Ivan Hlaváček; Raino Mäkinen

On the numerical solution of axisymmetric domain optimization problems

Applications of Mathematics, Vol. 36 (1991), No. 4, 284–304

Persistent URL: http://dml.cz/dmlcz/104467

36 (1991) APPLICATIONS OF MATHEMATICS No. 4, 284–304

ON THE NUMERICAL SOLUTION OF AXISYMMETRIC DOMAIN OPTIMIZATION PROBLEMS

IVAN HLAVÁČEK, RAINO MÄKINEN

(Received January 12, 1990)

Summary. An axisymmetric second order elliptic problem with mixed boundary conditions is considered. A part of the boundary has to be found so as to minimize one of four types of cost functionals. The numerical realization is presented in detail. The convergence of piecewise linear approximations is proved. Several numerical examples are given.

Keywords: shape optimization, axisymmetric elliptic problems, finite elements.

AMS Subject class: 65N99, 65N30, 49A22.

INTRODUCTION

One often meets elliptic problems in three-dimensional domains Ω which are generated by the rotation of a bounded plane domain D around an axis. Then the most suitable approach is to use the cylindrical coordinates. If the data of the problem are axisymmetric, the problem is then reduced to the two-dimensional domain D.

Let a part Γ of the boundary ∂D be optimized. In this work we present the numerical realization of the problem, which was given by Hlaváček in [5]. However, some changes in the set of admissible boundaries were made which were suggested by the numerical experiments done by the second author. Therefore the proofs of those results in [5] which need modification, are presented in a revised form.

1. THE STATE PROBLEM

We shall consider a class of admissible domains $D(\alpha)$, where

$$D(\alpha) = \{(r, z) \mid 0 < r < \alpha(z),\ 0 < z < 1\}$$

and the function $\alpha(z)$ – the design variable – belongs to the following set of admissible functions

$$U_{\mathrm{ad}} = \Big\{\alpha \in C^{(1),1}([0, 1]) \mid 0 < \alpha_{\min} \leqq \alpha(z) \leqq \alpha_{\max}\,,\quad |\alpha'(z)| \leqq C_1\,,$$
$$|\alpha''(z)| \leqq C_2 \text{ a.e. in } (0, 1)\,,\ \int_0^1 \alpha^2 \,\mathrm{d}z = C_3\Big\}$$

with given positive constants $\alpha_{\min}, \alpha_{\max}, C_1, C_2, C_3$. Here $C^{(1),1}([0, 1])$ denotes the space of functions with Lipschitz-continuous derivatives.

Lemma 1.1. *U_{ad} is compact in $C^{(1)}([0, 1])$.*

Proof. See the proof of Lemma 2 in [4] with a slight modification.

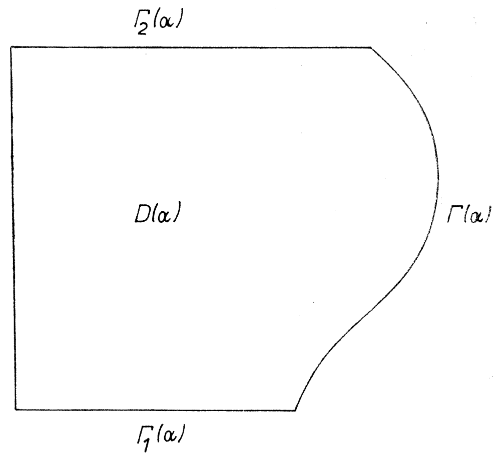


Figure 1.1

Corollary 1.1. *U_{ad} is compact in $C([0, 1])$.*

Proof. Proof follows immediately, since $\alpha_n \to \alpha$ in $C^{(1)}([0, 1])$ implies that $\alpha_n \to \alpha$ in $C([0, 1])$.

