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BAYES UNBIASED ESTIMATION IN A MODEL WITH THREE VARIANCE COMPONENTS

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Summary. In the paper necessary and sufficient conditions for the existence and an explicit expression for the Bayes invariant quadratic unbiased estimate of the linear function of the variance components are presented for the mixed linear model $\mathbf{t} = \boldsymbol{X\beta} + \varepsilon$, $\mathrm{E}(\mathbf{t}) = \boldsymbol{X\beta}$, $\mathrm{Var}(\mathbf{t}) = \theta_1 \boldsymbol{U}_1 + \theta_2 \boldsymbol{U}_2 + \theta_3 \boldsymbol{U}_3$, with three unknown variance components in the normal case.
An application to some examples from the analysis of variance is given.

Keywords: mixed linear model, Bayes estimate, variance components.

AMS classification: 62J99.

1. INTRODUCTION

In the paper the following linear model is considered

$$(1) \qquad \mathbf{t} = \boldsymbol{X\beta} + \varepsilon\,, \quad \boldsymbol{E}(\mathbf{t}) = \boldsymbol{X\beta}\,, \quad \mathrm{Var}(\mathbf{t}) = \theta_1 \boldsymbol{U}_1 + \theta_2 \boldsymbol{U}_2 + \theta_3 \boldsymbol{U}_3 = \boldsymbol{U}(\boldsymbol{\theta})\,,$$

where $\mathbf{t}$ is an N-dimensional, normally distributed random vector, $\boldsymbol{X}$ is a known $N \times m$ matrix of rank $\mathrm{R}(\boldsymbol{X}) = p$, $\boldsymbol{\beta} \in \mathrm{R}^m$ is a vector of unknown parameters, $\boldsymbol{U}_1, \boldsymbol{U}_2$ are known, positive semidefinite matrices, $\boldsymbol{U}_3 = \boldsymbol{I}_N$, and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)'$ is a vector of unknown variance components $\boldsymbol{\theta} \in \mathscr{T}$, $\mathscr{T} = \{\boldsymbol{\theta}: \theta_1 > 0,\ \theta_2 \geqq 0,\ \theta_3 \geqq 0\}$.

The problem is to estimate a linear function $\gamma = f_1\theta_1 + f_2\theta_2 + f_3\theta_3$ in the class of quadratic forms $\hat{\gamma}(\mathbf{t}) = \mathbf{t}'\mathbf{B}\mathbf{t}$, where $\mathbf{B} \in \mathscr{S}_N$, $\mathscr{S}_N$ is the class of all symmetric $N \times N$ matrices. We restrict our considerations to quadratic estimates $\hat{\gamma} = \mathbf{t}'\mathbf{B}\mathbf{t}$, which are invariant with respect to translations $\mathbf{t} \to \mathbf{t} + \boldsymbol{X\beta}$, i.e. $\hat{\gamma}(\mathbf{t}) = \hat{\gamma}(\mathbf{t} + \boldsymbol{X\beta})$ for all $\boldsymbol{\beta} \in \mathrm{R}^m$, unbiased, and minimize the Bayes risk function

$$\mathrm{r}(\hat{\gamma}) = \tfrac{1}{2} \int \mathrm{E}_\theta(\hat{\gamma} - \gamma)^2 \,\mathrm{dP}_\theta\,,$$

where P_θ is a priori distribution for the vector parameter $\boldsymbol{\theta}$ having the second order moments, i.e.

$$\mathrm{E}(\theta_i\theta_j) = \int \theta_i\theta_j \,\mathrm{dP}_\theta = c_{ij} \geqq 0\,, \quad i, j = 1, 2, 3\,.$$

We use the approach by Olsen, Seely and Birkes [3] to derive explicit expressions for Bayes invariant quadratic unbiased estimates (BAIQUE). These estimates have been introduced by Kleffe and Pincus [2]. Explicit easily computable expressions for BAIQUE in a model with two variance components have been given by Gnot and Kleffe [1] and Stuchlý [6].

2. A CANONICAL MODEL

Let $\mathbf{P}$ be an $(N - p) \times N$ matrix satisfying $\mathbf{P}'\mathbf{P} = \mathbf{I} - \mathbf{X}\mathbf{X}^+$, $\mathbf{P}\mathbf{P}' = \mathbf{I}_n$, $n = N - p$. Then the random vector $\mathbf{y} = \mathbf{P}\mathbf{t}$ satisfies the simplified canonical model

$$\mathbf{y} = \mathbf{P}\mathbf{t}\,, \quad \mathrm{E}(\mathbf{y}) = \mathbf{0}\,, \quad \mathrm{Var}\,(\mathbf{y}) = \theta_1\mathbf{V}_1 + \theta_2\mathbf{V}_2 + \theta_3\mathbf{V}_3 = \mathbf{V}(\boldsymbol{\theta})\,, \tag{2}$$

where $\mathbf{V}_1 = \mathbf{P}\mathbf{U}_1\mathbf{P}'$, $\mathbf{V}_2 = \mathbf{P}\mathbf{U}_2\mathbf{P}'$, $\mathbf{V}_3 = \mathbf{P}\mathbf{U}_3\mathbf{P}' = \mathbf{I}_n$. The vector $\mathbf{y}$ is a maximal invariant statistic and $\hat{\gamma} = \mathbf{t}'\mathbf{B}\mathbf{t}$ is BAIQUE for γ in the model (1) iff $\mathbf{B} = \mathbf{P}'\mathbf{A}\mathbf{P}$ and $\hat{\gamma} = \mathbf{y}'\mathbf{A}\mathbf{y}$ is Bayes quadratic unbiased estimate (BAQUE) for γ under the model (2).

3. SOLUTION

According to Theorem 7a in Kleffe and Pincus [2] $\mathbf{y}'\mathbf{A}\mathbf{y}$ is BAQUE for $f_1\theta_1 + f_2\theta_2 + f_3\theta_3$ under the model (2) iff

$$\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}\mathbf{V}_i\mathbf{A}\mathbf{V}_j = \sum_{i=1}^{3}\lambda_i\mathbf{V}_i\,, \tag{3}$$

where $c_{ij} = \mathrm{E}(\theta_i\theta_j)$, $i, j = 1, 2, 3$ and $\lambda_1, \lambda_2, \lambda_3$ satisfy the unbasedness conditions

$$\mathrm{tr}(\mathbf{A}\mathbf{V}_i) = f_i\,, \quad i = 1, 2, 3\,. \tag{4}$$

Let $c_{11} = \mathrm{E}(\theta_1^2) \neq 0$. Since the Bayes risk function $\mathrm{r}(\hat{\gamma})$ is linear in $\mathbf{C} = (c_{ij}) = \mathrm{E}(\boldsymbol{\theta}\boldsymbol{\theta}')$ we can put $c_{11} = 1$ without loss of generality. Under the notation $c_{12} = u \geqq 0$, $c_{13} = k \geqq 0$, $c_{22} = u^2 + v^2$, $c_{23} = ku + vl \geqq 0$, $c_{33} = k^2 + l^2 + m^2$ the matrix $\mathbf{C}$ takes the form

$$\mathbf{C} = \begin{pmatrix} 1 & u & k \\ u & u^2 + v^2 & ku + vl \\ k & ku + vl & k^2 + l^2 + m^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ u & v & 0 \\ k & l & m \end{pmatrix} \begin{pmatrix} 1 & u & k \\ 0 & v & l \\ 0 & 0 & m \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} 1 \\ u \\ k \end{pmatrix} (1 \;\; u \;\; k) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & v^2 & vl \\ 0 & vl & l^2 + m^2 \end{pmatrix}$$

and is positive semidefinite for every real numbers u, k, v, l, m. It follows that the class of a priori distributions with $\theta_1 = 1$ fixed and

$$\mathrm{E}\begin{pmatrix}\theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}u\\ k\end{pmatrix}, \quad \mathrm{var}\begin{pmatrix}\theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}v^2 & vl\\ v & l^2 + m^2\end{pmatrix}$$

forms an essentially complete class for our problem (cf. [7], Chap. 8).

