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Jiří Neustupa

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GLOBAL WEAK SOLVABILITY OF A REGULARIZED SYSTEM OF THE NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE FLUID

JIŘÍ NEUSTUPA

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Summary. The paper contains the proof of global existence of weak solutions to the mixed initial-boundary value problem for a certain modification of a system of equations of motion of viscous compressible fluid. The modification is based on an application of an operator of regularization to some terms appearing in the system of equations and it does not contradict the laws of fluid mechanics. It is assumed that pressure is a known function of density. The method of discretization in time is used and finally, a so called energy inequality is derived. The inequality is independent on the regularization used.

Keywords: Navier-Stoke equations for compressible fluid, method of a discretization in time, modifications of the Navier-Stokes equations, global existence of weak solutions.

AMS Classification: 35A05, 35Q10, 76N10.

INTRODUCTION

Many works proving the local (in time) existence of solutions of a system of equations of motion of viscous compressible fluid have been published (see e.g. [8] and references at the end of this paper). The global (in time) existence of solutions has been proved for example under assumptions of the type "the initial conditions are small enough" or "the initial and boundary conditions are near enough to some equilibrium state" (see e.g. [3]). In contradistinction to the Navier-Stokes equations for incompressible liquid, the question of global existence of weak or strong solutions in the case of viscous compressible fluid and general initial conditions remains still open both in two or three space dimensions.

In this paper we deal with a barotropic fluid and prove the global existence of weak solutions of the system of equations of motion with a certain regularization. We also derive an inequality of the energy type which does not depend on the parameter h used in the regularization.

We use the relation

$$p = \tilde{p}(\varrho) \tag{0.1}$$

between the pressure p and the density ϱ. We assume that p is a nondecreasing continuously differentiable function on $\langle 0, +\infty)$ such that $p(0) = 0$ and

a) there exist $\varrho_1 > 0$, $C_1 > 0$ and $\alpha > 0$ such that

$$p(\varrho) \leqq C_1 \varrho^\alpha \quad \text{for} \quad \varrho \in \langle 0, \varrho_1 \rangle , \tag{0.2}$$

b) there exist $\varrho_2 > 0$, $C_2 > 0$, $C_3 > 0$ and $\varkappa \in (1, 6)$ such that

$$C_2 \varrho^\varkappa \leqq p(\varrho) \leqq C_3 \varrho^\varkappa \quad \text{for} \quad \varrho \geqq \varrho_2 , \tag{0.3}$$

c) there exists $C_4 > 0$ such that

$$p'(\varrho) \leqq C_4 \varrho^{\varkappa - 1} \quad \text{for} \quad \varrho \geqq \varrho_2 . \tag{0.4}$$

The tilda denotes the regularization. Its exact meaning is explained in details in the following section, but already here we can say that $\tilde{p}(\varrho)(x, t)$ represents an average of $p(\varrho)(\cdot, t)$ considered with a certain smooth weight function on a neighbourhood $B_h(x)$ of x. The radius h of this neighbourhood is a constant which may be chosen arbitrarily small. The assumption concerning the barotropicity was used for example in papers [6], [9] dealing with the equations of motion of viscous compressible fluid. It is not quite correct from the physical point of view (see e.g. [2]) and so we must not forget that our mathematical model is only an approximate reflection of the real situation.

The system of equations we deal with has the form

$$\varrho_{,t} + (\varrho \tilde{u}_j)_{,j} = 0 , \tag{0.5}$$

$$(\varrho u_i)_{,t} + (\varrho \tilde{u}_j u_i)_{,j} = -\tilde{p}(\varrho)_{,i} + \tfrac{1}{3}\mu u_{j,ji} + \mu u_{i,jj} \quad (i = 1, 2, 3) \tag{0.6}$$

where $U = (u_1, u_2, u_3)$ is the velocity of the moving fluid and μ is the dynamic viscosity coefficient. We assume that μ is a positive constant. The notion of the velocity of the fluid at a point x is usually introduced in terms of an average of velocities of all particles contained in a small neighbourhood of x. So if h is small enough, $\tilde{u}_j$ is almost the same as u_j from the point of view of mechanics.

In [5], R. Rautmann used a similar regularization in the Navier-Stokes equations for the incompressible liquid in order to prove the global existence of strong solutions in three space dimensions.

1. FORMULATION OF AN INITIAL-BOUNDARY VALUE PROBLEM

Assume that Ω is a bounded region in R^3 with its boundary $\partial\Omega$ of the class $C^{2+(a)}$ for some $a \in (0, 1)$. Let us choose $h > 0$ and put

$$\Omega_h = \{x \in R^3;\ \operatorname{dist}(x, \Omega) < h\} .$$

Assume that Ω has such a form that h can be chosen so small that $\partial\Omega_h$ is also of the class $C^{2+(a)}$.

Put

$$\omega_h(\zeta) = K_h \exp\left(-\frac{|\zeta|^2}{h^2 - |\zeta|^2}\right) \quad \text{for} \quad \zeta \in R^3 , \quad |\zeta| < h ,$$

$$\omega_h(\zeta) = 0 \quad \text{for} \quad \zeta \in R^3 , \quad |\zeta| \geqq h .$$

K_h is a constant chosen so that

$$\int_{B_h(0)} \omega_h(\zeta) \, \mathrm{d}\zeta = 1 .$$

If $f \in L^1(\Omega_h)$, put

$$\tilde{f}(x) = \int_{\Omega_h} \omega_h(x - y) f(y) \, \mathrm{d}y .$$

It may be easily shown that

i) $$\int_{\Omega_h} \tilde{f}(x) \, g(x) \, \mathrm{d}x = \int_{\Omega_h} f(x) \, \tilde{g}(x) \, \mathrm{d}x$$

for $f, g \in L^1(\Omega_h)$;

ii) if $f = 0$ a.e. in $\Omega_h - \Omega$ then $\tilde{f} \equiv 0$ on $\partial\Omega_h$ and moreover, any derivative of $\tilde{f}$ is also identically equal to zero on $\partial\Omega_h$;

iii) if $f \in {}^\circ H^1(\Omega_h)$ then

$$\frac{\partial}{\partial x_i} \tilde{f} = \left(\frac{\partial f}{\partial x_i}\right)^{\sim}$$

iv) there exists $K_1 > 0$ such that if $f \in L^2(\Omega_h)$ and $D\tilde{f}$ represents any derivative of $\tilde{f}$ of an order less or equal to 3 then

(1.1) $$\max_{x \in \bar{\Omega}_h} |D \tilde{f}(x)| \leqq K_1 \|f\|_{L^2(\Omega_h)} .$$

If f is defined in $\Omega_h \times R^1$ then we denote by $\tilde{f}$ the function

$$\tilde{f}(x, t) = \int_{\Omega_h} \omega_h(x - y) f(y, t) \, \mathrm{d}y .$$

We shall often apply the regularization $\sim$ to the components of velocity of the fluid or to their approximations. Since they are defined only for the space variable in Ω, we shall always extend them by zero onto $\Omega_h - \Omega$ so that it will be possible to use the properties from the points ii), iii).

Let T be a given positive number. Denote $Q_T = \Omega \times (0, T)$ and $Q_{T,h} = \Omega_h \times \times (0, T)$. We shall solve the equation (0.5) on $Q_{T,h}$ and the system (0.6) on Q_T. We consider the boundary condition

(1.2) $$u_i|_{x \in \partial\Omega} \equiv 0 \quad (i = 1, 2, 3)$$

and the initial conditions

(1.3) $$\varrho|_{t=0} = \varrho_0 ,$$

(1.4) $$(\varrho u_i)|_{t=0} = \varrho_0 u_{0i} \quad (i = 1, 2, 3) .$$

Let us assume that ϱ_0 and $U_0 = (u_{01}, u_{02}, u_{03})$ are given functions such that $\varrho_0 \in H^1(\Omega_h)$, $\varrho_0 \geqq 0$ a.e. in Ω_h, $U_0 \in L^2(\Omega)^3$ and $\varrho_0 u_{0i} u_{0i} \in L^1(\Omega)$.

The reason why we shall solve (0.5) on $Q_{T,h}$ is that the functions $\tilde{u}_j$ appearing in (0.5) are equal to zero on $\partial\Omega_h \times \langle 0, T)$ (which we shall use) while it is not possible to say the same about their values on $\partial\Omega \times \langle 0, T)$.

By the weak solution of (0.5), (0.6), (1.2), (1.3), (1.4) we shall mean the couple of functions U, ϱ satisfying

$$(1.5) \qquad \begin{cases} U = (u_1, u_2, u_3) \in L^2(0, T; {}^\circ H^1(\Omega)^3)\,, \\ \varrho \in L^\infty(0, T; H^1(\Omega_h)),\ \varrho \geqq 0 \text{ a.e. in } Q_{T,h}\,; \end{cases}$$

$$(1.6) \qquad \int_0^T \int_\Omega \{\varrho u_i \varphi_{i,t} + \varrho \tilde{u}_j u_i \varphi_{i,j} + p(\varrho)\, \varphi_{i,i} - \tfrac{1}{3}\mu u_{j,j} \varphi_{i,i} - \mu u_{i,j} \varphi_{i,j}\}\, \mathrm{d}x\, \mathrm{d}t = -\int_\Omega \varrho_0 u_{0i} (\varphi_i|_{t=0})\, \mathrm{d}x$$

for all $\varphi \equiv (\varphi_1, \varphi_2, \varphi_3) \in C^\infty(\bar{Q}_T)^3$ such that $\varphi \equiv 0$ on $\partial\Omega \times \langle 0, T\rangle$ and $\varphi|_{t=T} \equiv 0$;

$$(1.7) \qquad \int_0^T \int_{\Omega_h} \{\varrho \psi_{,t} + \varrho \tilde{u}_j \psi_{,j}\}\, \mathrm{d}x\, \mathrm{d}t = -\int_{\Omega_h} \varrho_0 (\psi|_{t=0})\, \mathrm{d}x$$

for all $\psi \in C^\infty(\bar{Q}_{T,h})$ such that $\psi|_{t=T} \equiv 0$.