Let $\Gamma(\alpha)$ denote the graph of the function α,

$$\Gamma_1 = \partial D(\alpha) \cap \{z = 0\}, \quad \Gamma_2 = \partial D(\alpha) \cap \{z = 1\}.$$

We shall consider the following boundary value problem

$$(1.1) \qquad \begin{cases} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(A_i(x) \frac{\partial u}{\partial x_i}\right) = F \quad \text{in} \quad \Omega(\alpha), \\ \sum_{i=1}^{3} A_i \frac{\partial u}{\partial x_i} v_i = G \quad \text{on} \quad S_1(\alpha) \cup S(\alpha), \\ u = 0 \quad \text{on} \quad S_2(\alpha), \end{cases}$$

where $\Omega(\alpha)$ is generated by rotation of $D(\alpha)$ around the x_3-axis, $S_i(\alpha)$ by rotation of $\Gamma_i(\alpha)$, $i = 1, 2$ and $S(\alpha)$ by rotation of $\Gamma(\alpha)$, (see Figure 1.1), v_i are components of the unit outward normal with respect to $\partial\Omega(\alpha)$.

Let $\hat{\Omega}$ be the cylindrical domain generated by rotation of the rectangle $\hat{D} = (0, \delta) \times (0, 1)$, $\delta > \alpha_{\max}$.

Assume that the function F in (1.1) is determined as the restriction to $\Omega(\alpha)$ of an axisymmetric function $F \in L^2(\hat{\Omega})$,

$$G = \begin{cases} 0 & \text{on} \quad S(\alpha) \\ G_1 & \text{on} \quad S_1(\alpha), \end{cases}$$

where G_1 is determined as the restriction to $S_1(\alpha)$ of an axisymmetric function $G_1 \in L^2(\hat{S}_1)$, $\hat{S}_1 = \partial\hat{\Omega} \cap \{x_3 = 0\}$.

Assume that the coefficients A_i are restrictions to $\Omega(\alpha)$ of axisymmetric functions $A_i \in L^\infty(\hat{\Omega})$, $A_1 = A_2$ a.e. and a positive constant a_0 exists such that

$$A_i(x) \geqq a_0 \quad \text{a.e. in } \hat{\Omega}\,. \tag{1.2}$$

Let us denote $A_1 = A_2 = a_r$, $A_3 = a_z$.

Passing to the cylindrical coordinate system, we transform the standard variational formulation of the problem (1.1) to the following state problem:

$$\begin{cases} \text{Find} \quad y \in V(D(\alpha)) \quad \text{such that} \\ a(\alpha; y, v) = L(\alpha; v) \quad \forall v \in V(D(\alpha)), \end{cases} \tag{1.3}$$

where

$$V(D(\alpha)) = \{v \in W^1_{2,r}(D(\alpha)) \mid \gamma v = 0 \quad \text{on} \quad \Gamma_2(\alpha)\},$$

$$a(\alpha; y, v) = \int_{D(\alpha)} \left(a_r \frac{\partial y}{\partial r}\frac{\partial v}{\partial r} + a_z \frac{\partial y}{\partial z}\frac{\partial v}{\partial z} \right) r \, \mathrm{d}r \, \mathrm{d}z\,,$$

$$L(\alpha; v) = \int_{D(\alpha)} f v r \, \mathrm{d}r \, \mathrm{d}z + \int_{\Gamma_1(\alpha)} g \gamma v r \, \mathrm{d}s$$

and where the functions $f \in L_{2,r}(\hat{D})$ and $g \in L_{2,r}(\hat{\Gamma}_1)$, $\hat{\Gamma}_1 = \partial\hat{D} \cap \{z = 0\}$, are given. Here $W^1_{2,r}(D)$ denotes the weighted Sobolev space, with the following norm

$$\|u\|_{1,r,D} = \left(\int_D \left[u^2 + \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] r \, \mathrm{d}r \, \mathrm{d}z \right)^{1/2};$$

γ: $W^1_{2,r}(D) \to L_{2,r}(\Gamma)$ is the trace mapping. Finally, $a_r, a_z \in L^\infty(\hat{D})$,

$$a_r(r, z) \geqq a_0\,, \quad a_z(r, z) \geqq a_0 \text{ a.e. in } \hat{D}\,. \tag{1.4}$$