We shall use the notation

$$\mathbf{W} = \mathbf{V}_1 + u\mathbf{V}_2 + k\mathbf{I}\,, \quad \mathbf{V} = v\mathbf{V}_2 + l\mathbf{I}\,. \tag{6}$$

Then the equation (3) has the form

$$\mathbf{WAW} + \mathbf{VAV} + m^2\mathbf{A} = \lambda_1\mathbf{V}_1 + \lambda_2\mathbf{V}_2 + \lambda_3\mathbf{I}\,. \tag{7}$$

First let us prove some lemmas.

Lemma 1. *Let* $\mathbf{A}, \mathbf{B}$ *be positive definite (p.d.) matrices of order* n. *Then the Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ *is a p.d. matrix of order* n^2.

Proof. Since the matrices $\mathbf{A}$, $\mathbf{B}$ are p.d., there exists orthogonal matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{P}'\mathbf{AP} = \Lambda$, $\mathbf{Q}'\mathbf{BQ} = \mathbf{D}$, where the matrices Λ and $\mathbf{D}$ are diagonal with positive diagonal elements (cf. Theorem I, § lc. 3 in [4]). Then $(\mathbf{P} \otimes \mathbf{Q})(\mathbf{P}' \otimes \mathbf{Q}') = \mathbf{PP}' \otimes \mathbf{QQ}' = \mathbf{I}_n \otimes \mathbf{I}_n = \mathbf{I}_{n^2}$ holds. Hence $\mathbf{x}'(\mathbf{A} \otimes \mathbf{B})\,\mathbf{x} = \mathbf{x}'(\mathbf{P} \otimes \mathbf{Q})(\mathbf{P}' \otimes \mathbf{Q}')\,.\,(\mathbf{A} \otimes \mathbf{B})(\mathbf{P} \otimes \mathbf{Q})(\mathbf{P}' \otimes \mathbf{Q}')\,\mathbf{x} = [(\mathbf{P}' \otimes \mathbf{Q}')\,\mathbf{x}]'(\mathbf{P}'\mathbf{AP} \otimes \mathbf{Q}'\mathbf{BQ})(\mathbf{P}' \otimes \mathbf{Q}')\,\mathbf{x} = \mathbf{y}'(\Lambda \otimes \mathbf{D})\,\mathbf{y} > 0$ for all $\mathbf{x} \neq 0$.

Lemma 2. *The equation* (7) *is consistent (for all* λ_3 *real) iff the matrix* $\mathbf{W}$ *is regular.*

Proof. Using the operation vec, we get (7) in the form

$$[\mathbf{W} \otimes \mathbf{W} + \mathbf{V} \otimes \mathbf{V} + m^2(\mathbf{I} \otimes \mathbf{I})]\,\mathrm{vec}\,\mathbf{A} = \lambda_1\,\mathrm{vec}\,\mathbf{V}_1 + \lambda_2\,\mathrm{vec}\,\mathbf{V}_2 + \lambda_3\,\mathrm{vec}\,\mathbf{I}\,. \tag{8}$$

Sufficiency follows obvisouly from Lemma 1. Now we shall prove the necessity. Let $\mathbf{W}$ be singular and assume that the equation (8) is consistent. Since $\mathbf{W}$ has the form (6) we get $k = \mathrm{E}(\theta_1\theta_3) = 0$. Using the conditions $\theta_1 > 0$ $(c_{11} \neq 0)$ and $\theta_3 \geqq 0$, we have $\mathrm{E}(\theta_3) = 0$ and $\mathrm{D}(\theta_3) = \mathrm{E}(\theta_3^2) = l^2 + m^2 = 0$, which implies $l = m = 0$. Now the equation (8) has the form

$$[(\mathbf{W} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{W}) + (\mathbf{V} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{V})]\,\mathrm{vec}\,\mathbf{A} = \lambda_1\,\mathrm{vec}\,\mathbf{V}_1 + \lambda_2\,\mathrm{vec}\,\mathbf{V}_2 + \lambda_3\,\mathrm{vec}\,\mathbf{I}\,, \tag{9}$$

where $\mathbf{W} = \mathbf{V}_1 + u\mathbf{V}_2$, $\mathbf{V} = v\mathbf{V}_2$.

If $u \neq 0$ then $\mathscr{M}(\mathbf{V}_1) \subset \mathscr{M}(\mathbf{W})$, $\mathscr{M}(\mathbf{V}_2) \subset \mathscr{M}(\mathbf{W})$, $\mathscr{M}(\mathbf{V}) \subset \mathscr{M}(\mathbf{W})$ ($\mathscr{M}(\mathbf{W})$ denotes the vector space generated by the columns of $\mathbf{W}$) and there exist matrices $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}$

such that $\mathbf{V}_1 = \mathbf{W}\mathbf{Q}_1$, $\mathbf{V}_2 = \mathbf{W}\mathbf{Q}_2$, $\mathbf{V} = \mathbf{W}\mathbf{Q}$, i.e. $\operatorname{vec}\mathbf{V}_1 = (\mathbf{W}\otimes\mathbf{I})\operatorname{vec}\mathbf{Q}_1'$, $\operatorname{vec}\mathbf{V}_2 = (\mathbf{W}\otimes\mathbf{I})\operatorname{vec}\mathbf{Q}_2'$, $\mathbf{V}\otimes\mathbf{I} = (\mathbf{W}\otimes\mathbf{I})(\mathbf{Q}\otimes\mathbf{I})$. Under the notation $\operatorname{vec}\mathbf{A}_w = (\mathbf{I}\otimes\mathbf{W})\operatorname{vec}\mathbf{A}$, $\operatorname{vec}\mathbf{A}_v = (\mathbf{Q}\otimes\mathbf{I})(\mathbf{I}\otimes\mathbf{V})\operatorname{vec}\mathbf{A}$ the equation (9) takes the form $\lambda_3 \operatorname{vec}\mathbf{I} = (\mathbf{W}\otimes\mathbf{I})(\operatorname{vec}\mathbf{A}_w + \operatorname{vec}\mathbf{A}_v - \lambda_1\operatorname{vec}\mathbf{Q}_1' - \lambda_2\operatorname{vec}\mathbf{Q}_2')$. This equation is consistent for all λ_3 real iff there exists a matrix $\mathbf{Z}$ such that $\operatorname{vec}\mathbf{I} = (\mathbf{W}\otimes\mathbf{I})\operatorname{vec}\mathbf{Z} \Leftrightarrow \mathbf{I} = \mathbf{W}\mathbf{Z} \Leftrightarrow \mathcal{M}(\mathbf{I}) \subset \mathcal{M}(\mathbf{W})$, and this is a contradiction with singularity of $\mathbf{W}$. If $u = 0$ then $v = 0$ and $\mathbf{W} = \mathbf{V}_1$, $\mathbf{V} = \mathbf{0}$. So (9) has the form $(\mathbf{W}\otimes\mathbf{W})\operatorname{vec}\mathbf{A} = \lambda_1\operatorname{vec}\mathbf{V}_1 + \lambda_2\operatorname{vec}\mathbf{V}_2 + \lambda_3\operatorname{vec}\mathbf{I}$ and, similarly as above, we get a contradiction.