Remark 1.1. In the next sections, we shall often use the function $\mathscr{P}: \langle 0, +\infty) \to R^1$ which is defined in the following way:

$$(1.8) \qquad \mathscr{P}(\sigma) = \sigma \int_1^\sigma \frac{p(\vartheta)}{\vartheta^2}\, \mathrm{d}\vartheta \quad \text{for} \quad \sigma > 0\,,$$

$$\mathscr{P}(0) = \lim_{\sigma \to 0+} \mathscr{P}(\sigma) = 0\,.$$

It may be verified that the function $\mathscr{P}$ has the following properties:

I) $$\mathscr{P}(\sigma) > 0 \quad \text{for} \quad \sigma \in (1, +\infty)\,;$$

II) there exists $C_5 > 0$ such that

$$(1.9) \qquad \mathscr{P}(\sigma) \geqq -C_5 \quad \text{for} \quad \sigma \in \langle 0, +\infty)\,;$$

III) there exist $C_6 > 0$ and $C_7 > 0$ such that

$$(1.10) \qquad C_6 \sigma^\varkappa \leqq \mathscr{P}(\sigma) \leqq C_7 \sigma^\varkappa \quad \text{for} \quad \sigma \in \langle \varrho_2, +\infty)\,;$$

IV) there exist positive constants $C_8 - C_{11}$ such that if $\varrho \in L^\varkappa(\Omega_h)$, $\varrho(x) \geqq 0$ for a.a. $x \in \Omega_h$ then

$$(1.11) \qquad \int_{\Omega_h} \varrho^\varkappa\, \mathrm{d}x \leqq C_8 \int_{\Omega_h} \mathscr{P}(\varrho)\, \mathrm{d}x + C_9 \leqq C_{10} \int_{\Omega_h} \varrho^\varkappa\, \mathrm{d}x + C_{11}\,,$$

$$(1.12) \qquad \int_{\Omega_h} p(\varrho)\, \mathrm{d}x \leqq C_8 \int_{\Omega_h} \mathscr{P}(\varrho)\, \mathrm{d}x + C_9\,;$$

V) $$\lim_{\sigma \to 0+} \sigma\, \mathscr{P}'(\sigma) = 0\,;$$

VI) $$\sigma\, \mathscr{P}'(\sigma) - \mathscr{P}(\sigma) = p(\sigma) \quad \text{for} \quad \sigma \in (0, +\infty)\,;$$

VII) given $\eta > 0$, there exists $C_\eta > 0$ such that if $\zeta \in L^\varkappa(\Omega_h)$, $\zeta(x) \geqq \eta$ for a.a. $x \in \Omega_h$; then

(1.13) $$\int_{\Omega_h} |\mathscr{P}'(\zeta(x))|^{\varkappa/\varkappa-1} \, dx \leqq C_\eta \|\zeta\|_{L^\varkappa(\Omega_h)} ;$$

VIII) $$\mathscr{P}'(\sigma_1)(\sigma_1 - \sigma_2) \geqq \mathscr{P}(\sigma_1) - \mathscr{P}(\sigma_2) \quad \text{for} \quad \sigma_1, \sigma_2 \in (0, +\infty) ;$$

IX) $$\mathscr{P}''(\sigma) = p'(\sigma)/\sigma \geqq 0 \quad \text{for} \quad \sigma \in (0, +\infty) .$$

Let us point out that especially the property VI) is important. In fact, $\mathscr{P}$ was defined in order to satisfy VI). We could choose any positive number instead of 1 as the lower bound in the integral in (1.8) without any essential changes in the whole paper. Sometimes we could even choose it equal to zero. However, we could then have troubles with the integrability of $p(\vartheta)/\vartheta^2$ over $(0, \sigma)$ for example if $p(\vartheta) = $ const. ϑ for small ϑ.

Remark 1.2. We shall denote by $\|\cdot\|_0$, $\|\cdot\|_1$, $\|\cdot\|_{0,h}$ and $\|\cdot\|_{1,h}$ the norms in $L^2(\Omega)^3$, ${}^\circ H^1(\Omega)^3$, $L^2(\Omega_h)$ and $H^1(\Omega_h)$, respectively.

2. TIME DISCRETIZATION

In order to prove the existence of U, ϱ satisfying (1.5), (1.6) and (1.7), we shall use the method of time discretization. Let m be a natural number. Put $\tau = T/m$, $t_k = k\tau$ $(k = -1, 0, 1, \ldots, m)$. Denote $\varrho^{(-1)} = \varrho_0$, $u_i^{(0)} = u_{0i}$ $(i = 1, 2, 3)$ and let $U^{(k)} = (u_1^{(k)}, u_2^{(k)}, u_3^{(k)})$, $\varrho^{(k)}$ be the approximations of U, ϱ on the k-th time layer. We approximate (0.5), (0.6) in the following way:

$(2.1)_k$ $$\varrho^{(k)} - \varrho^{(k-1)} + \tau(\varrho^{(k)} \tilde{u}_j^{(k)})_{,j} = 0 \qquad (k = 0, 1, 2, \ldots, m) ,$$

$(2.2)_k$ $$\varrho^{(k-1)} u_i^{(k)} - \varrho^{(k-2)} u_i^{(k-1)} + \tau(\varrho^{(k-1)} \tilde{u}_j^{(k-1)} u_i^{(k)})_{,j} = - \tau \tilde{p}(\varrho^{(k)})_{,i} + \tfrac{1}{3}\tau\mu u_{j,ji}^{(k)} + \tau\mu u_{i,jj}^{(k)} \qquad (k = 1, 2, \ldots, m) .$$

The corresponding weak formulation (i.e. the approximation of (1.5), (1.6), (1.7)) reads as follows: We look for $\varrho^{(k)} \in H^1(\Omega_h)$, $\varrho^{(k)} \geqq 0$ a.e. in Ω_h $(k = 0, 1, 2, \ldots, m)$ and $U^{(k)} \in {}^\circ H^1(\Omega)^3$ $(k = 1, 2, \ldots, m)$ such that $(2.1)_k$ holds a.e. in Ω_h (for $k = 0, 1, 2, \ldots, m$) and

$(2.3)_k$ $$\int_\Omega \{\varrho^{(k-1)} u_i^{(k)} \Phi_i - \varrho^{(k-2)} u_i^{(k-1)} \Phi_i - \tau \varrho^{(k-1)} \tilde{u}_j^{(k-1)} u_i^{(k)} \Phi_{i,j} - \tau \, \tilde{p}(\varrho^{(k)}) \, \Phi_{i,i} + \tfrac{1}{3}\tau\mu u_{j,j}^{(k)} \Phi_{i,i} + \tau\mu u_{i,j}^{(k)} \Phi_{i,j}\} \, dx = 0$$

holds for all $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in {}^\circ C^\infty(\overline{\Omega})^3$ and $k = 1, 2, \ldots, m$.

We shall often use

$$(2.3\text{a})_k \qquad \int_\Omega \{\varrho^{(k-1)}u_i^{(k)}\Phi_i - \varrho^{(k-2)}u_i^{(k-1)}\Phi_i - - \tfrac{1}{2}[\varrho^{(k-1)} - \varrho^{(k-2)}]\, u_i^{(k)}\Phi_i + \tfrac{1}{2}\tau\varrho^{(k-1)}\tilde{u}_j^{(k-1)}u_{i,j}^{(k)}\Phi_i - - \tfrac{1}{2}\tau\varrho^{(k-1)}\tilde{u}_j^{(k-1)}u_j^{(k)}\Phi_{i,j} - \tau\tilde{p}(\varrho^{(k)})\,\Phi_{i,i} + + \tfrac{1}{3}\tau\mu u_{i,j}^{(k)}\Phi_{i,i} + \tau\mu u_{i,j}^{(k)}\Phi_{i,j}\}\,\mathrm{d}x = 0$$

instead of $(2.3)_k$. $(2.3\text{a})_k$ results from $(2.3)_k$ by adding the expression

$$\int_\Omega \{\tfrac{1}{2}\tau\varrho^{(k-1)}\tilde{u}_j^{(k-1)}u_i^{(k)}\Phi_{i,j} - \tfrac{1}{2}\varrho^{(k-1)}u_i^{(k)}\Phi_i + + \tfrac{1}{2}\varrho^{(k-2)}u_i^{(k)}\Phi_i + \tfrac{1}{2}\tau\varrho^{(k-1)}\tilde{u}_j^{(k-1)}u_{i,j}^{(k)}\Phi_i\}\,\mathrm{d}x$$

to $(2.3)_k$. $(2.1)_{k-1}$ implies that this expression is equal to zero.

We shall further proceed in such a way that we shall successively solve $(2.1)_0$, $(2.3)_1$ and $(2.1)_1$, $(2.3\text{a})_2$ and $(2.1)_2, \ldots, (2.3\text{a})_m$ and $(2.1)_m$. In order to do that, we shall solve some auxiliary problems first.

It will be useful to have $\tau < 1$. That is why we assume that $T < m$.

3. AUXILIARY PROBLEM I — A.P. I

This problem is given by the equation

$$(3.1) \qquad \varrho - \varrho^{(k-1)} + \tau(\varrho\tilde{u}_j)_{,j} = r\,\Delta\varrho \quad (\text{in } \Omega_h)$$

and by the boundary condition

$$(3.2) \qquad \left.\frac{\partial\varrho}{\partial\nu}\right|_{\partial\Omega_h} = 0\,.$$

ν denotes the outer normal vector on $\partial\Omega_h$. First we assume that $r > 0$ and $\varrho^{(k-1)}$ and $U \equiv (u_1, u_2, u_3)$ are known functions such that $\varrho^{(k-1)} \in C^\infty(\Omega_h)$, $\varrho^{(k-1)} \geqq 0$ in Ω_h, $U \in L^2(\Omega_h)^3$, $u_j = 0$ a.e. in $\Omega_h - \Omega$ $(j = 1, 2, 3)$. We also assume that

$$(3.3) \qquad \tau K_2 \|U\|_0 \leqq \tfrac{1}{4}\,.$$

K_2 is a positive constant which will be specified later in this section. Now we only suppose that

$$(3.4) \qquad K_2 \geqq \max\{\tfrac{3}{2}K_1, K_1 C_8\}\,.$$

It is a consequence of Theorem 3.2 in [1], p. 179 that (3.1), (3.2) has a unique solution $\varrho \in C^{2+(\alpha)}(\bar{\Omega}_h)$ if and only if the homogeneous boundary value problem given by the equation

$$(3.5) \qquad \varrho + \tau(\varrho\tilde{u}_j)_{,j} = r\,\Delta\varrho \quad (\text{in } \Omega_h)$$

and the boundary condition (3.2) has only the trivial solution in $C^{2+(\alpha)}(\bar{\Omega}_h)$. Suppose that ϱ is a solution of (3.5), (3.2) in $C^{2+(\alpha)}(\bar{\Omega}_h)$. If we multiply (3.5) by ϱ and integrate over Ω_h, we can derive the inequality

$$\int_{\Omega_h} (1 + \tfrac{1}{2}\tau \tilde{u}_{j,j})\, \varrho^2 \,\mathrm{d}x \leqq 0 . \tag{3.6}$$

(3.3) and (1.1) imply that

$$\tau \max_{x \in \bar{\Omega}_h} |\tilde{u}_{j,j}(x)| \leqq \tfrac{1}{2}$$

and so

$$1 + \tfrac{1}{2}\tau\, \tilde{u}_{j,j}(x) \geqq \tfrac{3}{4} \tag{3.7}$$

for all $x \in \bar{\Omega}_h$. Hence it follows from (3.6) that $\varrho \equiv 0$ in Ω_h and consequently, the problem (3.1), (3.2) has a unique solution $\varrho \in C^{2+(\alpha)}(\bar{\Omega}_h)$.

