We shall pases to the limit with $h \to 0$. Let again $\alpha_m = \alpha(z) - 1/m$, where $m = 2, 3, \ldots$,

$$G_m = \{(r, z)|\ 0 < r \leqq \alpha_m(z),\ 0 < z < 1\}\,.$$

Then

$$G_m \subset D_h$$

for $h < h_0(m)$ and we may write

$$\begin{aligned} &|a(\alpha_h;\, y_h^*,\, \pi_h v_\varkappa) - a(\alpha_m;\, y,\, v_\varkappa)| = \\ &= |a(\alpha_m;\, y_h^*,\, v_\varkappa) + a(\alpha_m;\, y_h^*,\, \pi_h v_\varkappa - v_\varkappa) + \\ &+ \tilde{a}(\alpha_h - \alpha_m;\, y_h^*,\, \pi_h v_\varkappa) - a(\alpha_m;\, y,\, v_\varkappa) \leqq \\ &\leqq |a(\alpha_m;\, y_h^* - y,\, v_\varkappa)| + |a(\alpha_m;\, y_h^*,\, \pi_h v_\varkappa - v_\varkappa)| + \\ &+ |\tilde{a}(\alpha_h - \alpha_m;\, y_h^*,\, \pi_h v_\varkappa)|\,, \end{aligned} \tag{77}$$

where

$$\tilde{a}(\alpha_h - \alpha_m;\, \cdot,\, \cdot) = a(\alpha_h;\, \cdot,\, \cdot) - a(\alpha_m;\, \cdot,\, \cdot)\,.$$

Consider a positive ε. From (74) we conclude that the first term on the right-hand side of (77) is not greater than $\varepsilon/6$ if $h < h_1(\varepsilon, m)$.

To estimate the second term, we employ the well-known inequality (see e.g. [5])

(78) $$\|\pi_h v_\varkappa - v_\varkappa\|_{1,D_h} \leqq Ch\|v_\varkappa\|_{2,D_h} \leqq Ch\|v_\varkappa\|_{2,\hat{D}}\,.$$

Combining (73) and (78) we obtain

(79) $$|a(\alpha_m; y_h^*, \pi_h v_\varkappa - v_\varkappa)| \leqq C_5\|y_h^*\|_{1,r,G_m} \|\pi_h v_\varkappa - v_\varkappa\|_{1,r,G_m} \leqq$$
$$\leqq Ch\|v_\varkappa\|_{2,\hat{D}} < \varepsilon/6 \quad \text{for} \quad h < h_2\,.$$

It remains to estimate the third term. To this end, we realize that

$$\|\pi_h v_\varkappa\|_{1,T} \leqq C\|v_\varkappa\|_{2,T} \quad \forall h$$

holds for all triangles $T \in \mathscr{T}_h(\alpha_h)$.

Let G_m^h be the smallest union U of triangles $T \in \mathscr{T}_h(\alpha_h)$ such that $D_h \dot{-} G_m \subset U$. Obviously, we have

(80) $$\operatorname{mes} G_m^h \leqq 1/m + 2h + \|\alpha_h - \alpha\|_\infty\,,$$

where $\|\cdot\|_\infty$ denotes the norm in $C([0, 1])$. Consequently,

$$\|\pi_h v_\varkappa\|_{1,D_h - G_m}^2 \leqq \|\pi_h v_\varkappa\|_{1,G_m{}^h}^2 = \sum_{T \in G_m{}^h} \|\pi_h v_\varkappa\|_{1,T}^2 \leqq C^2\|v_\varkappa\|_{2,G_m{}^h}^2\,.$$

Using again (73), we may write

(81) $$|\tilde{a}(\alpha_h - \alpha_m; y_h^*, \pi_h v_\varkappa)| \leqq C_5\|y_h^*\|_{1,r,D_h} \|\pi_h v_\varkappa\|_{1,r,D_h - G_m} \leqq C\|v_\varkappa\|_{2,G_m{}^h}\,.$$