Theorem 1. *Let $\mathbf{V}_3 = \mathbf{I}$ and let matrix $\mathbf{W} = \mathbf{V}_1 + u\mathbf{V}_2 + k\mathbf{V}_3$ be regular.*

a) *The BAQUE for the parametric function $\gamma = f_1\theta_1 + f_2\theta_2 + f_3\theta_3$ in the model* (2) *exists iff*

$$\mathbf{f} \in \mathcal{M}(\mathbf{S}\mathbf{H}^{-1}\mathbf{S}')\,, \tag{10}$$

where

$$\mathbf{S} = (\operatorname{vec}\mathbf{V}_1,\ \operatorname{vec}\mathbf{V}_2,\ \operatorname{vec}\mathbf{V}_3)'\,,\quad \mathbf{f} = (f_1, f_2, f_3)'\,,$$
$$\mathbf{H} = (\mathbf{W}\otimes\mathbf{W}) + (\mathbf{V}\otimes\mathbf{V}) + m^2(\mathbf{V}_3\otimes\mathbf{V}_3)\,,\quad \mathbf{V} = v\mathbf{V}_2 + l\mathbf{V}_3\,.$$

b) *The BAQUE from* a) *is uniquely given by*

$$\hat{\gamma} = \mathbf{y}'\mathbf{A}\mathbf{y}\,,\quad \operatorname{vec}\mathbf{A} = \mathbf{H}^{-1}\mathbf{S}'(\mathbf{S}\mathbf{H}^{-1}\mathbf{S}')^{+}\,\mathbf{f}\,. \tag{11}$$

(Here the symbol $\mathbf{Z}^+$ stands for the Moore-Penrose inverse of the matrix $\mathbf{Z}$.)

Proof. a) We get the BAQUE as a solution of the equations (7) and (4). Since $\mathbf{W}$ is regular, (7) is consistent by Lemma 2. Using the notation as above, we transform these equation to the form

$$\mathbf{H}\operatorname{vec}\mathbf{A} = \mathbf{S}'\boldsymbol{\lambda}\,,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)'$, and

$$\mathbf{S}\operatorname{vec}\mathbf{A} = \mathbf{f}\,,$$

respectively. From the regularity of $\mathbf{W}$ the regularity of $\mathbf{H}$ follows, and therefore

$$\operatorname{vec}\mathbf{A} = \mathbf{H}^{-1}\mathbf{S}'\boldsymbol{\lambda}$$

and

$$\mathbf{S}\mathbf{H}^{-1}\mathbf{S}'\boldsymbol{\lambda} = \mathbf{f}\,.$$

The last equation is consistent iff (10) holds.

b) A solution of the last equation is (cf. Theorem IId, § 16.5 in [4])

$$\boldsymbol{\lambda} = (\mathbf{S}\mathbf{H}^{-1}\mathbf{S}')^{-}\,\mathbf{f} + [\mathbf{I} - (\mathbf{S}\mathbf{H}^{-1}\mathbf{S}')^{-}\,\mathbf{S}\mathbf{H}^{-1}\mathbf{S}']\,\mathbf{x}\,,$$

where $\boldsymbol{x}$ is an arbitrary vector. So

$$\operatorname{vec} \boldsymbol{A} = \boldsymbol{H}^{-1}\boldsymbol{S}'(\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')^{-} \boldsymbol{f} + \boldsymbol{H}^{-1}\boldsymbol{S}' \left[\boldsymbol{I} - (\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')^{-} \boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}'\right] \boldsymbol{x} = $$
$$= \boldsymbol{H}^{-1}\boldsymbol{S}'(\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')^{-} \boldsymbol{f},$$

since $\mathscr{M}(\boldsymbol{S}\boldsymbol{H}^{-1}) \subset \mathscr{M}(\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')$ (cf. Th. VIb, § 1b.5 in [4]). Using the condition (10), we see that the expression $\boldsymbol{H}^{-1}\boldsymbol{S}'(\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')^{-} \boldsymbol{f}$ is invariant for any choice of the g-inverse (cf. Th. VIc, § 1b.5 in [4]), and we can use the Moore-Penrose inverse. Hence the BAQUE is uniquely given by (11).

Remark 1. If the matrix $\boldsymbol{W}$ is singular then $k = 0$ and also $l = m = 0$. Therefore $\mathrm{P}(\theta_3 = 0) = 1$ and, instead of the model (2), we can use the model with two variance components considered in [6].

Remark 2. If the matrices $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ as elements of the linear space $\mathscr{S}_n$ are linearly independent then the matrix $\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}'$ is regular. In this case we can use in (11) the usual inverse instead of the Moore-Penrose inverse. Alternatively, the case when the matrices $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ are not linearly independent can be solved by reducing the number of the variance components.

Now we rewrite Theorem 1 for the model (1) as follows.

Corollary 1. *Let* $\boldsymbol{U}_3 = \boldsymbol{I}_N$, $\boldsymbol{Z} = \boldsymbol{U}_1 + u\boldsymbol{U}_2 + k\boldsymbol{U}_3$, $\boldsymbol{M} = \boldsymbol{P}'\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{X}\boldsymbol{X}^{+}$, $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{I}$ *and rank* $\mathrm{r}(\boldsymbol{M}\boldsymbol{Z}\boldsymbol{M}) = \mathrm{N} - \mathrm{r}(\boldsymbol{X})$.