Suppose that $x_0 \in \bar{\Omega}_h$ is such a point that

$$\varrho(x_0) = \min_{x \in \bar{\Omega}_h} \varrho(x) .$$

It is possible to show that

$$\varrho(x_0) - \varrho^{(k-1)}(x_0) + \tau\, \varrho(x_0)\, \tilde{u}_{j,j}(x_0) = r\Delta\, \varrho(x_0) \geqq 0 ,$$

$$\varrho(x_0)\, [1 + \tau\, \tilde{u}_{j,j}(x_0)] \geqq \varrho^{(k-1)}(x_0) \geqq 0 .$$

Due to (3.7) we have $\varrho(x_0) \geqq 0$, which means that $\varrho(x) \geqq 0$ for all $x \in \bar{\Omega}_h$.

If we multiply (3.1) by ϱ, integrate over Ω_h and use the Green formula, we get

$$\int_{\Omega_h} \left\{ \left[\frac{1}{2} + \frac{\tau}{2} \tilde{u}_{j,j} \right] \varrho^2 + \frac{1}{2} (\varrho - \varrho^{(k-1)})^2 + r\varrho_{,j}\varrho_{,j} \right\} \mathrm{d}x = \int_{\Omega_h} \frac{1}{2}\, \varrho^{(k-1)2} \,\mathrm{d}x .$$

Since

$$\frac{1}{2} + \frac{\tau}{2}\, \tilde{u}_{j,j}(x) \geqq \frac{1}{2} - \frac{3}{2} \tau K_1 \|U\|_0 \geqq \frac{1}{2} - \tau K_2 \|U\|_0 ,$$

we have

$$\begin{aligned} &[\tfrac{1}{2} - \tau\, K_2 \|U\|_0] \int_{\Omega_h} \varrho^2 \,\mathrm{d}x + \int_{\Omega_h} \tfrac{1}{2}(\varrho - \varrho^{(k-1)})^2 \,\mathrm{d}x + \\ &\quad + r \int_{\Omega_h} \varrho_{,j}\varrho_{,j} \,\mathrm{d}x \leqq \int_{\Omega_h} \tfrac{1}{2} \varrho^{(k-1)2} \,\mathrm{d}x . \end{aligned} \tag{3.8}$$

Similarly, multiplying (3.1) by $-\Delta\varrho$ and integrating over Ω_h, we can derive that there exist $K_3 > 0$ and $K_4 > 0$ (depending on K_1) such that

$$\begin{aligned} &\int_{\Omega_h} \tfrac{1}{2} \varrho_{,i}\varrho_{,i} \,\mathrm{d}x + \int_{\Omega_h} \tfrac{1}{2} (\varrho_{,i} - \varrho_{,i}^{(k-1)}) (\varrho_{,i} - \varrho_{,i}^{(k-1)}) \,\mathrm{d}x + \\ &\quad + r \int_{\Omega_h} (\Delta\varrho)^2 \,\mathrm{d}x \leqq K_3 \tau \int_{\Omega_h} \varrho_{,i}\varrho_{,i} \,\mathrm{d}x\, \|U\|_0 + \\ &\quad + \int_{\Omega_h} \tfrac{1}{2} \varrho_{,i}^{(k-1)} \varrho_{,i}^{(k-1)} \,\mathrm{d}x + K_4 \tau \int_{\Omega_h} \varrho^2 \,\mathrm{d}x\, \|U\|_0 . \end{aligned}$$

Assume now that

$$K_2 = \max \{ \tfrac{3}{2} K_1 + K_4, K_3, K_1 C_8 \} . \tag{3.9}$$

If we add the last inequality to (3.8) and use (3.9), we obtain

$$(3.10)\qquad [\tfrac{1}{2} - \tau K_2\|U\|_0]\,\|\varrho\|_{1,h}^2 + \tfrac{1}{2}\|\varrho - \varrho^{(k-1)}\|_{1,h}^2 + + r\int_{\Omega_h}(\Delta\varrho)^2\,\mathrm{d}x \leqq \tfrac{1}{2}\|\varrho^{(k-1)}\|_{1,h}^2 .$$

Suppose that $c > 0$. Let us multiply (3.1) by $\mathscr{P}'(\varrho + c)$ and integrate over Ω_h. We have

$$\int_{\Omega_h}\mathscr{P}'(\varrho + c)(\varrho - \varrho^{(k-1)})\,\mathrm{d}x + \tau\int_{\Omega_h}\mathscr{P}'(\varrho + c)(\varrho\tilde{u}_j)_{,j}\,\mathrm{d}x = = r\int_{\Omega_h}\mathscr{P}'(\varrho + c)\,\Delta\varrho\,\mathrm{d}x ,$$

$$\int_{\Omega_h}\mathscr{P}'(\varrho + c)(\varrho - \varrho^{(k-1)})\,\mathrm{d}x + \tau\int_{\Omega_h}\mathscr{P}'(\varrho + c)\,\varrho\tilde{u}_{j,j}\,\mathrm{d}x + + \tau\int_{\Omega_h}\mathscr{P}'(\varrho + c)\,\varrho_{,j}u_j\,\mathrm{d}x = -r\int_{\Omega_h}\mathscr{P}''(\varrho + c)\,\varrho_{,j}\varrho_{,j}\,\mathrm{d}x \leqq 0 ,$$

$$\int_{\Omega_h}\mathscr{P}'(\varrho + c)\,[(\varrho + c) - (\varrho^{(k-1)} + c)]\,\mathrm{d}x + + \tau\int_{\Omega_h}\mathscr{P}'(\varrho + c)\,\varrho\tilde{u}_{j,j}\,\mathrm{d}x - \tau\int_{\Omega_h}\mathscr{P}(\varrho + c)\,\tilde{u}_{j,j}\,\mathrm{d}x \leqq 0 .$$

According to VI) and VIII) (see Sec. 1), we have

$$\int_{\Omega_h}\mathscr{P}(\varrho + c)\,\mathrm{d}x - \int_{\Omega_h}\mathscr{P}(\varrho^{(k-1)} + c)\,\mathrm{d}x + + \tau\int_{\Omega_h}p(\varrho + c)\,\tilde{u}_{j,j}\,\mathrm{d}x - \tau\int_{\Omega_h}c\mathscr{P}'(\varrho + c)\,\tilde{u}_{j,j}\,\mathrm{d}x \leqq 0 .$$

If $c \to 0+$, we get

$$(3.11)\qquad \int_{\Omega_h}\mathscr{P}(\varrho)\,\mathrm{d}x + \int_{\Omega_h}\tau\,p(\varrho)\,\tilde{u}_{j,j}\,\mathrm{d}x \leqq \int_{\Omega_h}\mathscr{P}(\varrho^{(k-1)})\,\mathrm{d}x .$$

Using (1.1), (1.12 abd (3.3), we can also derive

$$(3.12)\qquad \tfrac{3}{4}\int_{\Omega_h}\mathscr{P}(\varrho)\,\mathrm{d}x \leqq \int_{\Omega_h}\mathscr{P}(\varrho^{(k-1)})\,\mathrm{d}x + \frac{K_1C_9}{4K_2} .$$

Remark 3.1. The dependence of the solution ϱ of A.P. I on U may be expressed by the relation $\varrho = A(U)$. It is possible to show that A is a continuous operator from the set $\{U \in L^2(\Omega)^3;\ U$ satisfies (3.3)$\}$ into $H^1(\Omega_h)$.

Remark 3.2. If $\varrho^{(k-1)} \in H^1(\Omega_h)$, $\varrho^{(k-1)} \geqq 0$ and $r = 0$, we can use a sequence $\{\varrho_s^{(k-1)}\}$ of functions from $C^\infty(\bar{\Omega}_h)$ such that $\varrho_s^{(k-1)} \geqq 0$ in $\bar{\Omega}_h$, $\varrho_s^{(k-1)} \to \varrho^{(k-1)}$ in $H^1(\Omega_h)$ as $s \to +\infty$, and a sequence $\{r_s\}$ of real numbers such that $r_s \searrow 0$ as $s \to +\infty$, in order to get a sequence $\{\varrho_s\}$ of solutions of (3.1) and (3.2), corresponding to $\{\varrho_s^{(k-1)}\}$ and $\{r_s\}$. ϱ_s satisfies (3.10) (with r_s instead of r), hence there exists a subsequence (we shall denote it by $\{\varrho_s\}$ again in order not to complicate the notation) and $\varrho \in H^1(\Omega_h)$ such that $\varrho_s \rightharpoonup \varrho$ as $s \to +\infty$. It can be easily shown that ϱ satisfies (3.1) (with $r = 0$), $\varrho \geqq 0$ in Ω_h and moreover, ϱ satisfies (3.8) and (3.10) (with $r = 0$).

Now we are going to show that ϱ satisfies (3.12). In order to do that, it is sufficient to prove that

$$(3.13)\qquad \lim_{s\to+\infty}\int_{\Omega_h}\mathscr{P}(\varrho_s)\,\mathrm{d}x = \int_{\Omega_h}\mathscr{P}(\varrho)\,\mathrm{d}x .$$

Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $g \in L^\varkappa(\Omega_h)$, $0 \leqq g \leqq \delta$ a.e. in Ω_h then

$$\int_{\Omega_h} |\mathscr{P}(g(x))| \, dx < \varepsilon .$$

There also exists $\eta > 0$ such that if $\sigma_1 \geqq 0$, $\sigma_2 \geqq 0$, $\sigma_1 \geqq \delta$ or $\sigma_2 \geqq \delta$, $\xi \in R^1$ is such a number that

$$\mathscr{P}(\sigma_2) - \mathscr{P}(\sigma_1) = \mathscr{P}'(\xi)(\sigma_2 - \sigma_1) ,$$

then $\xi \geqq \eta$. Let us now divide Ω_h into two parts: $\Omega^s_{h,1}$ (where $\varrho_s < \delta$ and $\varrho < \delta$) and $\Omega^s_{h,2}$ (where $\varrho_s \geqq \delta$ or $\varrho \geqq \delta$). We can write

$$\left|\int_{\Omega_h} \mathscr{P}(\varrho_s) \, dx - \int_{\Omega_h} \mathscr{P}(\varrho) \, dx\right| \leqq \int_{\Omega_{h,1}{}^s} |\mathscr{P}(\varrho_s) - \mathscr{P}(\varrho)| \, dx +$$
$$+ \int_{\Omega_{h,2}{}^s} |\mathscr{P}(\varrho_s) - \mathscr{P}(\varrho)| \, dx \leqq 2\varepsilon + \int_{\Omega_{h,2}{}^s} |\mathscr{P}'(\zeta_s)| \, |\varrho_s - \varrho| \, dx \leqq$$
$$\leqq 2\varepsilon + \left(\int_{\Omega_{h,2}{}^s} |\mathscr{P}'(\zeta_s)|^{\varkappa/(\varkappa-1)} \, dx\right)^{(\varkappa-1)/\varkappa} \left(\int_{\Omega_h} |\varrho_s - \varrho|^\varkappa \, dx\right)^{1/\varkappa} .$$

$\varrho_s \to \varrho$ in $L^\varkappa(\Omega_h)$. ζ_s is a function with values between the values of ϱ_s and ϱ and such that $\zeta_s \geqq \eta$ a.e. in $\Omega^s_{h,2}$. Hence, using (1.13), we get

$$\limsup_{s \to +\infty} \left|\int_{\Omega_h} \mathscr{P}(\varrho_s) \, dx - \int_{\Omega_h} \mathscr{P}(\varrho) \, dx\right| \leqq 2\varepsilon .$$

Since ε was chosen arbitrarily, we get (3.13).