Combining (77), (79), (81), we deduce the following inequality

$$|a(\alpha_h; y_h^*, \pi_h v_\varkappa) - a(\alpha; y, v_\varkappa)| \leqq$$
$$\leqq |a(\alpha_h; y_h^*, \pi_h v_\varkappa) - a(\alpha_m; y, v_\varkappa)| + |\tilde{a}(\alpha - \alpha_m; y, v_\varkappa)| \leqq$$
$$\leqq \varepsilon/3 + C\|v_\varkappa\|_{2,G_m{}^h} + C_5\|y\|_{1,r,D} \|v_\varkappa\|_{1,r,D - G_m} \quad \text{for} \quad h < h_3(\varepsilon, m)\,.$$

Making use of (80), we conclude that

(82) $$\lim_{h \to 0} a(\alpha_h; y_h^*, \pi_h v_\varkappa) = a(\alpha; y, v_\varkappa)\,.$$

Next we may write

$$|L(\alpha_h; \pi_h v_\varkappa) - L(\alpha; v_\varkappa)| \leqq$$
$$\leqq |L(\alpha_h; \pi_h v_\varkappa - v_\varkappa)| + |L(\alpha_h; v_\varkappa) - L(\alpha; v_\varkappa)| = \mathscr{L}_1 + \mathscr{L}_2\,,$$
$$|\mathscr{L}_1| \leqq C_6\|\pi_h v_\varkappa - v_\varkappa\|_{1,r,D_h} \leqq Ch\|v_\varkappa\|_{2,D}\,,$$
$$|\mathscr{L}_2| \leqq \int_{\triangle(D_h,D)} |f v_\varkappa|\, r\, dr\, dz + \int_{\triangle(\Gamma_1(\alpha_h),\Gamma_1)} |g v_\varkappa|\, r\, dr$$

and since

$$\triangle(D_h, D) \to 0\,, \quad \triangle(\Gamma_1(\alpha_h), \Gamma_1) \to 0$$

for $h \to 0$, we conclude that

$$\lim_{h \to 0} L(\alpha_h; \pi_h v_\varkappa) = L(\alpha; v_\varkappa)\,. \tag{83}$$

Passing to the limit with $h \to 0$ in (76) and using (82), (83), we arrive at

$$a(\alpha; y, v_\varkappa) = L(\alpha; v_\varkappa)\,.$$

Passing to the limit with $\kappa \to 0$ and using Lemma 5, Lemma 6 and (75), we obtain

$$a(\alpha; y, v) = L(\alpha; v)\,.$$

The space $V(\hat{D})$ is weakly closed in $W_{2,r}^{(1)}(\hat{D})$ and every function Ey_h^* belongs to $V(\hat{D})$. Hence the weak limit $y \in V(\hat{D})$ and its restriction to $D \equiv D(\alpha)$ belongs to $V(D)$. Consequently, $y|_D$ is a solution of (18). Since the solution is unique, $y|_D = y(\alpha)$ and (74) implies that

$$Ey_h^*|_D \rightharpoonup y(\alpha) \quad \text{(weakly)} \quad \text{in} \quad W_{2,r}^{(1)}(D)\,; \tag{84}$$

the latter convergence holds for the whole sequence $\{Ey_h^*|_D\}$.
The remainder of the proof is an easy consequence of (72). Q.E.D.

For a fixed parameter h, we define the Approximate Domain Optimization Problems:

$$\text{find} \quad \alpha_h^0 \in U_{\text{ad}}^h \quad \text{such that}$$

$$j_i(\alpha_h^0, y_h(\alpha_h^0)) \leqq j_i(\alpha_h, y_h(\alpha_h)) \quad \forall \alpha_h \in U_{\text{ad}}^h\,, \tag{85i}$$

where $i \in \{1, 2, 3\}$ and $y_h(\alpha_h)$ is the solution of the approximate state problem (49).

Proposition 4. *The Approximate Domain Optimization Problems have at least one solution for any $i \in \{1, 2, 3\}$ and any $h = 1/N$, $N = 2, 3, \ldots$.*

Proof. It is readily seen that

$$\alpha_h \in U_{\text{ad}}^h \Leftrightarrow \boldsymbol{a} \in \mathscr{A}\,,$$

if $\boldsymbol{a} \in \mathbb{R}^{N+1}$ denotes the vector of $\alpha_h(jh)$, $j = 0, 1, \ldots, N$, and $\mathscr{A}$ is a compact set in $\mathbb{R}^{N+1}$. One can show that the nodal values of $y_h(\alpha_h)$ depend continuously on $\boldsymbol{a}$; the same assertion can be then verified for $j_i(\alpha_h, y_h(\alpha_h)) \equiv J_i(\boldsymbol{a})$. Consequently, the function $J_i(\boldsymbol{a})$ attains its minimum on the set $\mathscr{A}$.