a) *The BAIQUE for the parametric function* $\gamma = f_1\theta_1 + f_2\theta_2 + f_3\theta_3$ *in the model* (1) *exists iff*

$$\boldsymbol{f} \in \mathscr{M}(\boldsymbol{R}(\boldsymbol{M}^*\boldsymbol{G}\boldsymbol{M}^*)^{+} \boldsymbol{R}'),$$

where

$$\boldsymbol{R} = (\operatorname{vec} \boldsymbol{U}_1, \operatorname{vec} \boldsymbol{U}_2, \operatorname{vec} \boldsymbol{U}_3)', \quad \boldsymbol{f} = (f_1, f_2, f_3)', \quad \boldsymbol{M}^* = \boldsymbol{M} \otimes \boldsymbol{M},$$
$$\boldsymbol{G} = \boldsymbol{Z} \otimes \boldsymbol{Z} + \boldsymbol{U} \otimes \boldsymbol{U} + m^2(\boldsymbol{U}_3 \otimes \boldsymbol{U}_3), \quad \boldsymbol{U} = v\boldsymbol{U}_2 + l\boldsymbol{U}_3.$$

b) *The BAIQUE from* a) *is uniquely given by*

$$\hat{\gamma} = \boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}, \quad \operatorname{vec} \boldsymbol{B} = (\boldsymbol{M}^*\boldsymbol{G}\boldsymbol{M}^*)^{+} \boldsymbol{R}'[\boldsymbol{R}(\boldsymbol{M}^*\boldsymbol{G}\boldsymbol{M}^*)^{+} \boldsymbol{R}']^{+} \boldsymbol{f}.$$

Proof. Since the BAQUE $\hat{\gamma} = \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}$ in the model (2) is simultaneously the BAIQUE $\hat{\gamma} = \boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}$ in the model (1) and $\boldsymbol{B} = \boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}$, we can write $\operatorname{vec} \boldsymbol{B} = (\boldsymbol{P}' \otimes \boldsymbol{P}') \operatorname{vec} \boldsymbol{A}$. Substituting $\boldsymbol{V}_i = \boldsymbol{P}\boldsymbol{U}_i\boldsymbol{P}'$, $i = 1, 2, 3$, in the expressions for $\operatorname{vec} \boldsymbol{A}$, we get the BAIQUE expressed in terms of the original model (1)

$$\hat{\gamma} = \boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}, \quad \operatorname{vec} \boldsymbol{B} = (\boldsymbol{P}' \otimes \boldsymbol{P}')\, \boldsymbol{H}^{-1}\boldsymbol{S}'(\boldsymbol{S}\boldsymbol{H}^{-1}\boldsymbol{S}')^{+} \boldsymbol{f},$$

where

$$\boldsymbol{S} = (\operatorname{vec}(\boldsymbol{P}\boldsymbol{U}_1\boldsymbol{P}'), \quad \operatorname{vec}(\boldsymbol{P}\boldsymbol{U}_2\boldsymbol{P}'), \quad \operatorname{vec}(\boldsymbol{P}\boldsymbol{U}_3\boldsymbol{P}'))',$$

$$\mathbf{H} = (\mathbf{PZP}') \otimes (\mathbf{PZP}') + (\mathbf{PUP}') \otimes (\mathbf{PUP}') + m^2(\mathbf{PU}_3\mathbf{P}') \otimes (\mathbf{PU}_3\mathbf{P}'),$$
$$\mathbf{Z} = \mathbf{U}_1 + u\mathbf{U}_2 + k\mathbf{U}_3, \quad \mathbf{U} = v\mathbf{U}_2 + l\mathbf{U}_3, \quad \mathbf{U}_3 = \mathbf{I}_N.$$

Since for every matrix $\mathbf{B} \in \mathscr{S}_N$ we have

$$\mathbf{P}'(\mathbf{PBP}')^+ \mathbf{P} = (\mathbf{MBM})^+ \tag{12}$$

we obtain

$$\begin{aligned}\operatorname{vec} \mathbf{B} &= (\mathbf{P}' \otimes \mathbf{P}') [(\mathbf{P} \otimes \mathbf{P}) \mathbf{G}(\mathbf{P}' \otimes \mathbf{P}')]^{-1} (\mathbf{P} \otimes \mathbf{P}) \mathbf{R}'\{\mathbf{R}(\mathbf{P}' \otimes \mathbf{P}') \times \\ &\times [(\mathbf{P} \otimes \mathbf{P}) \mathbf{G}(\mathbf{P}' \otimes \mathbf{P}')]^{-1} (\mathbf{P} \otimes \mathbf{P}) \mathbf{R}'\}^+ \mathbf{f} = \\ &= [(\mathbf{M} \otimes \mathbf{M}) \mathbf{G}(\mathbf{M} \otimes \mathbf{M})]^+ \mathbf{R}'\{\mathbf{R}[(\mathbf{M} \otimes \mathbf{M}) \mathbf{G}(\mathbf{M} \otimes \mathbf{M})]^+ \mathbf{R}'\}^+ \mathbf{f} = \\ &= (\mathbf{M}^*\mathbf{G}\mathbf{M}^*)^+ \mathbf{R}'[\mathbf{R}(\mathbf{M}^*\mathbf{G}\mathbf{M}^*)^+ \mathbf{R}']^+ \mathbf{f},\end{aligned}$$

where $\mathbf{G} = \mathbf{Z} \otimes \mathbf{Z} + \mathbf{U} \otimes \mathbf{U} + m^2(\mathbf{U}_3 \otimes \mathbf{U}_3)$, $\mathbf{R} = (\operatorname{vec} \mathbf{U}_1, \operatorname{vec} \mathbf{U}_2, \operatorname{vec} \mathbf{U}_3)'$. For $\mathbf{W}$ regular we have $\mathrm{r}(\mathbf{MZM}) = \mathrm{r}(\mathbf{P}'\mathbf{PZP}'\mathbf{P}) = \mathrm{r}(\mathbf{P}'\mathbf{WP}) = \mathrm{N} - \mathrm{r}(\mathbf{X})$.

In the special case that the matrices $\mathbf{V}_1, \mathbf{V}_2$ are commutative we shall derive more explicit expressions for the BAQUE.

Theorem 2. *Let* $\mathbf{V}_1\mathbf{V}_2 = \mathbf{V}_2\mathbf{V}_1$, $\mathbf{V}_3 = \mathbf{I}$ *and let the matrix* $\mathbf{W} = \mathbf{V}_1 + \mathrm{u}\mathbf{V}_2 + \mathrm{k}\mathbf{V}$ *be regular.*

a) *The BAQUE for the parametric function* $\gamma = \mathbf{f}'\theta$ *in the model* (2) *exists iff*

$$\mathbf{f} \in \mathscr{M}(\mathbf{Q}), \tag{13}$$

where

$$\mathbf{f} = (f_1, f_2, f_3)', \quad \theta = (\theta_1, \theta_2, \theta_3)',$$

$$\mathbf{Q} = \begin{pmatrix} \operatorname{tr}(\mathbf{M}_1\mathbf{V}_1), & \operatorname{tr}(\mathbf{M}_2\mathbf{V}_1), & \operatorname{tr}(\mathbf{M}_3\mathbf{V}_1) \\ \operatorname{tr}(\mathbf{M}_1\mathbf{V}_2), & \operatorname{tr}(\mathbf{M}_2\mathbf{V}_2), & \operatorname{tr}(\mathbf{M}_3\mathbf{V}_2) \\ \operatorname{tr}(\mathbf{M}_1), & \operatorname{tr}(\mathbf{M}_2), & \operatorname{tr}(\mathbf{M}_3) \end{pmatrix},$$