It is also possible to show that

$$\int_{\Omega_h} \tau \, p(\varrho_s) \, \tilde{u}_{j,j} \, dx \to \int_{\Omega_h} \tau \, p(\varrho) \, \tilde{u}_{j,j} \, dx \quad \text{as} \quad s \to +\infty$$

in an analogous way. It is only necessary to use (0.4) instead of (1.13). Thus, ϱ satisfies (3.11), too.

4. AUXILIARY PROBLEM II — A.P. II

We look for $U \equiv (u_1, u_2, u_3) \in H^1(\Omega)^3$ such that

(4.1)
$$\int_\Omega \{\varrho^{(k-1)} u_i \Phi_i - \varrho^{(k-2)} u_i^{(k-1)} \Phi_i -$$
$$- \tfrac{1}{2}[\varrho^{(k-1)} - \varrho^{(k-2)}] \, u_i \Phi_i + \tfrac{1}{2}\tau \varrho^{(k-1)} \, \tilde{u}_j^{(k-1)} u_{i,j} \Phi_i -$$
$$- \tfrac{1}{2}\tau \varrho^{(k-1)} \tilde{u}_j^{(k-1)} u_i \Phi_{i,j} - \tau \, \tilde{p}(\varrho) \, \Phi_{i,i} +$$
$$+ \tfrac{1}{3}\tau\mu u_{j,j} \Phi_{i,i} + \tau\mu u_{i,j} \Phi_{i,j}\} \, dx = 0$$

for all $\Phi \equiv (\Phi_1, \Phi_2, \Phi_3) \in {}^\circ H^1(\Omega)^3$. We assume that ϱ, $\varrho^{(k-1)}$, $\varrho^{(k-2)}$, $U^{(k-1)} \equiv \equiv (u_1^{(k-1)}, u_2^{(k-1)}, u_3^{(k-1)})$ are known functions such that $\varrho^{(k-1)}, \varrho^{(k-2)} \in C^\infty(\bar{\Omega}_h)$, $\varrho^{(k-1)} \geqq 0$ in Ω_h, $\varrho^{(k-2)} \geqq 0$ in Ω_h, $\varrho \in L^\varkappa(\Omega_h)$, $\varrho \geqq 0$ a.e. in Ω_h, $U^{(k-1)} \in L^2(\Omega)^3$.

Denote

$$F_1(U)(\Phi) = \int_\Omega \{\varrho^{(k-1)} u_i \Phi_i - \tfrac{1}{2}[\varrho^{(k-1)} - \varrho^{(k-2)}] \, u_i \Phi_i +$$
$$+ \tfrac{1}{2}\tau \varrho^{(k-1)} \tilde{u}_j^{(k-1)} u_{i,j} \Phi_i - \tfrac{1}{2}\tau \varrho^{(k-1)} \tilde{u}_j^{(k-1)} u_i \Phi_{i,j} +$$
$$+ \tfrac{1}{3}\tau\mu u_{j,j} \Phi_{i,i} + \tau\mu u_{i,j} \Phi_{i,j}\} \, dx .$$

It can be proved that

$$F_1(U)(\Phi) \leqq \text{const. } \|U\|_1 \|\Phi\|_1 . \tag{4.2}$$

Due to the Riesz theorem, given $U \in {}^\circ H^1(\Omega)^3$, there exists $G(U) \in {}^\circ H^1(\Omega)^3$ such that $F_1(U)(\Phi) = ((G(U), \Phi))$ (where $((\cdot, \cdot))$ denotes the scalar product in ${}^\circ H^1(\Omega)^3$). (4.2) implies that G is a bounded linear operator from ${}^\circ H^1(\Omega)^3$ into itself. Moreover, the inequality

$$((G(U), U)) = \int_\Omega \{\tfrac{1}{2}\varrho^{(k-1)} u_i u_i + \tfrac{1}{2}\varrho^{(k-2)} u_i u_i + \\ + \tfrac{1}{3}\tau\mu u_{j,j} u_{i,i} + \tau\mu u_{i,j} u_{i,j}\} \,\mathrm{d}x \geqq \text{const. } \|U\|_1^2$$

implies that the range of G is the whole space ${}^\circ H^1(\Omega)^3$.

Denote

$$F_2(\Phi) = \int_\Omega \{\varrho^{(k-2)} u_i^{(k-1)} \Phi_i + \tau\, \tilde{p}(\varrho)\, \Phi_{i,i}\} \,\mathrm{d}x .$$

F_2 is a bounded linear functional on ${}^\circ H^1(\Omega)^3$, hence there exists $V \in {}^\circ H^1(\Omega)^3$ such that $F_2(\Phi) = ((V, \Phi))$. Denote by U such an element of ${}^\circ H^1(\Omega)^3$ that $G(U) = V$. Then $((G(U), \Phi)) = ((V, \Phi))$ for all $\Phi \in {}^\circ H^1(\Omega)^3$. It means that U is a solution of (4.1).

If we put $\Phi = U$ in (4.1), we obtain

$$\int_\Omega \{\tfrac{1}{2}\varrho^{(k-1)} u_i u_i + \\ + \tfrac{1}{2}\varrho^{(k-2)}[u_i - u_i^{(k-1)}]\,[u_i - u_i^{(k-1)}] - \\ - \tfrac{1}{2}\varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)} - \tau\, \tilde{p}(\varrho)\, u_{i,i} + \\ + \tfrac{1}{3}\tau\mu(u_{j,j})^2 + \tau\mu u_{i,j} u_{i,j}\} \,\mathrm{d}x = 0. \tag{4.3}$$

Using the estimates (0.3) and (1.11), we can write

$$\int_\Omega \tau\, \tilde{p}(\varrho)\, u_{i,i} \,\mathrm{d}x \leqq \int_\Omega \tfrac{1}{3}\tau\mu(u_{i,i})^2 \,\mathrm{d}x + \\ + \tau \text{ const. } \int_\Omega \tilde{p}(\varrho)^2 \,\mathrm{d}x \leqq \int_\Omega \tfrac{1}{3}\tau\mu(u_{i,i})^2 \,\mathrm{d}x + \\ + \tau K_5 (\int_{\Omega_h} \mathscr{P}(\varrho) \,\mathrm{d}x)^2 + K_5$$

for some $K_5 > 0$. Substituting this estimate into (4.3), we get

$$\int_\Omega \{\tfrac{1}{2}\varrho^{(k-1)} u_i u_i + \\ + \tfrac{1}{2}\varrho^{(k-2)}[u_i - u_i^{(k-1)}]\,[u_i - u_i^{(k-1)}] + \tau\mu u_{i,j} u_{i,j}\} \,\mathrm{d}x \leqq \\ \leqq \int_\Omega \tfrac{1}{2}\varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)} \,\mathrm{d}x + \tau K_5 (\int_{\Omega_h} \mathscr{P}(\varrho) \,\mathrm{d}x)^2 + \tau K_5 . \tag{4.4}$$

Remark 4.1. It follows from (4.4) that there exists a constant $K_6 > 0$ such that

$$\|U\|_1 \leqq \frac{K_6}{\sqrt{\tau}} \{1 + \int_\Omega \tfrac{1}{2}\varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)} \mathrm{d}x + \tfrac{3}{4}\int_{\Omega_h} \mathscr{P}(\varrho) \,\mathrm{d}x + \sqrt{\tau}\} . \tag{4.5}$$

Since $\|U\|_1 \geqq \|U\|_0$, U satisfies (3.3) if

$$(4.6) \quad \sqrt{(\tau)}\, K_2 K_6 \{1 + \int_\Omega \tfrac{1}{2} \varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)}\, dx + \tfrac{3}{4} \int_{\Omega_h} \mathscr{P}(\varrho)\, dx + \sqrt{(\tau)}\} \leqq \tfrac{1}{4}\,.$$

Remark 4.2. Let us denote by $U = B(\varrho)$ the dependence of the solution U of A.P. II on the function ϱ. B is an operator from $L^\alpha(\Omega_h)$ into ${}^\circ H^1(\Omega)^3$. It follows from (4.5) and (1.11) that B is bounded. It is possible to show that B is continuous, too.

5. AUXILIARY PROBLEM III — A.P. III

We look for $U \equiv (u_1, u_2, u_3) \in {}^\circ H^1(\Omega)^3$ and $\varrho \in H^1(\Omega_h)$ such that (4.1) holds for all $\Phi \equiv (\Phi_1, \Phi_2, \Phi_3) \in {}^\circ H^1(\Omega)^3$, $\varrho \geqq 0$ a.e. in Ω_h and

$$(5.1) \qquad \varrho - \varrho^{(k-1)} + \tau(\varrho \tilde{u}_j)_{,j} = r\, \Delta\varrho \quad (\text{in } \Omega_h)\,,$$

$$(5.2) \qquad \left.\frac{\partial \varrho}{\partial \nu}\right|_{\partial\Omega_h} = 0\,.$$

First we assume that $r > 0$ and $U^{(k-1)} \equiv (u_1^{(k-1)}\, u_2^{(k-1)}, u_3^{(k-1)})$, $\varrho^{(k-1)}, \varrho^{(k-2)}$ are known functions such that $U^{(k-1)} \in L^2(\Omega)^3$, $\varrho^{(k-2)} \in C^\infty(\bar{\Omega}_h)$, $\varrho^{(k-1)} \in C^\infty(\bar{\Omega}_h)$, $\varrho^{(k-2)} \geqq 0$ in $\bar{\Omega}_h$, $\varrho^{(k-1)} \geqq 0$ in $\bar{\Omega}_h$. We also assume that

$$(5.3) \qquad \sqrt{(\tau)}\, K_2 K_6 \Big\{1 + \int_\Omega \tfrac{1}{2} \varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)}\, dx + \int_{\Omega_h} \mathscr{P}(\varrho^{(k-1)})\, dx + \frac{K_1 C_9}{4K_2} + \sqrt{\tau}\Big\} \leqq \tfrac{1}{4}\,.$$

Let U_1 be such an element of ${}^\circ H^1(\Omega)^3$ that

$$(5.4) \qquad \|U_1\|_1 \leqq \frac{1}{4\tau K_2} \equiv K_7\,.$$

Denote by E the compact embedding of ${}^\circ H^1(\Omega)^3$ into $L^2(\Omega)^3$. (5.4) implies that $\|EU_1\|_0 \leqq K_7$.