Remark 1.1. The variational formulation (1.3) corresponds with the following "classical one":

$$\begin{cases} -\dfrac{1}{r}\dfrac{\partial}{\partial r} r a_r \left(\dfrac{\partial y}{\partial r}\right) - \dfrac{\partial}{\partial z}\left(a_z \dfrac{\partial y}{\partial z}\right) = f \quad \text{in} \quad D(\alpha)\,, \\ a_r \dfrac{\partial y}{\partial r} \nu_r + a_z \dfrac{\partial y}{\partial z} \nu_z = 0 \quad \text{on} \quad \Gamma(\alpha)\,, \\ a_r \dfrac{\partial y}{\partial r} \nu_r + a_z \dfrac{\partial y}{\partial z} \nu_z = g \quad \text{on} \quad \Gamma_1(\alpha)\,, \\ y = 0 \quad \text{on} \quad \Gamma_2(\alpha)\,. \end{cases} \tag{1.5}$$

Next we shall show that the solution of the problem (1.3) depends continuously on the design variable α. To do this we shall construct an extension $Eu \in W^1_{2,r}(\hat{D})$ of the function $u \in V(D(\alpha))$ as follows:

$$Eu(r, z) = \begin{cases} u(2\alpha(z) - r, z) & \text{on} \quad \hat{D} \setminus D(\alpha)\,, \\ u & \text{on} \quad D(\alpha) \end{cases} \tag{1.6}$$

Proposition 1.1. *Assume that a sequence* $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n \in U_{\text{ad}}$, *converges to a function* α *in* $C([0, 1])$. *Then*

$$Ey(\alpha_n)|_{D(\alpha)} \to y(\alpha) \quad (\textit{weakly}) \;\; \textit{in} \quad W^1_{2,r}(D(\alpha))\,,$$

where $y(\alpha)$ *is the solution of the state problem* (1.3) *on* $D(\alpha)$.

Proof. See [5] pp. 221–223.

2. SETTING OF DOMAIN OPTIMIZATION PROBLEMS AND THE EXISTENCE OF OPTIMAL SOLUTIONS

We shall consider the following four types of the cost functional:

$$j_1(\alpha, y) = \int_{D(\alpha)} (y - y_d)^2 \, r \, dr \, dz\,,$$

$$j_2(\alpha, y) = \int_0^1 (y(\alpha(z), z) - y_\gamma)^2 \, dz\,,$$

$$j_3(\alpha, y) = a(\alpha; y, y)\,,$$

$$j_4(\alpha, y) = \int_{D(\alpha)} \left[\left(a_r \frac{\partial y}{\partial r} - K_1\right)^2 + \left(a_z \frac{\partial y}{\partial z} - K_2\right)^2\right] r \, dr \, dz\,,$$

where $y(\alpha(z), z) \equiv \gamma y$ denotes the trace of y on the curve $\Gamma(\alpha)$, $y_d \in L_{2,r}(\hat{D})$, $y_\gamma \in L^2([0, 1])$ and $K_i \in L_{2,r}(\hat{D})$, $i = 1, 2$ are given functions.

We define the *Domain Optimization Problems*:

$$(\mathrm{P}_i) \qquad \begin{cases} \text{Find} \quad \alpha^* \in U_{\text{ad}} \quad \text{such that} \\ j_i(\alpha^*, y(\alpha^*)) \leqq j_i(\alpha, y(\alpha)) \quad \forall \alpha \in U_{\text{ad}}\,, \quad i \in \{1, 2, 3, 4\}\,, \end{cases}$$

where $y(\alpha)$ denotes the solution of the state problem (1.3).

To be able to prove the existence of an optimal α^* we need the following

Proposition 2.1. *Let the assumptions of Proposition* 1.1 *be satisfied. Then*

$$\lim_{n\to\infty} j_i(\alpha_n, y(\alpha_n)) = j_i(\alpha, y(\alpha))\,, \quad i = 1, 2, 3$$

$$\liminf_{n\to\infty} j_4(\alpha_n, y(\alpha_n)) \geqq j_4(\alpha, y(\alpha))\,.$$

Proof. See [5].

Theorem 2.1. *There exists at least one solution of the problem* (P_i), $i \in \{1, 2, 3, 4\}$.