$$\mathbf{M}_i = \mathbf{V}_i(\mathbf{W}^2 + \mathbf{V}^2 + m^2\mathbf{I})^{-1}, \quad i = 1, 2, 3, \tag{14}$$
$$\mathbf{V} = v\mathbf{V}_2 + l\mathbf{V}_3.$$

b) *The BAQUE is uniquely given by*

$$\hat{\gamma} = \mathbf{y}'\mathbf{A}\mathbf{y} = \lambda_1\mathbf{y}'\mathbf{M}_1\mathbf{y} + \lambda_2\mathbf{y}'\mathbf{M}_2\mathbf{y} + \lambda_3\mathbf{y}'\mathbf{M}_3\mathbf{y}, \tag{15}$$

where $\boldsymbol{\lambda}$ *satisfies the unbiasedness condition*

$$\mathbf{Q}\boldsymbol{\lambda} = \mathbf{f}. \tag{16}$$

Proof. We suppose that $\mathbf{W}$ is regular. By Lemma 2 the equation (7) is consistent. First we shall find its solution.

For $v \neq 0$ we get from (6)

$$\mathbf{V}_1 = \mathbf{W} - \frac{u}{v}\mathbf{V} - \frac{kv - ul}{v}\mathbf{I}\,, \quad \mathbf{V}_2 = \frac{1}{v}\mathbf{V} - \frac{l}{v}\mathbf{I}\,.$$

By Theorem II, § lc. 3 in [4], there exists an orthogonal matrix $\mathbf{C}$ and diagonal matrices Λ and $\mathbf{D}$ with non negative diagonal elements such that

$$\mathbf{C}'\mathbf{W}\mathbf{C} = \Lambda\,, \quad \text{i.e.} \quad \mathbf{W} = \mathbf{C}\Lambda\mathbf{C}'\,,$$

$$\mathbf{C}'\mathbf{V}\mathbf{C} = \mathbf{D}\,, \quad \text{i.e.} \quad \mathbf{V} = \mathbf{C}\mathbf{D}\mathbf{C}'\,.$$

Therefore the equation (7) has the form

$$\mathbf{C}\Lambda\mathbf{C}'\mathbf{A}\mathbf{C}\Lambda\mathbf{C}' + \mathbf{C}\mathbf{D}\mathbf{C}'\mathbf{A}\mathbf{C}\mathbf{D}\mathbf{C}' + m^2\mathbf{A} =$$
$$= \lambda_1\left(\mathbf{W} - \frac{u}{v}\mathbf{V} - \frac{kv - ul}{v}\mathbf{I}\right) + \lambda_2\left(\frac{1}{v}\mathbf{V} - \frac{l}{v}\mathbf{I}\right) + \lambda_3\mathbf{I}\,.$$

Multiplying this equation by $\mathbf{C}'$ from the left and by $\mathbf{C}$ from the right, we get

$$\Lambda\mathbf{C}'\mathbf{A}\mathbf{C}\Lambda + \mathbf{D}\mathbf{C}'\mathbf{A}\mathbf{C}\mathbf{D} + m^2\mathbf{C}'\mathbf{A}\mathbf{C} =$$
$$= \lambda_1\left(\Lambda - \frac{u}{v}\mathbf{D} - \frac{kv - ul}{v}\mathbf{I}\right) + \lambda_2\left(\frac{1}{v}\mathbf{D} - \frac{l}{v}\mathbf{I}\right) + \lambda_3\mathbf{I}\,.$$

Puting $\mathbf{Z} = \mathbf{C}'\mathbf{A}\mathbf{C}$ we obtain

$$\Lambda\mathbf{Z}\Lambda + \mathbf{D}\mathbf{Z}\mathbf{D} + m^2\mathbf{Z} = \lambda_1\left(\Lambda - \frac{u}{v}\mathbf{D} - \frac{kv - ul}{v}\mathbf{I}\right) + \lambda_2\left(\frac{1}{v}\mathbf{D} - \frac{l}{v}\mathbf{I}\right) + \lambda_3\mathbf{I}\,.$$

Since the matrices on the right hand side are diagonal, the matrix $\mathbf{Z}$ must be so, too. Therefore

$$\mathbf{A} = \left[\lambda_1\left(\mathbf{W} - \frac{u}{v}\mathbf{V} - \frac{kv - ul}{v}\mathbf{I}\right) + \lambda_2\left(\frac{1}{v}\mathbf{V} - \frac{l}{v}\mathbf{I}\right) + \right.$$
$$\left. + \lambda_3\mathbf{I}\right](\mathbf{W}^2 + \mathbf{V}^2 + m^2\mathbf{I})^{-1} =$$
$$= (\lambda_1\mathbf{V}_1 + \lambda_2\mathbf{V}_2 + \lambda_3\mathbf{I})(\mathbf{W}^2 + \mathbf{V}^2 + m^2\mathbf{I})^{-1}$$

and the BAQUE has the form (14)–(15).

If $v = 0$ we must apply Theorem II, § lc. 3 in [4] to the matrices $\mathbf{W}$ and $\mathbf{V}_3$. The equation (7) has the form

$$\Lambda\mathbf{C}'\mathbf{A}\mathbf{C}\Lambda + (l^2 + m^2)\,\mathbf{C}'\mathbf{A}\mathbf{C} = \lambda_1(\Lambda - u\mathbf{D} - k\mathbf{I}) + \lambda_2\mathbf{D} + \lambda_3\mathbf{I}\,.$$

Hence we get, in the same way,

$$\mathbf{A} = (\lambda_1\mathbf{V}_1 + \lambda_2\mathbf{V}_2 + \lambda_3\mathbf{V}_3)\left[\mathbf{W}^2 + (l^2 + m^2)\,\mathbf{I}\right]^{-1},$$

i.e. (14)–(15) are again valid.

Here $\lambda_1, \lambda_2, \lambda_3$ satisfy the unbiasedness condition (16), which has a solution iff (13) holds.