Put $\varrho_1 = A(EU_1)$ and $U_2 = B(\varrho_1) = B(A(EU_1))$. It follows from (4.5), (3.12) and (5.3) that

$$\|U_2\|_1 \leqq \frac{K_6}{\sqrt{\tau}} \{1 + \int_\Omega \tfrac{1}{2} \varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)}\, dx + \tfrac{3}{4} \int_{\Omega_h} \mathscr{P}(\varrho_1)\, dx + \sqrt{\tau}\} \leqq$$

$$\leqq \frac{K_6}{\sqrt{\tau}} \Big\{1 + \int_\Omega \tfrac{1}{2} \varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)}\, dx + \int_{\Omega_h} \mathscr{P}(\varrho^{(k-1)})\, dx + \frac{K_1 C_9}{4K_2} + \sqrt{\tau}\Big\} \leqq$$

$$\leqq \frac{1}{4\tau K_2} = K_7\,.$$

Hence $B * A * E$ is the compact operator from the ball $\overline{B_{K_7}(0)}$ in ${}^\circ H^1(\Omega)^3$ into itself. According to the Brouwer theorem, this operator has a fixed point U in this ball. The functions U and $\varrho = A(EU)$ are the solution of A.P. III.

It follows from (3.10) and (3.11) that

$$\text{(5.5)}\qquad [\tfrac{1}{2} - \tau K_2 \|U\|_0] \, \|\varrho\|_{1,h}^2 + \tfrac{1}{2}\|\varrho - \varrho^{(k-1)}\|_{1,h}^2 + \\ + r\|\Delta\varrho\|_{0,h}^2 \leqq \tfrac{1}{2}\|\varrho^{(k-1)}\|_{1,h}^2 ,$$

$$\text{(5.6)}\qquad \int_{\Omega_h} \mathscr{P}(\varrho)\,\mathrm{d}x + \tau \int_{\Omega_h} p(\varrho)\,\tilde{u}_{j,j}\,\mathrm{d}x \leqq \int_{\Omega_h} \mathscr{P}(\varrho^{(k-1)})\,\mathrm{d}x \, .$$

We can obtain one more estimate putting $\Phi = U$ in (4.1) and adding (5.6):

$$\text{(5.7)}\qquad \int_\Omega \{\tfrac{1}{2}\varrho^{(k-1)} u_i u_i = \\ + \tfrac{1}{2}\varrho^{(k-2)}[u_i - u_i^{(k-1)}[\,[u_i - u_i^{(k-1)}] + \\ + \tfrac{1}{3}\tau\mu(u_{j,j})^2 + \tau\mu u_{i,j} u_{i,j}\}\,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho)\,\mathrm{d}x \leqq \\ \leqq \int_\Omega \tfrac{1}{2}\varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)}\,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho^{(k-1)})\,\mathrm{d}x \, .$$

Remark 5.1. Using a similar approach as in Remark 3.2, we can show that A.P. III has a solution U, ϱ also in the case when $\varrho^{(k-2)}$ and $\varrho^{(k-1)}$ belong to $H^1(\Omega_h)$ only instead of $C^\infty(\bar{\Omega}_h)$, $\varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)} \in L^1(\Omega)$ and $r = 0$. U, ϱ satisfy (5.5) (with $r = 0$), (5.7) and

$$\text{(5.8)}\qquad \|U\|_1 \leqq \frac{1}{4\tau K_2} = K \, .$$

6. SOLUTION OF $(2.1)_0$ AND $(2.3a)_k$, $(2.1)_k$ FOR $k = 1, 2, \ldots, m$

Suppose that τ is so small that

$$\text{(6.1)}\qquad \tau K_2 \|U_0\|_0 \leqq \tfrac{1}{4} ,$$

$$\text{(6.2)}\qquad \sqrt{(\tau)}\, K_2 K_6 \left\{1 + \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i}\,\mathrm{d}x + \tfrac{4}{3}\int_{\Omega_h} \mathscr{P}(\varrho_0)\,\mathrm{d}x + \frac{7K_1C_9}{12K_2} + \sqrt{\tau}\right\} \leqq \tfrac{1}{4} .$$

The results of Section 3 imply that $(2.1)_0$ has a solution $\varrho^{(0)} \in H^1(\Omega_h)$, satisfying

$$\text{(6.3)}\qquad [\tfrac{1}{2} - \tau K_2 \|U_0\|_0] \, \|\varrho^{(0)}\|_{1,h} + \tfrac{1}{2}\|\varrho^{(0)} - \varrho_0\|_{1,h}^2 \leqq \tfrac{1}{2}\|\varrho_0\|_{1,h}^2 ,$$

$$\text{(6.4)}\qquad \tfrac{3}{4} \int_{\Omega_h} \mathscr{P}(\varrho^{(0)})\,\mathrm{d}x \leqq \int_{\Omega_h} \mathscr{P}(\varrho_0)\,\mathrm{d}x + \frac{K_1C_9}{4K_2} .$$

It follows from (6.2) and (6.4) that

$$\text{(6.5)}\qquad \sqrt{(\tau)}\, K_2 K_6 \left\{1 + \int_\Omega \tfrac{1}{2}\varrho^{(-1)} u_i^{(0)} u_i^{(0)}\,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho^{(0)})\,\mathrm{d}x + \frac{K_1C_9}{4K_2} + \sqrt{\tau}\right\} \leqq \tfrac{1}{4} .$$

By virtue of the results of Section 5, the problem $(2.3a)_1$, $(2.1)_1$ has a solution $U^{(1)} \equiv (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})$, $\varrho^{(1)}$, satisfying

$$(6.6) \qquad [\tfrac{1}{2} - \tau K_2 \|U^{(1)}\|_0] \, \|\varrho^{(1)}\|_{1,h}^2 + \\ + \tfrac{1}{2}\|\varrho^{(1)} - \varrho^{(0)}\|_{1,h}^2 \leqq \tfrac{1}{2}\|\varrho^{(0)}\|_{1,h}^2 ,$$

$$(6.7) \qquad \int_\Omega \{\tfrac{1}{2}\varrho^{(0)} u_i^{(1)} u_i^{(1)} + \\ + \tfrac{1}{2}\varrho^{(-1)}[u_i^{(1)} - u_i^{(0)}] \, [u_i^{(1)} - u_i^{(0)}] + \\ + \tfrac{1}{3}\tau\mu (u_{j,j}^{(1)})^2 + \tau\mu u_{i,j}^{(1)} u_{i,j}^{(1)}\} \, \mathrm{d}x + \\ + \int_{\Omega_h} \mathscr{P}(\varrho^{(1)}) \, \mathrm{d}x \leqq \int_\Omega \tfrac{1}{2}\varrho^{(-1)} u_i^{(0)} u_i^{(0)} \, \mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho^{(0)}) \, \mathrm{d}x$$

$$(6.8) \qquad \tau K_2 \|U^{(1)}\|_1 \leqq \tfrac{1}{4} .$$

It follows from (6.5) and (6.7) that

$$(6.9) \quad \sqrt{(\tau)} \, K_2 K_6 \left\{1 + \int_\Omega \tfrac{1}{2}\varrho^{(0)} u_i^{(1)} u_i^{(1)} \, \mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho^{(1)}) \, \mathrm{d}x + \frac{K_1 C_9}{4K_2} + \sqrt{\tau}\right\} \leqq \tfrac{1}{4}$$

and hence the problem $(2.3\mathrm{a})_2$, $(2.1)_2$ has a solution $U^{(2)} \equiv (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})$, $\varrho^{(2)}$. We can continue in this way so that we finally get sequences $\{U^{(k)}\}$, $\{\varrho^{(k)}\}$ of solutions of $(2.3\mathrm{a})_k$, $(2.1)_k$, and the estimates

$$(6.10)_k \qquad [\tfrac{1}{2} - \tau K_2 \|U^{(k)}\|_0] \, \|\varrho^{(k)}\|_{1,h}^2 + \\ + \tfrac{1}{2}\|\varrho^{(k)} - \varrho^{(k-1)}\|_{1,h}^2 \geqq \tfrac{1}{2}\varrho^{(k-1)}\|_{1,h}^2 ,$$

$$(6.11)_k \qquad \int_\Omega \{\tfrac{1}{2}\varrho^{(k-1)} u_i^{(k)} u_i^{(k)} + \\ + \tfrac{1}{2}\varrho^{(k-2)}[u_i^{(k)} - u_i^{(k-1)}] \, [u_i^{(k)} - u_i^{(k-1)}] + \\ + \tfrac{1}{3}\tau\mu (u_{j,j}^{(k)})^2 + \tau\mu u_{i,j}^{(k)} u_{i,j}^{(k)}\} \, \mathrm{d}x + \\ + \int_{\Omega_h} \mathscr{P}(\varrho^{(k)}) \, \mathrm{d}x \leqq \int_\Omega \tfrac{1}{2}\varrho^{(k-2)} u_i^{(k-1)} u_i^{(k-1)} \, \mathrm{d}x + \\ + \int_{\Omega_h} \mathscr{P}(\varrho^{(k-1)}) \, \mathrm{d}x ,$$

$$(6.12)_k \qquad \tau K_2 \|U^{(k)}\|_1 \leqq \tfrac{1}{4}$$

for $k = 1, 2, \ldots, m$.