Proposition 5. *Let the assumptions of Proposition 3 be satisfied. Then*

$$\lim_{h \to 0} j_i(\alpha_h, y_h(\alpha_h)) = j_i(\alpha, y(\alpha))$$

holds for $i \in \{1, 2, 3\}$, where $y_h(\alpha_h)$ and $y(\alpha)$ is the solution of the problem (49) and (18), respectively.

Proof is parallel to that of Proposition 2. We replace α_n by α_h, y_n by y_h, D_n by D_h, instead of Proposition 1 and Lemma 6 we make use of Proposition 3 and Lemma 6, respectively. The boundedness of all $\|y_h\|_{1,r,D_h}$ follows from (72) and (73).

In proving the assertion for $j_3(\alpha_h, y_h)$, we have moreover to estimate the following term by means of Lemma 8 (50)

$$|L_h(\alpha_h; y_h) - L(\alpha_h; y_h)| \leqq C_8 h \|y_h\|_{1,r,D_h} \leqq Ch .$$

Theorem 2. *Let the assumptions of Lemma* 8 *hold. Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of solutions of the Approximate Domain Optimization Problem* (85i), $i \in \{1, 2, 3\}$.

Then a subsequence $\{\alpha_{\hat{h}}\}$ *exists such that*

$$\alpha_{\hat{h}} \to \alpha^0 \quad in \quad C([0, 1]) , \tag{86}$$

$$Ey_{\hat{h}}|_{D(\alpha^0)} \rightharpoonup y(\alpha^0) \quad (weakly) \quad in \quad W^{(1)}_{2,r}(D(\alpha^0)) , \tag{87}$$

where α^0 *is a solution of the Domain Optimization Problem* (29i), $Ey_{\hat{h}}$ *are the solutions* $y_{\hat{h}}(\alpha_{\hat{h}})$, *extended according to the formula* (12), $y(\alpha^0)$ *is the solution of the state problem* (18) *on* $D(\alpha^0)$.

The limit of any uniformly convergent subsequence of $\{\alpha_h\}$ *represents a solution of* (29i) *and an analogue of* (87) *holds.*

Proof. Since U_{ad} is compact in $C([0, 1])$, a subsequence $\{\alpha_{\hat{h}}\} \subset \{\alpha_h\}$ exists such that (86) holds and $\alpha^0 \in U_{\text{ad}}$.

Let $\alpha \in U_{\text{ad}}$ be given. There exists a sequence $\{\beta_h\}$, $\beta_h \in U^h_{\text{ad}}$, such that $\beta_h \to \alpha$ in $C([0, 1])$. This follows from Lemma A1 below (Appendix).

We have

$$j_i(\alpha_{\hat{h}}, y_{\hat{h}}(\alpha_{\hat{h}})) \leqq j_i(\beta_{\hat{h}}, y_{\hat{h}}(\beta_{\hat{h}})) \quad \forall \hat{h}$$

by definition.

Passing to the limit with $\hat{h} \to 0$ and using Proposition 5 on both sides, we obtain

$$j_i(\alpha^0, y(\alpha^0)) \leqq j_i(\alpha, y(\alpha)) .$$

Consequently, α^0 is a solution of the problem (29i).

The convergence (87) follows from Proposition 3. The rest of the Theorem is obvious.

APPENDIX

Lemma A1. *To any* $\alpha \in U_{\text{ad}}$ *there exists a sequence* $\{\alpha_h\}$, $h \to 0$, $\alpha_h \in U^h_{\text{ad}}$, *such that* $\alpha_h \to \alpha$ *in* $C([0, 1])$.