To prove uniqueness, let us put $\mathbf{m} = (\mathbf{y}'\mathbf{M}_1\mathbf{y}, \mathbf{y}'\mathbf{M}_2\mathbf{y}, \mathbf{y}'\mathbf{M}_3\mathbf{y})'$, $\mathbf{M}_i = \mathbf{V}_i\mathbf{K}^{-1}$, $i = 1, 2, 3$ and $\mathbf{K} = \mathbf{W}^2 + \mathbf{V}^2 + m^2\mathbf{I}$. The BAQUE is $\hat{\gamma} = \mathbf{m}'\boldsymbol{\lambda} = \mathbf{m}'[\mathbf{Q}^-\mathbf{f} - (\mathbf{I} - \mathbf{Q}^-\mathbf{Q})\,\mathbf{x}]$, where $\mathbf{x}$ is an arbitrary vector. We have

$$\mathbf{m}' = \left(\mathrm{tr}(\mathbf{K}^{-1/2}\mathbf{y}\mathbf{y}'\mathbf{V}_1\mathbf{K}^{-1/2}),\ \mathrm{tr}(\mathbf{K}^{-1/2}\mathbf{y}\mathbf{y}'\mathbf{V}_2\mathbf{K}^{-1/2}),\ \mathrm{tr}(\mathbf{K}^{-1/2}\mathbf{y}\mathbf{y}'\mathbf{V}_3\mathbf{K}^{-1/2})\right) =$$

$$= [\mathrm{vec}(\mathbf{y}\mathbf{y}'\mathbf{K}^{-1/2})]'\left(\mathrm{vec}(\mathbf{V}_1\mathbf{K}^{-1/2}),\ \mathrm{vec}(\mathbf{V}_2\mathbf{K}^{-1/2}),\ \mathrm{vec}(\mathbf{K}^{-1/2})\right),$$

$$\mathbf{Q} = \begin{pmatrix} [\mathrm{vec}(\mathbf{V}_1\mathbf{K}^{-1/2})]' \\ [\mathrm{vec}(\mathbf{V}_2\mathbf{K}^{-1/2})]' \\ [\mathrm{vec}(\mathbf{K}^{-1/2})]' \end{pmatrix}\left(\mathrm{vec}(\mathbf{V}_1\mathbf{K}^{-1/2}),\ \mathrm{vec}(\mathbf{V}_2\mathbf{K}^{-1/2}),\ \mathrm{vec}(\mathbf{K}^{-1/2})\right)$$

and therefore $\mathscr{M}(\mathrm{vec}(\mathbf{V}_1\mathbf{K}^{-1/2}), \mathrm{vec}(\mathbf{V}_2\mathbf{K}^{-1/2}), \mathrm{vec}(\mathbf{K}^{-1/2}))' \subset \mathscr{M}(\mathbf{Q})$. Now, similarly as in the proof of Theorem 1, it follows that $\mathbf{m}'(\mathbf{I} - \mathbf{Q}^-\mathbf{Q})\,\mathbf{x} = 0$. Since $\mathbf{f} \in \mathscr{M}(\mathbf{Q})$, the expression $\mathbf{m}'\mathbf{Q}^-\mathbf{f}$ is invariant for any choice of the g-inverse and we can use the Moore-Penrose inverse. Hence the BAQUE is uniquely given by (15)–(16).

Remark 3. The case $v = l = m = 0$ characterizes the situation that a priori distribution is concentrated at a point $\boldsymbol{\theta} = (1, u, k)'$. The equation (7) has the form

$$\mathbf{W}\mathbf{A}\mathbf{W} = \lambda_1\mathbf{V}_1 + \lambda_2\mathbf{V}_2 + \lambda_3\mathbf{V}_3\,, \quad \mathbf{W} = \mathbf{V}_1 + u\mathbf{V}_2 + k\mathbf{I}\,.$$

The solution is

$$\mathbf{A} = \mathbf{W}^{-1}(\lambda_1\mathbf{V}_1 + \lambda_2\mathbf{V}_2 + \lambda_3\mathbf{V}_3)\,\mathbf{W}^{-1} = \lambda_1\mathbf{M}_1 + \lambda_2\mathbf{M}_2 + \lambda_3\mathbf{M}_3\,.$$

Here $\mathbf{M}_i = \mathbf{W}^{-1}\mathbf{V}_i\mathbf{W}^{-1}$, $i = 1, 2, 3$ and $\lambda_1, \lambda_2, \lambda_3$ satisfy the conditions (16). In this case the BAQUE is the locally best estimate at the point $(\theta_1, \theta_2, \theta_3) = (1, u, k)$, and coincides with the MINQUE (cf. [5]).

Further, we reformulate Theorem 2 for the model (1). If we use (12), we get:

Corollary 2. *Let* $\mathbf{M}\mathbf{U}_1\mathbf{M}\mathbf{U}_2\mathbf{M} = \mathbf{M}\mathbf{U}_2\mathbf{M}\mathbf{U}_1\mathbf{M}$, $\mathbf{U}_3 = \mathbf{I}_N$, $\mathbf{Z} = \mathbf{U}_1 + u\mathbf{U}_2 + k\mathbf{U}_3$ *and rank* $\mathrm{r}(\mathbf{MZM}) = \mathrm{N} - \mathrm{r}(\mathbf{X})$.

a) *The BAIQUE for the parametric function* $\gamma = \mathbf{f}'\boldsymbol{\theta}$ *in the model* (1) *exists iff*

$$\mathbf{f} \in \mathscr{M}(\mathbf{Q}^*)\,,$$

where $\mathbf{f} = (f_1, f_2, f_3)'$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)'$,

$$\mathbf{Q}^* = \begin{pmatrix} \mathrm{tr}(\mathbf{N}_1\mathbf{U}_1), & \mathrm{tr}(\mathbf{N}_2\mathbf{U}_1), & \mathrm{tr}(\mathbf{N}_3\mathbf{U}_1) \\ \mathrm{tr}(\mathbf{N}_1\mathbf{U}_2), & \mathrm{tr}(\mathbf{N}_2\mathbf{U}_2), & \mathrm{tr}(\mathbf{N}_3\mathbf{U}_2) \\ \mathrm{tr}(\mathbf{N}_1), & \mathrm{tr}(\mathbf{N}_2), & \mathrm{tr}(\mathbf{N}_3) \end{pmatrix},$$

$$\mathbf{N}_i = \mathbf{M}\mathbf{V}_i\mathbf{M}[(\mathbf{MZM})^2 + (\mathbf{MUM})^2 + m^2\mathbf{M}]^+\,\mathbf{M}\,, \quad i = 1, 2, 3\,,$$

$$\mathbf{U} = v\mathbf{U}_2 + l\mathbf{I}\,.$$

b) *The BAIQUE from* a) *is uniquely given by*

$$\hat{\gamma} = \mathbf{t}'\mathbf{B}\mathbf{t} = \lambda_1 \mathbf{t}'\mathbf{N}_1\mathbf{t} + \lambda_2 \mathbf{t}'\mathbf{N}_2\mathbf{t} + \lambda_3 \mathbf{t}'\mathbf{N}_3\mathbf{t}\,,$$

where $\boldsymbol{\lambda}$ *satisfies the condition*

$$\mathbf{Q}^*\boldsymbol{\lambda} = \mathbf{f}\,.$$

4. EXAMPLE

The following experiment was made to judge the precision of p measuring instruments of the same type: q observers measured by each of p instruments the unknown value of the given variable. The value measured by the i-th instrument and the j-th observer is the realisation of the random variable $t_{ij} = c + a_i + b_j + e_{ij}$, where c is the expected value of the measured variable, a_i and b_j characterize the i-th instrument and the j-th observator, respectively, e_{ij} is the error of the observations. Using the transformation from Section 2, we get the model

$$y_{ij} = \alpha_i + \beta_j + \varepsilon_{ij}\,, \quad i = 1, \ldots, p\,, \quad j = 1, \ldots, q\,.$$