Starting from these inequalities and using the standard technique, we can derive also the following inequalities:

$$(6.13) \qquad \int_\Omega \{\tfrac{1}{2}\varrho^{(k-1)} u_i^{(k)} u_i^{(k)} + \\ + \tfrac{1}{2}\sum_{s=1}^{k} \varrho^{(s-2)}[u_i^{(s)} - u_i^{(s-1)}] \, [u_i^{(s)} - u_i^{(s-1)}] + \\ + \tfrac{1}{3}\tau\mu \sum_{s=1}^{k} (u_{j,j}^{(s)})^2 + \tau\mu \sum_{s=1}^{k} u_{i,j}^{(s)} u_{i,j}^{(s)}\} \, \mathrm{d}x + \\ + \int_{\Omega_h} \mathscr{P}(\varrho^{(k)}) \, \mathrm{d}x \leqq \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i} \, \mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho_0) \, \mathrm{d}x \\ (k = 1, 2, \ldots, m) ,$$

$$\|\varrho^{(k)}\|_{1,h}^2 + \sum_{s=0}^{k} \|\varrho^{(s)} - \varrho^{(s-1)}\|_{1,h}^2 \leqq \tag{6.14}$$

$$\leqq K_8 \|U_0\|_0, \int_\Omega \varrho_0 u_{0i} u_{0i} \, dx, \|\varrho_0\|_{1,h}) \quad (k = 0, 1, 2, \ldots, m).$$

K_8 is a positive function of three variables, nondecreasing in each of these variables.

7. AN APPROXIMATE SOLUTION OF (1.6), (1.7) AND ITS ESTIMATES

Put

$$\begin{aligned} {}^m\varrho(t) &= \varrho^{(k)} \quad \text{for } t \in (t_k, t_{k+1}\rangle \quad (k = -1, 0, 1, \ldots, m-1) \\ {}^mU(t) &= U^{(k+1)} \text{ for } t \in (t_k, t_{k+1}\rangle \quad (k = 0, 1, 2, \ldots, m-1). \end{aligned} \tag{7.1}$$

It follows from (6.14) that there exists $K_9 > 0$ such that

$$\|{}^m\varrho\|_{L^\infty(0,T;H^1(\Omega_h))} \leqq K_9. \tag{7.2}$$

If $t' \in (0, T\rangle$ then there exists $k \in \{1, 2, \ldots, m\}$ such that $t' \in (t_{k-1}, t_k\rangle$. Using (6.13) and (7.1), we can derive the inequality

$$\begin{aligned} &\int_\Omega \tfrac{1}{2}\, {}^m\varrho\, {}^mu_i\, {}^mu_i|_{t=t'}\, dx + \int_{\Omega_h} \mathscr{P}({}^m\varrho)|_{t=t'}\, dx + \\ &+ \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu({}^mu_{j,j})^2 + \mu\, {}^mu_{i,j}\, {}^mu_{i,j}\}\, dx\, dt \leqq \\ &\leqq \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i}\, dx + \int_{\Omega_h} \mathscr{P}(\varrho_0)\, dx + \\ &+ \int_{\Omega_h} [\mathscr{P}(\varrho^{(k-1)}) - \mathscr{P}(\varrho^{(k)})]\, dx. \end{aligned} \tag{7.3}$$

Due to (1.11), the inequality $\varkappa < 6$ and (6.14), the term

$$\int_{\Omega_h} [\mathscr{P}(\varrho^{(k-1)}) - \mathscr{P}(\varrho^{(k)})]\, dx$$

may be estimated from above by some constant which does not depend on k or m. This fact, (7.3), (1.9) and (1.10) imply that there exists $K_{10} > 0$ such that

$$\|{}^mU\|_{L^2(0,T;{}^0H^1(\Omega)^3))} \leqq K_{10}, \tag{7.4}$$

$$\|{}^m\varrho|{}^mU|^2\|_{L^\infty(0,T;L^1(\Omega))} \leqq K_{10}, \tag{7.5}$$

$$\|{}^m\varrho\|_{L^\infty(0,T;L^\varkappa(\Omega_h))} \leqq K_{10}. \tag{7.6}$$

By means of the Hölder inequality, it can be also shown that there exists $K_{11} > 0$ such that

$$\|{}^m\varrho\, {}^mU\|_{L^\infty(0,T;L^{12/7}(\Omega)^3)} \leqq K_{11}, \tag{7.7}$$

$$\|{}^m\varrho\, {}^mU\|_{L^2(0,T;W_{3/2}{}^1(\Omega)^3)} \leqq K_{11}. \tag{7.8}$$

K_9, K_{10} and K_{11} are constants which do not depend on m.

Before we derive further estimates, we introduce some functional spaces. If B_0 and B_1 are Banach spaces then we put

$$\mathscr{H}^\gamma(R^1; B_0, B_1) = \{v; v \in L^2(R^1; B_0), |\vartheta|^\gamma \hat{v} \in L^2(R^1; B_1)\}$$

where

$$\hat{v}(\vartheta) = \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi i t \vartheta} v(t) \, \mathrm{d}t .$$

$\mathscr{H}^\gamma(R^1; B_0, B_1)$ is the Banach space with the norm

$$\|v\|_{\mathscr{H}^\gamma(R^1;B_0,B_1)} = \left[\|v\|^2_{L^2(R^1;B_0)} + \| |\vartheta|^\gamma \hat{v}\|^2_{L^2(R^1;B_1)}\right]^{1/2} .$$

We denote by $\mathscr{H}^\gamma(0, T; B_0, B_1)$ the Banach space of restrictions of all functions from $\mathscr{H}^\gamma(R^1; B_0, B_1)$ onto the interval $(0, T)$ with the norm

$$\|v\|_{\mathscr{H}^\gamma(0,T;B_0,B_1)} = \inf \|w\|_{\mathscr{H}^\gamma(R^1;B_0,B_1)}$$

(where the infimum is considered over the set of all functions $w \in \mathscr{H}^\gamma(R^1; B_0, B_1)$ such that $w = v$ a.e. in $(0, T)$). The spaces $\mathscr{H}^\gamma(R^1; B_0, B_1)$ and $\mathscr{H}^\gamma(0, T; B_0, B_1)$ are used for example also in [7].

Lemma 7.1. *There exists $K_{12} > 0$ (not depending on m) such that if $0 < \gamma < 1/2$ then*

$$\|{}^m\varrho \, {}^mU\|_{\mathscr{H}^\gamma(0,T;W_{3/2}{}^1(\Omega)^3,H^{-1}(\Omega)^3)} \leqq K_{12} . \tag{7.9}$$

Proof. Due to (7.8), it is sufficient to show that there exists $K_{13} > 0$ such that

$$\| |\vartheta|^\gamma \, {}^m\hat{w}\|^2_{L^2(R^1;H^{-1}(\Omega)^3)} \leqq K_{13} , \tag{7.10}$$

where ${}^mw$ coincides with ${}^m\varrho \, {}^mU$ a.e. in $(0, T)$ and ${}^mw = 0$ a.e. in $R^1 - (0, T)$. We can write

$$\begin{aligned} {}^m\hat{w}(\vartheta) &= \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi i t\vartheta} \, {}^mw(t) \, \mathrm{d}t = \int_0^T \mathrm{e}^{-2\pi i t\vartheta} \, {}^m\varrho(t) \, {}^mU(t) \, \mathrm{d}t = \\ &= \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \mathrm{e}^{-2\pi i t\vartheta} \varrho^{(k-1)} U^{(k)} \, \mathrm{d}t = \\ &= \sum_{k=1}^m \left[- \frac{1}{2\pi i\vartheta} \left(\mathrm{e}^{-2\pi i t_k\vartheta} - \mathrm{e}^{-2\pi i t_{k-1}\vartheta}\right)\right] \varrho^{(k-1)} U^{(k)} . \end{aligned}$$

We shall use this form of ${}^m\hat{w}(\vartheta)$ if $|\vartheta| \leqq 1$. ${}^m\hat{w}(\vartheta)$ can be easily expressed also in the following way:

$$\begin{aligned} {}^m\hat{w}(\vartheta) &= \frac{1}{2\pi i\vartheta} \left[\mathrm{e}^{-2\pi i t_0\vartheta} \varrho^{(0)} U^{(1)} - \mathrm{e}^{-2\pi i t_m\vartheta}\right] \varrho^{(m-1)} U^{(m)} + \\ &+ \frac{1}{2\pi i\vartheta} \sum_{k=1}^{m-1} \mathrm{e}^{-2\pi i t_k\vartheta} \left[\varrho^{(k)} U^{(k+1)} - \varrho^{(k-1)} U^{(k)}\right] . \end{aligned} \tag{7.11}$$

If $|\vartheta| \leqq 1$ then we have

$$\| |\vartheta|^{\gamma}\, {}^m\hat{w}(\vartheta)\|_{H^{-1}(\Omega)^3} = \| |\vartheta|^{\gamma} \sum_{k=1}^{m} \frac{1}{2\pi i \vartheta} [e^{-2\pi i t_k \vartheta} - e^{-2\pi i t_{k-1}\vartheta}]\, \varrho^{(k-1)} U^{(k)} \|_{H^{-1}(\Omega)^3} \leqq$$

$$\leqq \text{const.}\, |\vartheta|^{\gamma}\, \tau \| \sum_{k=1}^{m} |\varrho^{(k-1)} U^{(k)}| \, \|_{L^{12/7}(\Omega)^3} \leqq$$

$$\leqq \text{const.}\, |\vartheta|^{\gamma} \sum_{k=1}^{m} \tau [(\textstyle\int_\Omega \varrho^{(k-1)6}\, dx)^{1/7} (\int_\Omega \varrho^{(k-1)} |U^{(k)}|^2\, dx)^{6/7}]^{7/12} \leqq$$

$$\leqq \text{const.}\, |\vartheta|^{\gamma} \leqq K_{14}\,.$$

If $|\vartheta| > 1$ then by means of (9.11) we get

$$\| |\vartheta|^{\gamma}\, {}^m\hat{w}(\vartheta)\|_{H^{-1}(\Omega)^3} \leqq \text{const.}\, |\vartheta|^{\gamma-1} \, \|\varrho^{(0)} U^{(1)}\|_{H^{-1}(\Omega)^3} +$$

$$+ \text{const.}\, |\vartheta|^{\gamma-1} \, \|\varrho^{(m-1)} U^{(m)}\|_{H^{-1}(\Omega)^3} +$$

$$+ \text{const.}\, |\vartheta|^{\gamma-1} \sum_{k=1}^{m-1} \|\varrho^{(k)} U^{(k+1)} - \varrho^{(k-1)} U^{(k)}\|_{H^{-1}(\Omega)^3} \leqq$$

$$\leqq \text{const.}\, |\vartheta|^{\gamma-1} + \text{const.}\, |\vartheta|^{\gamma-1} \sum_{k=2}^{m} \|\varrho^{(k-1)} U^{(k)} -$$

$$- \varrho^{(k-2)} U^{(k-1)}\|_{H^{-1}(\Omega)^3} \leqq \text{const.}\, |\vartheta|^{\gamma-1} +$$

$$+ \text{const.}\, |\vartheta|^{\gamma-1} \sum_{k=2}^{m} \{ \tau \| (\varrho^{(k-1)}\, \tilde{u}_j^{(k-1)} U^{(k)})_{,j} \|_{H^{-1}(\Omega)^3} +$$

$$+ \tau \| \nabla \tilde{p}(\varrho^{(k)}) \|_{H^{-1}(\Omega)^3} + \tfrac{1}{3} \tau\mu \| \nabla u_{j,j}^{(k)} \|_{H^{-1}(\Omega)^3} +$$

$$+ \tau\mu \| \Delta U^{(k)} \|_{H^{-1}(\Omega)^3} \} \leqq K_{14} |\vartheta|^{\gamma-1}\,.$$

K_{14} is a positive constant.