Proof. Let $\{\alpha_n\}$, $\alpha_n \in U_{ad}$, be a minimizing sequence of $j_i(\alpha, y(\alpha))$, $i \in \{1, 2, 3, 4\}$, i.e.

$$\lim_{n \to \infty} j_i(\alpha_n, y(\alpha_n)) = \inf_{\alpha \in U_{ad}} j_i(\alpha, y(\alpha)) . \tag{2.1}$$

By the Corollary 1.1 the set U_{ad} is compact in $C([0, 1])$. Hence there exist a subsequence $\{\alpha_{n_k}\} \subset \{\alpha_n\}$ and $\alpha^* \in U_{ad}$ such that

$$\lim_{k \to \infty} \alpha_{n_k} \to \alpha^* \quad \text{in} \quad C([0, 1]) .$$

Then Proposition 2.1 and (2.1) imply that

$$j_i(\alpha^*, y(\alpha^*)) \leqq \liminf_{k \to \infty} j_i(\alpha_{n_k}, y(\alpha_{n_k})) = \inf_{\alpha \in U_{ad}} j_i(\alpha, y(\alpha)) .$$

Consequently, at α^* a minimum is attained. Q.E.D.

3. APPROXIMATION BY FINITE ELEMENTS

In the present Section we propose an approximate solution of the domain optimization problems (P_i), $i \in \{1, 2, 3\}$, making use of piecewise linear design variable and linear triangular finite elements for solving the state problem. In the case $i = 4$, it is more suitable to use dual variational formulation for the state problem. For the dual approach we refer to the paper [7].

Let N be a positive integer and $h = 1/N$. We denote by Δ_j, $j = 1, 2, \ldots, N$, the subintervals $[(j - 1)h, jh]$ and introduce the set

$$U_{ad}^h = \Big\{ \alpha_h \in C^{(0),1}([0, 1]) \mid 0 < \alpha_{\min} \leqq \alpha_h(z) \leqq \alpha_{\max} ,$$

$$\alpha_h|_{\Delta_j} \in P_1(\Delta_j) \ \forall \Delta_j , \quad j = 1, 2, \ldots, N ,$$

$$|\alpha_h'| \leqq C_1, \ |\delta_h^2 \alpha_h| \leqq C_2, \ \int_0^1 \alpha_h^2 \, dz = C_3 \Big\} ,$$

where $C^{(0),1}([0, 1])$ denotes the set of Lipschitz-functions, $P_1(\Delta_j)$ is the set of linear functions defined on Δ_j and $\delta_h^2 \alpha_h$ denotes the second difference

$$\delta_h^2 \alpha_h(jh) = \frac{1}{h^2} [\alpha_h((j + 1)h) - 2\alpha_h(jh) + \alpha_h((j - 1)h)] ,$$

$$j = 1, 2, \ldots, N - 1 .$$

Let $D_h = D(\alpha_h)$ denote the domain bounded by the graph $\Gamma_h = \Gamma(\alpha_h)$ of the function $\alpha_h \in U_{ad}^h$. The polygonal domain D_h will be carved into triangles by the following way. We choose $\alpha_0 \in (0, \alpha_{\min})$ and introduce a uniform triangulation of the rectangle $\mathscr{R} = [0, \alpha_0] \times [0, 1]$, independent of α_h, if h is fixed.

In the remaining part $D_h \setminus \mathscr{R}$ let the nodal points divide the segments $[\alpha_0, \alpha_h(jh)]$, $j = 0, 1, 2, \ldots, N$, into M equal segments (see Figure 3.1), where

$$M = 1 + [(\alpha_{\max} - \alpha_0) N]$$

and the brackets [ ] denote the integer part of the number inside. Consequently, we obtain a strongly regular family $\{\mathscr{T}_h(\alpha_h)\}$, $h \to 0$, $\alpha_h \in U^h_{ad}$ of triangulations (cf. [5]).

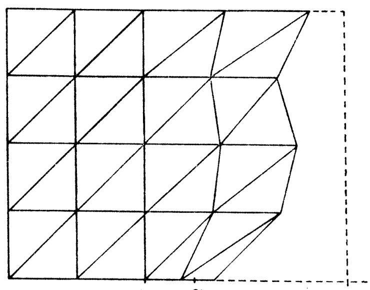


Figure 3.1

Let us consider the standard space V_h of linear finite elements

$$V_h = \{v_h \in C(\bar{D}_h) \cap V(D_h) \; |v_h|_T \in P_1(T) \; \forall T \in \mathscr{T}_h(\alpha_h)\} .$$