Here $\mathrm{E}(y_{ij}) = 0$ and $\alpha_i, \beta_j, \varepsilon_{ij}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$ are assumed to be independent random variables from normal populations with zero means and unknown variances $\theta_1, \theta_2, \theta_3$, respectively. We shall look for the BAQUE for the parameter function $\gamma = f_1\theta_1 + f_2\theta_2 + f_2\theta_3$. Suppose that the vector of observations is written in the lexicographic order

$$\mathbf{y} = (y_{11}, \ldots, y_{1q}, y_{21}, \ldots, y_{2q}, y_{31}, \ldots, y_{pq})'\,,$$

as well as the error vector

$$\varepsilon = (\varepsilon_{11}, \ldots, \varepsilon_{1q}, \varepsilon_{21}, \ldots, \varepsilon_{2q}, \varepsilon_{31}, \ldots, \varepsilon_{pq})'\,.$$

Then we have the model

$$\mathbf{y} = (\mathbf{I}_p \otimes \mathbf{1}_q) + (\mathbf{1}_p \otimes \mathbf{I}_q) + \varepsilon\,,$$

where $\mathbf{1}_r = (1, \ldots, 1)'$ is the r-vector of ones, $\alpha = (\alpha_1, \ldots, \alpha_p)'$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)'$ with $\mathrm{E}(\mathbf{y}) = \mathbf{0}$,

$$\mathrm{Var}(\mathbf{y}) = \theta_1(\mathbf{I}_p \otimes \mathbf{J}_q) + \theta_2(\mathbf{J}_p \otimes \mathbf{I}_q) + \theta_3(\mathbf{I}_p \otimes \mathbf{I}_q)\,,$$

where $\mathbf{J}_r = \mathbf{1}_r \otimes \mathbf{1}_r'$ is an $r \times r$-matrix of ones.

We have the model (2) with

$$\mathbf{V}_1 = \mathbf{I}_p \otimes \mathbf{J}_q\,, \quad \mathbf{V}_2 = \mathbf{J}_p \otimes \mathbf{I}_q\,, \quad \mathbf{V}_3 = \mathbf{I}_p \otimes \mathbf{I}_q\,.$$

According to Theorem 2, the BAQUE for $\gamma = f_1\theta_1 + f_2\theta_2 + f_3\theta_3$ is $\hat{\gamma} = \mathbf{y}'\mathbf{A}\mathbf{y}$, where $\mathbf{A} = \lambda_1\mathbf{M}_1 + \lambda_2\mathbf{M}_2 + \lambda_3\mathbf{M}_3$, $\mathbf{M}_i = \mathbf{V}_i\mathbf{K}^{-1}$, $i = 1, 2, 3$ and $\mathbf{W} = \mathbf{V}_1 + u\mathbf{V}_2 + k\mathbf{V}_3$, $\mathbf{K} = \mathbf{W}^2 + \mathbf{V}^2 + m^2\mathbf{I}$, $\mathbf{V} = v\mathbf{V}_2 + l\mathbf{I}_{pq}$. In our case

$$\mathbf{W} = k(\mathbf{I}_p \otimes \mathbf{I}_q) + (\mathbf{I}_p \otimes \mathbf{J}_q) + u(\mathbf{J}_p \otimes \mathbf{I}_q)\,.$$

The BAQUE for the function γ exists iff the matrix $\boldsymbol{W}$ is regular. Then we get

$$\begin{aligned}
\boldsymbol{V} &= l(\boldsymbol{I}_p \otimes \boldsymbol{I}_q) + v(\boldsymbol{J}_p \otimes \boldsymbol{I}_q)\,,\\
\boldsymbol{W} &= \boldsymbol{I}_p \otimes (k\boldsymbol{I}_q + \boldsymbol{J}_q) + u(\boldsymbol{J}_p \otimes \boldsymbol{I}_q)\,,\\
\boldsymbol{V}^2 &= l^2(\boldsymbol{I}_p \otimes \boldsymbol{I}_q) + (2lv + pv^2)(\boldsymbol{J}_p \otimes \boldsymbol{I}_q)\,,\\
\boldsymbol{W}^2 &= \boldsymbol{I}_p \otimes [k^2\boldsymbol{I}_q + (2k+q)\boldsymbol{J}_q] + \boldsymbol{J}_p \otimes [(2ku + pu^2)\boldsymbol{I}_q + 2u\boldsymbol{J}_q]\,,\\
\boldsymbol{K} &= \boldsymbol{I}_p \otimes [(l^2 + k^2 + m^2)\boldsymbol{I}_q + (2k+q)\boldsymbol{J}_q] + \boldsymbol{J}_p \otimes\\
&\quad \otimes \{[u(2k + pu) + v(2l + pv)]\boldsymbol{I}_q + 2u\boldsymbol{J}_q\}\,.
\end{aligned}$$

Denoting

$$\begin{aligned}
h_0 &= k^2 + l^2 + m^2\,,\\
h_1 &= (k+q)^2 + l^2 + m^2 = q(2k+q) + h_0\,,\\
h_2 &= (k+pu)^2 + (l+pv)^2 + m^2 = pu(2k+pu) + pv(2l+pv) + h_0\,,\\
h_3 &= (k+pu+q)^2 + (l+pv)^2 + m^2 = (2k+pu+q)(pu+q) +\\
&\quad + pv(2l+pv) + h_0 = h_1 + h_2 - h_0 + 2pqu\,,
\end{aligned}$$

after routine computation we arrive at

$$\boldsymbol{K} = \boldsymbol{I}_p \otimes \left(h_0\boldsymbol{I}_q + \frac{h_1 - h_0}{q}\boldsymbol{J}_q\right) + \boldsymbol{J}_q \otimes \left(\frac{h_2 - h_0}{p}\boldsymbol{I}_q + 2u\boldsymbol{J}_q\right),$$

$$\begin{aligned}
\boldsymbol{K}^{-1} &= \boldsymbol{I}_p \otimes \frac{1}{h_0}\left(\boldsymbol{I}_q - \frac{h_1 - h_0}{qh_1}\boldsymbol{J}_q\right) - \boldsymbol{J}_p \otimes \left[\frac{h_2 - h_0}{ph_0h_2}\left(\boldsymbol{I}_q - \frac{h_1 - h_0}{qh_1}\boldsymbol{J}_q\right) +\right.\\
&\left. + \frac{(h_1 - h_0)(h_2 - h_0) - 2h_0pqu}{pqh_1h_2h_3}\boldsymbol{J}_q\right] = \frac{1}{h_0}(\boldsymbol{I}_p \otimes \boldsymbol{I}_q) - \frac{h_1 - h_0}{qh_0h_1}(\boldsymbol{I}_p \otimes \boldsymbol{J}_q) -\\
&\quad - \frac{h_2 - h_0}{ph_0h_2}(\boldsymbol{J}_p \otimes \boldsymbol{I}_q) \times \frac{1}{pq}\left(\frac{1}{h_0} - \frac{1}{h_1} - \frac{1}{h_2} + \frac{1}{h_3}\right)(\boldsymbol{J}_p \otimes \boldsymbol{J}_q)\,.
\end{aligned}$$