Proof. 1° If $\alpha = \text{const}$, then $\alpha_h = \alpha$. Consequently, we assume that $\alpha \neq \text{const}$. Let the mean value of α be denoted by S, i.e.

$$S = \int_0^1 \alpha(z)\, dz .$$

Let us define the sets $I = [0, 1]$,

$$I^+ = \{z \in [0, 1] | \alpha - S \geqq 0\},$$
$$I^- = \{z \in [0, 1] | \alpha - S < 0$$

and the following modified function

$$Z_{\mu k}\alpha = \begin{cases} S + (1 - k\mu)(\alpha - S), & z \in I^+, \\ S + (1 - \mu)(\alpha - S), & z \in I^-, \end{cases}$$

where k and μ are positive real parameters, k is fixed $(k = k(\alpha))$ and $\mu \to 0$. Note that $\operatorname{mes}(I^+) > 0$, $\operatorname{mes}(I^-) > 0$.

It is easy to see that

(A.1) $$\|Z_{\mu k}\alpha - \alpha\|_{\infty,I} \leqq \mu \max(1, k) \|\alpha - S\|_{\infty,I},$$

(A.2) $$\left\|\frac{\mathrm{d}}{\mathrm{d}z}(Z_{\mu k}\alpha)\right\|_{\infty,I} \leqq \|\mathrm{d}\alpha/\mathrm{d}z\|_{\infty,I} \leqq C_1,$$

(A.3) $$\alpha_{\max} - Z_{\mu k}\alpha \geqq \alpha_{\max} - \bar{\alpha} + k\mu(\bar{\alpha} - S) \geqq k\mu(\bar{\alpha} - S),$$

where $\bar{\alpha}$ is the maximum of α on $[0, 1]$,

(A.4) $$Z_{\mu k}\alpha - \alpha_{\min} \geqq \alpha^* - \alpha_{\min} + \mu(S - \alpha^*) \geqq \mu(S - \alpha^*),$$

where α^* is the minimum of α on $[0,1]$.

2° We apply Lemma 7.1 of [1] to obtain a sequence $\{\beta_h\}$, $h \to 0$, such that

(A.5) $$\beta_h \to Z_{\mu k}\alpha \quad \text{in} \quad C([0, 1]) \quad \text{for} \quad h \to 0,$$

(A.6) $$\begin{gathered} \beta_h|_{\triangle_j} \in P_1(\triangle_j), \quad j = 1, 2, \ldots, N, \quad \beta_h \in C([0, 1]), \\ \min_{z\in I}(Z_{\mu k}\alpha) \leqq \beta_h(z) \leqq \max_{z\in I}(Z_{\mu k}\alpha), \quad |\mathrm{d}\beta_h/\mathrm{d}z| \leqq C_1, \\ \int_0^1 \beta_h \,\mathrm{d}z = \int_0^1 Z_{\mu k}\alpha \,\mathrm{d}z. \end{gathered}$$

Since

$$\int_0^1 \beta_h^2 \,\mathrm{d}z \neq \int_0^1 \alpha^2 \,\mathrm{d}z$$

in general, we define a constant a_h by the relation

(A.7) $$\int_0^1 (\beta_h + a_h)^2 \,\mathrm{d}z = \int_0^1 \alpha^2 \,\mathrm{d}z$$

and show that $k = k(\alpha)$ exists such that

$$\alpha_h = \beta_h + a_h$$

satisfies the conditions of the Lemma.

Denoting

$$c_{\mu h} = \int_0^1 (\beta_h^2 - \alpha^2)\,\mathrm{d}z\,, \quad \int_0^1 Z_{\mu k}\alpha\,\mathrm{d}z = u\,,$$

we obtain from (A.7)

$$a_h = -u + (u^2 - c_{\mu h})^{1/2}\,, \tag{A.8}$$

$$c_{\mu h} = c_{\mu h}^{(1)} + c_{\mu}^{(2)}\,, \quad c_{\mu h}^{(1)} = \int_0^1 (\beta_h^2 - (Z_{\mu k}\alpha)^2)\,\mathrm{d}z\,,$$

$$c_{\mu}^{(2)} = \int_0^1 ((Z_{\mu k}\alpha)^2 - \alpha^2)\,\mathrm{d}z = \int_0^1 (Z_{\mu k}\alpha - \alpha)(Z_{\mu k}\alpha + \alpha)\,\mathrm{d}z = \tag{A.9}$$

$$= -\mu\, 2S\left(k\int_{I^+} \gamma\,\mathrm{d}z + \int_{I^-} \gamma\,\mathrm{d}z\right) - k\mu(2 - k\mu)\int_{I^+} \gamma^2\,\mathrm{d}z - \mu(2 - \mu)\int_{I^-} \gamma^2\,\mathrm{d}z\,,$$

where $\gamma = \alpha - S$.