These estimates yield

$$\| |\vartheta|^{\gamma}\, {}^m\hat{w}(\vartheta)\|^2_{H^{-1}(\Omega)^3} \leqq g^2(\vartheta)$$

where $g(\vartheta) = K_{14}$ for $|\vartheta| \leqq 1$ and $g(\vartheta) = K_{14}|\vartheta|^{\gamma-1}$ for $|\vartheta| > 1$. Since $\gamma \in (0, 1/2)$, the function g^2 is integrable over R^1. Thus, (7.10) and consequently also (7.9) are proved.

By a similar method it is possible to show that there exists $K_{15} > 0$ (not depending on m) such that

$$\|{}^m\varrho\|_{\mathscr{H}^{\gamma}(0,T;H^1(\Omega_h),L^2(\Omega_h))} \leqq K_{15}\,. \tag{7.12}$$

Remark 7.1. It is proved e.g. in [7] that if B_0, B and B_1 are Hilbert spaces such that $B_0 \subset B \subset B_1$, the embedding of B_0 into B is compact, the embedding of B into B_1 is continuous and $\gamma > 0$, then the injection of $\mathscr{H}^{\gamma}(0, T; B_0, B_1)$ into $L^2(0, T; B)$ is compact. The proof can be modified in such a way that this result remains valid also if B_0 is merely a reflexive Banach space.

It follows from (7.4) that there exist $U \in L^2(0, T; H^1(\Omega)^3)$ and a subsequence of $\{{}^mU\}$ (which will be denoted by $\{{}^mU\}$ again such that ${}^mU \rightharpoonup U$ in $L^2(0, T; {}^\circ H^1(\Omega)^3)$.

Similarly, it follows from (7.2) that there exist $\varrho \in L^\infty(0, T; H^1(\Omega_h))$ and a subsequence of $\{{}^m\varrho\}$ (which will be denoted by $\{{}^m\varrho\}$ again) such that $\{{}^m\varrho\}$ converges weakly-$*$ to ϱ in $L^\infty(0, T; H^1(\Omega_h))$. Denote $q = \max\{\varkappa; 2\}$. Since $H^1(\Omega_h)$ and $L^2(\Omega_h)$ are Hilbert spaces, $X_1 = \mathscr{H}^\gamma(0, T; H^1(\Omega_h), L^2(\Omega_h))$ is a Hilbert space, too. So, due to (7.12), there exists a subsequence of $\{{}^m\varrho\}$ (denoted by $\{{}^m\varrho\}$ again) weakly convergent to ϱ in X_1. The embedding of $H^1(\Omega_h)$ into $L^q(\Omega_h)$ is compact, hence also the embedding of X_1 into $L^2(0, T; L^q(\Omega_h))$ is compact. Thus, ${}^m\varrho \to \varrho$ in$L^2(0, T; L^q(\Omega_h))$.

The sequence $\{{}^m\varrho\, {}^mU\}$ is uniformly bounded in $X_2 = \mathscr{H}^\gamma(0, T; W^1_{3/2}(\Omega)^3, H^{-1}(\Omega)^3)$. X_2 is a reflxive Banach space, hence there exists a subsequence of $\{{}^m\varrho\, {}^mU\}$(which will be denoted by $\{{}^m\varrho\, {}^mU\}$ again) and $V \in X_2$ such that ${}^m\varrho\, {}^mU \rightharpoonup V$ in X_2. The injection of X_2 into $L^2(0, T; L^2(\Omega)^3) = L^2(Q_T)^3$ is compact, hence $\{{}^m\varrho\, {}^mU\}$ converges strongly to V in $L^2(Q_T)^3$.

It may be easily shown that ${}^m\varrho\, {}^mU \to \varrho U$ in $L^2(Q_T)^3$. It means that $V = \varrho U$ and so ${}^m\varrho\, {}^mU \to \varrho U$ in $L^2(Q_T)^3$.

The functions ${}^m\varrho, {}^mU$ satisfy (1.6) and (1.7) approximately with some errors (which will be denoted E_1 and E_2):

$$
\begin{aligned}
(8.1) \qquad & \textstyle\int_0^T \int_\Omega \{{}^m\varrho\, {}^mu_i\varphi_{i,t} + {}^m\varrho\, {}^m\tilde{u}_j {}^mu_i\varphi_{i,j} + \tilde{p}({}^m\varrho)\, \varphi_{i,i} - \\
& - \tfrac{1}{3}\mu\, {}^mu_{j,j}\varphi_{i,i} - \mu\, {}^mu_{i,j}\varphi_{i,j}\}\, \mathrm{d}x\, \mathrm{d}t = \\
& = -\textstyle\int_\Omega \varrho_0 u_{0i}(\varphi_i|_{t=0})\, \mathrm{d}x + E_1({}^m\varrho, {}^mU, \varphi)
\end{aligned}
$$

for all $\varphi \in C^\infty(\bar{Q}_T)^3$ such that $\varphi \equiv 0$ on $\partial\Omega \times \langle 0, T\rangle$, $\varphi|_{t=T} \equiv 0$, and

$$
\begin{aligned}
(8.2) \qquad & \textstyle\int_0^T \int_{\Omega_h} \{{}^m\varrho\psi_{,t} + {}^m\varrho\, {}^m\tilde{u}_j\psi_{,j}\}\, \mathrm{d}x\, \mathrm{d}t = \\
& = -\textstyle\int_{\Omega_h} \varrho_0(\psi|_{t=0})\, \mathrm{d}x + E_2({}^m\varrho, {}^mU, \psi)
\end{aligned}
$$

for all $\psi \in C^\infty(\bar{Q}_{T,h})$ such that $\psi|_{t=T} \equiv 0$.

If we use all types of convergences of $\{{}^m\varrho\}$ to ϱ, $\{{}^mU\}$ to U and $\{{}^m\varrho\, {}^mU\}$ to ϱU mentioned in this section we can prove that if φ, ψ are given functions with all the required properties then the left hand sides of (8.1) and (8.2) converge to the same expressions containing ϱ, U instead of ${}^m\varrho, {}^mU$ as $m \to +\infty$. In order to prove that ϱ, U satisfy (1.6) and (1.7), it is sufficient to show that if φ and ψ are given functions with the properties mentioned above then $E_1({}^m\varrho, {}^mU, \varphi) \to 0$ and $E_2({}^m\varrho, {}^mU, \psi) \to 0$ as $m \to +\infty$. Indeed, the functions E_1 and E_2 satisfy these conditions, although the complete proof is long and labourious from the technical point of view. However, its main idea is quite simple. For example, (8.1) can be written in the form

$$\sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_\Omega \{\varrho^{(k-1)} u_i^{(k)} \varphi_{i,t} + \varrho^{(k-1)} \tilde{u}_j^{(k)} u_i^{(k)} \varphi_{i,j} + \tag{8.3}$$
$$+ \tilde{p}(\varrho^{(k-1)}) \varphi_{i,i} - \tfrac{1}{3}\mu u_{j,j}^{(k)} \varphi_{i,i} - \mu u_{i,j}^{(k)} \varphi_{i,j}\} \,\mathrm{d}x\,\mathrm{d}t +$$
$$+ \int_\Omega \varrho_0 u_{0i} (\varphi_i|_{t=0}) \,\mathrm{d}x = E_1({}^m\varrho, {}^mU, \varphi) .$$

The left hand side of (8.3) may be decomposed into two parts – the first part has the form

$$\sum_{k=1}^{m-1} \int_\Omega \{\varrho^{(k-1)} u_i^{(k)} (\varphi_i|_{t=t_k}) - \varrho^{(k)} u_i^{(k+1)} (\varphi_i|_{t=t_k}) +$$
$$+ \tau \varrho^{(k-1)} \tilde{u}_j^{(k-1)} u_i^{(k)} (\varphi_{i,j}|_{t=t_k}) + \tau \tilde{p}(\varrho^{(k)}) (\varphi_{i,i}|_{t=t_k}) -$$
$$- \tfrac{1}{3}\mu u_{j,j}^{(k)} (\varphi_{i,i}|_{t=t_k}) - \mu u_i^{(k)} (\varphi_{i,j}|_{t=t_k})\} \,\mathrm{d}x\,\mathrm{d}t .$$

It is equal to zero due to $(2.3)_k$ $(k = 1, 2, \ldots, m - 1)$. The second part is of the type $O(\tau^{1/2})$ as $\tau \to 0+$, i.e. of the type $O(m^{-1/2})$ as $m \to +\infty$. It is necessary to use (6.13) and (6.14) in order to verify this fact. We believe that the reader interested in details can do it himself.

It is also obvious that U, ϱ satisfy (1.5). The condition $\varrho \geqq 0$ a.e. in $Q_{T,h}$ follows immediately from the inequalities $\varrho^{(k)} \geqq 0$ a.e. in Ω_h $(k = 0, 1, \ldots, m)$.