Therefore

$$\boldsymbol{M}_1 = \boldsymbol{V}_1\boldsymbol{K}^{-1} = \frac{1}{h_1}(\boldsymbol{I}_p \otimes \boldsymbol{J}_q) - \frac{h_3 - h_1}{ph_1h_3}(\boldsymbol{J}_p \otimes \boldsymbol{J}_q)\,,$$

$$\boldsymbol{M}_2 = \boldsymbol{V}_2\boldsymbol{K}^{-1} = \frac{1}{h_2}(\boldsymbol{J}_p \otimes \boldsymbol{I}_q) - \frac{h_3 - h_2}{qh_2h_3}(\boldsymbol{J}_p \otimes \boldsymbol{J}_q)\,,$$

$$\boldsymbol{M}_3 = \boldsymbol{K}^{-1}\,.$$

The BAQUE is

$$\hat{\gamma} = \lambda_1^* \mathbf{y}'\mathbf{M}_1\mathbf{y} + \lambda_2^* \mathbf{y}'\mathbf{M}_2\mathbf{y} + \lambda_3^* \mathbf{y}'\mathbf{M}_3\mathbf{y} =$$

$$= \lambda_1^* \left[\frac{q^2}{h_1} \sum_{i=1}^{p} y_{i.}^2 - \frac{(h_3 - h_1)\, p^2 q^2}{p h_1 h_3}\, y_{..}^2\right] +$$

$$+ \lambda_2^* \left[\frac{p^2}{h_2} \sum_{j=1}^{q} y_{.j}^2 - \frac{(h_3 - h_2)\, p^2 q^2}{q h_2 h_3}\, y_{..}^2\right] +$$

$$+ \lambda_3^* \left[\frac{1}{h_0} \sum_{i=1}^{p} \sum_{j=1}^{q} y_{ij}^2 - \frac{(h_1 - h_0)\, q^2}{q h_0 h_1} \sum_{i=1}^{p} y_{i.}^2 -\right.$$

$$\left. - \frac{(h_2 - h_0)\, p^2}{p h_0 h_2} \sum_{j=1}^{q} y_{.j}^2 + \frac{1}{pq}\left(\frac{1}{h_0} - \frac{1}{h_1} - \frac{1}{h_2} + \frac{1}{h_3}\right) p^2 q^2 y_{..}^2\right] =$$

$$= \lambda_3^* \frac{1}{h_0} \sum_{i=1}^{p} \sum_{j=1}^{q} y_{ij}^2 + \left[\lambda_1^* \frac{q^2}{h_1} - \lambda_3^* \frac{(h_1 - h_0)\, q}{h_0 h_1}\right] \sum_{i=1}^{p} y_{i.}^2 +$$

$$+ \left[\lambda_2^* \frac{p^2}{h_2} - \lambda_3^* \frac{(h_2 - h_0)\, p}{h_0 h_2}\right] \sum_{j=1}^{q} y_{.j}^2 + \left[\lambda_3^* \left(\frac{1}{h_0} - \frac{1}{h_1} - \frac{1}{h_2} +\right.\right.$$

$$\left.\left. + \frac{1}{h_3}\right) pq - \lambda_1^* \frac{(h_3 - h_1)\, pq^2}{h_1 h_3} - \lambda_2^* \frac{(h_3 - h_2)\, p^2 q}{h_2 h_3}\right] y_{..}^2\,,$$

where $\lambda_1^*, \lambda_2^*, \lambda_3^*$ are solutions of the linear system

$$\frac{pq^2}{h_3}\lambda_1 + \frac{pq}{h_3}\lambda_2 + \left[\frac{(p-1)\, q}{h_1} + \frac{q}{h_3}\right]\lambda_3 = f_1\,,$$

$$\frac{pq}{h_3}\lambda_1 + \frac{p^2 q}{h_3}\lambda_2 + \left[\frac{p(q-1)}{h_2} + \frac{p}{h_3}\right]\lambda_3 = f_2\,,$$

$$\left[\frac{(p-1)\, q}{h_1} + \frac{q}{h_3}\right]\lambda_1 + \left[\frac{p(q-1)}{h_2} + \frac{p}{h_3}\right]\lambda_2 +$$

$$+ \left[\frac{(p-1)(q-1)}{h_0} + \frac{p-1}{h_1} + \frac{q-1}{h_2} + \frac{1}{h_3}\right]\lambda_3 = f_3\,,$$

and

$$y_{.j} = \frac{1}{p} \sum_{i=1}^{p} y_{ij}\,, \quad y_{i.} = \frac{1}{q} \sum_{j=1}^{q} y_{ij}\,, \quad y_{..} = \frac{1}{pq} \sum_{i=1}^{p} \sum_{j=1}^{q} y_{ij}\,.$$

In the case $v = l = m = 0 \Rightarrow h_0 = k^2$, $h_1 = (k + q)^2$, $h_2 = (k + pu)^2$, $h_3 = (k + pu + q)^2$, we get the locally best estimate at the point $\boldsymbol{\theta} = (1, u, k)'$.

If $k = 0$ then $l = m = 0$ and the matrices $\mathbf{W}, \mathbf{K}$ can be singular. The BAQUE is not uniqualy given or must not exist.

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Souhrn

BAYESOVSKÉ NEVYCHÝLENÉ ODHADY V MODELU S TŘEMI VARIAČNÍMI KOEFICIENTY

JAROSLAV STUCHLÝ

V článku jsou uvedeny nutné a postačující podmínky pro existenci a odvozeny explicitní vzorce pro Bayesův invariantní kvadratický nevychýlený odhad lineární funkce varianěních koeficientů v lineárním smíšeném modelu $\mathbf{t} = \mathbf{X}\beta + \varepsilon$, $\mathrm{E}(\mathbf{t}) = \mathbf{X}\beta$, $\mathrm{Var}(\mathbf{t}) = \theta_1 \mathbf{U}_1 + \theta_2 \mathbf{U}_2 + \theta_3 \mathbf{U}_3$ s třemi neznámými varianěními koeficienty v normálním případě. V závěru jsou výsledky aplikovány v analýze rozptylu.

Резюме

НЕСМЕЩЕНАЯ ОЦЕНКА БАЙЕСА В МОДЕЛИ С ТРЕМЬЯ ДИСПЕРСИОННЫМИ КОМПОНЕНТАМИ

JAROSLAV STUCHLÝ

В статье приводится необходимое и достаточное условие для инвариантной квадратичной чесмещеной оценки Байеса линейной функции от параметров ковариационной матрицы в слусае линейной нормальной модели $\mathbf{t} = \mathbf{X}\beta + \varepsilon$, $\mathrm{E}(\mathbf{t}) = \mathbf{X}\beta$, $\mathrm{Var}(\mathbf{t}) = \theta_1 \mathbf{U}_1 + \theta_2 \mathbf{U}_2 + \theta_3 \mathbf{U}_3$ трэмья дисперсионными компонентами.

В заключении статьи приведен пример применения изложенной теории к дисперсионному анализу.

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