3° Let us show that a positive constant $k = k(\alpha)$ exists such that

$$0 < a_h < k\mu(\bar{\alpha} - S) \tag{A.10}$$

holds for sufficiently small μ and $h < h_0(\mu)$.

First we choose k such that $c_{\mu}^{(2)} < 0$ for μ sufficiently small. To this end we distinguish two cases.

1. Case:

$$S\bar{\gamma} \geqq \int_{I^+} \gamma\alpha\,\mathrm{d}z\,, \quad (\bar{\gamma} = (\bar{\alpha} - S))\,.$$

Let us put $k = 1$ to obtain

$$c_{\mu}^{(2)} = -\mu(2 - \mu)\int_0^1 \gamma^2\,\mathrm{d}z < 0 \quad \forall \mu \in (0, 2)\,.$$

2. Case:

$$S\bar{\gamma} < \int_{I^+} \gamma\alpha\,\mathrm{d}z\,.$$

Then if

$$k > 1 - \int_0^1 \gamma^2\,\mathrm{d}z \left(\int_{I^+} \gamma\alpha\,\mathrm{d}z\right)^{-1}\,, \tag{A.11}$$

we derive easily that

$$\lim_{\mu\to 0+} c_{\mu}^{(2)}/(2\mu) = -S\left(k\int_{I^+} \gamma\,\mathrm{d}z + \int_{I^-} \gamma\,\mathrm{d}z\right) - \tag{A.12}$$

$$- k\int_{I^+} \gamma^2\,\mathrm{d}z - \int_{I^-} \gamma^2\,\mathrm{d}z = -\int_0^1 \gamma^2\,\mathrm{d}z - (k - 1)\int_{I^+} \gamma\alpha\,\mathrm{d}z < 0\,.$$

Consequently, we have

$$c_\mu^{(2)} < 0 \quad \text{for} \quad \mu \in (0, \mu_1).$$

By virtue of (A.5) we have

(A.13) $$c_{\mu h}^{(1)} \to 0 \quad \text{for} \quad h \to 0$$

and

(A.14) $$c_{\mu h} = c_{\mu h}^{(1)} + c_\mu^{(2)} < 0 \quad \forall h < h_1(\mu), \quad \mu < \mu_1.$$

Then from (A.7) we conclude that

(A.15) $$0 < a_h < -c_{\mu h}/(2u).$$

Let us show that k exists such that

(A.16) $$0 < -c_\mu^{(2)} < 2uk(\bar{\alpha} - S)\mu$$

holds for μ sufficiently small.

In fact, let us first consider the case

$$S\bar{\gamma} \geqq \int_{I^+} \gamma\alpha \, dz$$

and set $k = 1$. Then

$$u = S,$$

$$\lim_{\mu \to 0} (u(\bar{\alpha} - S) + c_\mu^{(2)}/(2\mu)) = S\bar{\gamma} - \int_0^1 \gamma^2 \, dz > \int_{I^+} \gamma\alpha \, dz + \int_{I^-} \gamma\alpha \, dz - \int_0^1 \gamma^2 \, dz = 0.$$

Consequently,

$$2\mu u(\bar{\alpha} - S) + c_\mu^{(2)} > 0, \quad 0 < \mu < \mu_2$$

and (A.16) is fulfilled.