Thus, the following theorem holds:

Theorem 8.1. *Let $\varrho_0 \in H^1(\Omega_h)$, $\varrho_0 \geqq 0$ a.e. in Ω_h, $U_0 \equiv (u_{01}, u_{02}, u_{03}) \in L^2(\Omega)^3$, $\varrho_0 u_{0i} u_{0i} \in L^1(\Omega)$. Then there exists a solution U, ϱ of the problem given by (1.5), (1.6) and (1.7).*

9. THE ENERGY INEQUALITY

Let us now turn our attention to the inequality (7.3). If we put ${}^m\varrho(x, t) = \varrho^{(m)}(x)$ for $x \in \Omega_h$ and $t \in (T, T + \tau\rangle$, we write it in the form

$$\int_\Omega \tfrac{1}{2}\, {}^m\varrho\, {}^mu_i\, {}^mu_i|_{t=t'} \,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}({}^m\varrho)|_{t=t'+\tau} \,\mathrm{d}x + \tag{9.1}$$
$$+ \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu({}^mu_{j,j})^2 + \mu\, {}^mu_{i,j}\, {}^mu_{i,j}\} \,\mathrm{d}x\,\mathrm{d}t \leqq$$
$$\leqq \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i} \,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho_0) \,\mathrm{d}x .$$

Lemma 9.1. *Suppose that $M \subset \langle 0, T\rangle$, $\mu_1(M) > 0$ (where μ_1 denotes the one-dimensional Lebesgue measure). Then*

$$\int_M \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu(u_{j,j})^2 + \mu u_{i,j} u_{i,j}\} \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' \leqq \tag{9.2}$$
$$\leqq \limsup_{m \to +\infty} \int_M \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu({}^mu_{j,j})^2 + \mu\, {}^mu_{i,j}\, {}^mu_{i,j}\} \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' .$$

Proof. Assume that (9.2) does not hold. Then there exist $\Delta > 0$ and $m_0 \in N$ such that

$$\int_M \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu(u_{j,j})^2 + \mu u_{i,j} u_{i,j}\} \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}t' > \tag{9.3}$$
$$> \int_M \left(\int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu({}^mu_{j,j})^2 + \mu\, {}^mu_{i,j}\, {}^mu_{i,j}\} \,\mathrm{d}x\,\mathrm{d}t + \Delta\right) \mathrm{d}t'$$

for all $m \geqq m_0$. It means that if $m \in N$, $m \geqq m_0$ then there exists $M_m \subset M$, M_m measurable, so that

$$(9.4) \qquad \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu(u_{j,j})^2 + \mu u_{i,j} u_{i,j}\} \,\mathrm{d}x \,\mathrm{d}t > \\ > \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu({}^m u_{j,j})^2 + \mu\, {}^m u_{i,j}\, {}^m u_{i,j}\} \,\mathrm{d}x \,\mathrm{d}t + \tfrac{1}{2}\Delta$$

for all $t' \in M_m$ while the converse inequality holds for a.a. $t' \in M - M_m$. There are two possibilities:

(a) There exists $\delta > 0$ such that $\mu_1(M_m) > \delta$ for all $m \geqq m_0$,
or
(b) $\liminf_{m \to +\infty} \mu_1(M_m) = 0$.

We can easily get a contradiction with (9.3) in the case that (b) is valid. It is a consequence of (a) that there exists an increasing sequence of integers $\{m_k\}_{k=1}^{+\infty}$ such that $m_k \geqq m_0$ for all $k \in N$ and

$$\bigcap_{k=1}^{+\infty} M_{m_k} \neq \emptyset .$$

Let t' be an element of this intersection. Then (9.4) holds for this t' and all $m = m_k$ ($k = 1, 2, \ldots$). But this is not possible due to the weak convergence of $\{{}^m U\}$ to U in $L^2(0, T; {}^\circ H^1(\Omega)^3)$. Hence (9.2) holds.

Theorem 9.1. *Let all assumption of Theorem* 8.1 *be fulfilled. Then there exists a solution* U, ϱ *of the problem given by* (1.5), (1.6) *and* (1.7) *satisfying the inequality*

$$(9.5) \qquad \int_\Omega \tfrac{1}{2}\varrho u_i u_i \big|_{t=t'} \,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho)\big|_{t=t'} \,\mathrm{d}x + \\ + \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\mu(u_{j,j})^2 + \mu u_{i,j} u_{i,j}\} \,\mathrm{d}x \,\mathrm{d}t \leqq \\ \leqq \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i} \,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho_0) \,\mathrm{d}x$$

for a.a. $t' \in \langle 0, T\rangle$.

Proof. Let U, ϱ be the solution of (1.5), (1.6), (1.7) constructed in this and the preceding sections. Assume that (9.5) is not true. Then there exists $M \subset \langle 0, T\rangle$ such that $\mu_1(M) > 0$, and the converse inequality to (9.5) holds for all $t' \in M$. Thus, there exists $\delta > 0$ and $M_\delta \subset M$ such that $\mu_1(M_\delta) > \frac{1}{2}\,\mu_1(M)$ and

$$\int_\Omega \tfrac{1}{2}\varrho u_i u_i \big|_{t=t'} \,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho)\big|_{t=t'} \,\mathrm{d}x + \\ + \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\,\mu(u_{j,j})^2 + \mu u_{i,j} u_{i,j}\} \,\mathrm{d}x \,\mathrm{d}t > \\ > \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i} \,\mathrm{d}x + \int_{\Omega_h} \mathscr{P}(\varrho_0) \,\mathrm{d}x + \delta$$

for all $t' \in M_\delta$. Integrating over M_δ, we get

$$(9.6) \qquad \int_{M_\delta} \int_\Omega \tfrac{1}{2}\varrho u_i u_i \big|_{t=t'} \,\mathrm{d}x \,\mathrm{d}t' + \int_{M_\delta} \int_{\Omega_h} \mathscr{P}(\varrho)\big|_{t=t'} \,\mathrm{d}x \,\mathrm{d}t' + \\ + \int_{M_\delta} \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\,\mu(u_{j,j})^2 + \mu u_{i,j} u_{i,j}\} \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}t' > \\ > \mu_1(M_\delta) \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i} \,\mathrm{d}x + \mu_1(M_\delta) \int_{\Omega_h} \mathscr{P}(\varrho_0) \,\mathrm{d}x + \mu_1(M_\delta)\, \delta .$$

As a consequence of the strong convergence of $\{^m\varrho^mU\}$ to ϱU in $L^2(Q_T)^3$ and the weak convergence of $\{^mU\}$ to U in $L^2(0, T; {}^\circ H^1(\Omega)^3)$, we have

$$(9.7) \qquad \int_{M_\delta} \int_\Omega \tfrac{1}{2}\varrho u_i u_i\big|_{t=t'} \, \mathrm{d}x \, \mathrm{d}t' = \\ = \lim_{m\to+\infty} \int_{M_\delta} \int_\Omega \tfrac{1}{2}\, {}^m\varrho\, {}^mu_i\, {}^mu_i\big|_{t=t'} \, \mathrm{d}x \, \mathrm{d}t' .$$

Using a similar approach as in the proof of (3.13), we can also obtain

$$(9.8) \qquad \int_{M_\delta} \int_{\Omega_h} \mathscr{P}(\varrho)\big|_{t=t'} \, \mathrm{d}x \, \mathrm{d}t' = \lim_{m\to+\infty} \int_{M_\delta} \int_{\Omega_h} \mathscr{P}({}^m\varrho)\big|_{t=t'+T/m} \, \mathrm{d}x \, \mathrm{d}t' .$$

By virtue of Lemma 9.1, we get

$$(9.9) \qquad \int_{M_\delta} \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\,\mu(u_{j,j})^2 + \mu u_{i,j}u_{i,j}\} \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}t' \leqq \\ \leqq \limsup_{m\to+\infty} \int_{M_\delta} \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\,\mu({}^mu_{j,j})^2 + \mu\, {}^mu_{i,j}\, {}^mu_{i,j}\} \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}t' .$$

It follows from (9.6), (9.7), (9.8) and (9.9) that there exists infinitely many $m \in N$ such that

$$(9.10) \qquad \int_{M_\delta} \int_\Omega \tfrac{1}{2}\, {}^m\varrho\, {}^mu_i\, {}^mu_i\big|_{t=t'} \, \mathrm{d}x \, \mathrm{d}t' + \int_{M_\delta} \int_{\Omega_h} \mathscr{P}({}^m\varrho)\big|_{t=t'+T/m} \, \mathrm{d}x \, \mathrm{d}t' + \\ + \int_{M_\delta} \int_0^{t'} \int_\Omega \{\tfrac{1}{3}\,\mu({}^mu_{j,j})^2 + \mu\, {}^mu_{i,j}\, {}^mu_{i,j}\} \, \mathrm{d}x \, \mathrm{d}t \, \mathrm{d}t' > \\ > \mu_1(M_\delta) \int_\Omega \tfrac{1}{2}\varrho_0 u_{0i} u_{0i} \, \mathrm{d}x + \mu_1(M_\delta) \int_{\Omega_h} \mathscr{P}(\varrho_0) \, \mathrm{d}x + \tfrac{1}{2}\,\mu_1(M_\delta)\,\delta .$$

However, if we integrate (9.1) over M_δ, we obtain an inequality showing that (9.10) cannot be true. It is a contradiction with the assumption that (9.5) is false.

Remark 9.1. The inequality (9.5) may be called "the energy inequality" in accordance with an analogous inequality which is valid in the case of the viscous incompressible liquid.

Remark 9.2. The presence of a body force in the system (0.6) does not cause any difficulties if it is smooth enough. Hence also in this case results analogous to Theorems 8.1 and 9.1 could be derived.

Remark 9.3. We will discuss what can happen if the parameter h in the regularization $\sim$ (see Section 1) tends to zero in the paper [4], which is in preparation.

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Souhrn

GLOBÁLNÍ SLABÁ ŘEŠITELNOST REGULARIZOVANÉHO SYSTÉMU NAVIEROVÝCH-STOKESOVÝCH ROVNIC PRO STLAČITELNOU TEKUTINU

JIŘÍ NEUSTUPA

V práci je dokázána globální existence slabých řešení smíšené počáteční-okrajové úlohy pro jistou modifikaci systému pohybových rovnic vazké stlačitelné tekutiny. Modifikace spočívá v aplikaci regularizačního operátoru na některé členy vyskytující se v systému rovnic a není v rozporu se zákony mechaniky tekutin. Předpokládáme, že tlak je známou funkcí hustoty. Je použita metoda časové diskretizace a v závěru je odvozena tzv. energetická nerovnost. Tato nerovnost je nezávislá na použité regularizaci.

Резюме

ГЛОБАЛЬНАЯ СЛАБАЯ РАЗРЕШИМОСТЬ РЕГУЛЯРИЗИРОВАННОЙ СИСТЕМЫ УРАВНЕНИЙ НАВЬЕ-СТОКСА ДЛЯ СЖИМАЕМОЙ ЖИДКОСТИ

JIŘÍ NEUSTUPA

В работе доказано глобальное существование слабых решений смешанной начально-краевой задачи для определенной модификации системы уравнений движения вязкой сжимаемой жидкости. Модификация состоит в применении оператора регуляризации к некоторым членам системы уравнений и не противоречит законам механики жидкостей. Предполагается, что давление является известной функцией плотности. Применяется метод дискретизации по времени и в заключении выведено так называемое энергетическое неравенство, которое оказывается независимым от использованной регуляризации.

Author's address: Doc. RNDr. *Jiří Neustupa*, CSc., strojní fakulta ČVUT - K 205, Suchbátarova 4, 166 07 Praha 6.