Second, let

(A.17) $$S\bar{\gamma} < \int_{I^+} \gamma\alpha \, dz$$

and set

(A.18) $$k < -\int_{I^-} \gamma\alpha \, dz \left(\int_{I^+} \gamma\alpha \, dz - S\bar{\gamma}\right)^{-1}.$$

Note, that (A.18) is compatible with (A.11), since

$$1 - \int_0^1 \gamma^2 \, dz \left(\int_{I^+} \gamma\alpha \, dz\right)^{-1} = \left(\int_{I^+} \gamma\alpha \, dz - \int_0^1 \gamma^2 \, dz\right) \cdot$$

$$\cdot \left(\int_{I^+} \gamma\alpha \, dz\right)^{-1} < \left(\int_{I^+} \gamma\alpha \, dz - \int_0^1 \gamma^2 \, dz\right) \cdot$$

$$\cdot \left(\int_{I^+} \gamma\alpha \, dz - S\bar{\gamma}\right)^{-1} = \left(-\int_{I^-} \gamma\alpha \, dz\right)\left(\int_{I^+} \gamma\alpha \, dz - S\bar{\gamma}\right)^{-1}.$$

Then

$$\lim_{\mu\to 0} [ku(\bar{\alpha} - S) + c_\mu^{(2)}/(2\mu)] =$$

$$= kS\bar{\gamma} - S\left(k\int_{I^+} \gamma \, dz + \int_{I^-} \gamma \, dz\right) - k\int_{I^+} \gamma^2 \, dz - \int_{I^-} \gamma^2 \, dz =$$

$$= k\left(S\bar{\gamma} - S\int_{I^+} \gamma \, dz - \int_{I^+} \gamma^2 \, dz\right) - \left(S\int_{I^-} \gamma \, dz + \int_{I^-} \gamma^2 \, dz\right) =$$

$$= k\left(S\bar{\gamma} - \int_{I^+} \gamma\alpha \, dz\right) - \int_{I^-} \gamma\alpha \, dz > 0 .$$

Consequently, we easily deduce

$$2\mu ku(\bar{\alpha} - S) + c_\mu^{(2)} > 0 \quad \text{for} \quad \mu < \mu_3$$

and (A.16) holds.

Since

$$\text{(A.19)} \qquad -c_{\mu h}/(2u) = -c_{\mu h}^{(1)}/(2u) - c_\mu^{(2)}/(2u) < k\mu(\bar{\alpha} - S)$$

follows from (A.4), (A.13) and (A.16) for $h < h_2(\mu)$, we arrive at (A.10), making use of (A.15).

4° Combining (A.10) with (A.6), (A.3), we obtain

$$\alpha_h \equiv \beta_h + a_h \leqq \max (Z_{\mu k}\alpha) + k\mu(\bar{\alpha} - S) \leqq \alpha_{\max} .$$

The lower bound is obvious.

Finally, we may write

$$\|\alpha_h - \alpha\|_{\infty,I} \leqq a_h + \|\beta_h - Z_{\mu k}\alpha\|_{\infty,I} + \|Z_{\mu k}\alpha - \alpha\|_{\infty,I} .$$

It is easy to see that (A.15), (A.19) imply that

$$\text{(A.20)} \qquad \lim_{h\to 0} a_h = 0 .$$

Then the uniform convergence of $\{\alpha_h\}$ to α follows from (A.20), (A.5) and (A.1).

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Souhrn

OPTIMALIZACE OBLASTI V OSOVĚ SYMETRICKÝCH ELIPTICKÝCH ÚLOHÁCH METODOU KONEČNÝCH ELEMENTŮ

IVAN HLAVÁČEK

Uvažuje se osově symetrická eliptická úloha druhého řádu s kombinovanými okrajovými podmínkami. Je třeba nalézt část hranice oblasti tak, aby minimalizovala jeden ze čtyř typů účelového funkcionálu. Dokazuje se existence optimální hranice a konvergence přibližných, po částech lineárních řešení, a to prostřednictvím teorie Sobolevových prostorů s vahou.

Резюме

ОПТИМИЗАЦИЯ ОБЛАСТИ В ОСЕСИММЕТРИЧЕСКИХ ЗАДАЧАХ МЕТОДОМ КОНЕЧНЫХ ЭЛЕМЕНТОВ

IVAN HLAVÁČEK

Рассматривается осесимметрическая эллиптическая задача второго порядка с смешанными краевыми условиями. Требуется найти часть границы области так, чтобы минимизировать один из четырех типов целевого функционала. Доказывается существование оптимальной границы и сходимость приближенных по частям линейных решений. В анализе используются пространства Соболева с весом.

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