We define the approximate state problem:

$$(3.1) \qquad \begin{cases} \text{Find} \quad y_h \equiv y_h(\alpha_h) \in V_h \quad \text{such that} \\ a(\alpha_h; y_h, v_h) = L_h(\alpha_h; v_h) \quad \forall v_h \in V_h . \end{cases}$$

Here $L_h(\alpha_h; v_h)$ denotes a suitable approximation of $L(\alpha_h; v_h)$. For instance, let us define

$$(3.2) \qquad \sum_{T \in \mathscr{T}_h(\alpha_h)} [frv_h]_{G(T)} \operatorname{meas}(T) + \sum_{I \in \mathscr{T}_h(\alpha_h)} [grv_h]_{G(I)} \operatorname{meas}(I) ,$$

where $G(T)$ denotes the center of gravity of the triangle T and $G(I)$ the midpoint of the interval $I = T \cap \Gamma_1(\alpha)$. Henceforth we shall assume that

$$(3.3) \qquad f \in H^1(\hat{D}) \cap C(\hat{D}) , \quad r^2 D^p f \in L^2(\hat{D}) \quad \text{for} \quad |p| = 2 ,$$

$$(3.4) \qquad g \text{ is piecewise from } C^2 .$$

One can show, see [5] – Lemma 9, that for any $\alpha_h \in U^h_{ad}$ the problem (3.1) has a unique solution.

Proposition 3.1. *Let (3.3), (3.4) hold. Assume that $\{\alpha_h\}$, $h \to 0$, is a sequence of $\alpha_h \in U^h_{\mathrm{ad}}$ converging to α in $C([0, 1])$. Then*

$$\lim_{h \to 0} j_i(\alpha_h, y_h(\alpha_h)) = j_i(\alpha, y(\alpha))$$

holds for $i \in \{1, 2, 3\}$.

Proof. See [5].

Lemma 3.1. *For any $\alpha \in U_{\mathrm{ad}}$ there exists a sequence $\{\alpha_h\}$, $h \to 0$, $\alpha_h \in U^h_{\mathrm{ad}}$, such that $\alpha_h \to \alpha$ in $C([0, 1])$.*

Proof. We use some results of [5]. Since our new set U_{ad} is contained in the set U^*_{ad}, defined in [5] – p. 214, Lemma A.1 of [5] yields that for any $\alpha \in U_{\mathrm{ad}}$ there exists a sequence $\{\alpha_h\}$, $h \to 0$, $\alpha_h \in U^{*h}_{\mathrm{ad}}$, such that $\alpha_h \to \alpha$ in $C([0, 1])$, where

$$U^{*h}_{\mathrm{ad}} = \{\alpha_h \in U^*_{\mathrm{ad}} |\ \alpha_h|_{\Delta_j} \in P_1(\Delta_j)\ \forall j\}\ .$$

Consequently, it would be sufficient to prove that the sequence $\{\alpha_h\}$ from Lemma A.1 satisfies also the condition for the second difference.

We may write (see point 2° of the Proof of Lemma A.1 in [5] – p. 240)

$$\delta^2_h\ \alpha_h(z_j) = \delta^2_h\ \beta_h(z_j) =$$
$$= \frac{1}{h^2}\left[\beta_h(z_{j+1}) - 2\ \beta_h(z_j) + \beta_h(z_{j-1})\right],\quad j = 1, 2, \dots, N - 1\ ,$$

where $z_j = jh$. Recalling the definition of β_h (see [1] – Proof of Lemma 7.1) we have

$$\beta_h(z_j) = \frac{1}{h}\int_{z_j-(h/2)}^{z_j+(h/2)} \beta(z)\,\mathrm{d}z\ ,\quad j = 0, 1, \dots, N \equiv 1/h \tag{3.5}$$

where $\beta \equiv Z_{\mu k}\alpha$.

In order that β_h can be defined also for $j = 0$ and $j = N$, we extend β to the interval $(-\varepsilon, 1 + \varepsilon)$, $\varepsilon > 0$ in the "antisymmetric" way, i.e.

$$\beta(z) - \beta(0) = -[\beta(-z) - \beta(0)]\ ,\quad z \in (-\varepsilon, 0)$$

and similarly in the interval $z \in (1, 1 + \varepsilon)$.

Consequently, we may write

$$|\delta^2_h\ \alpha_h(z_j)| = \frac{1}{h^3}\left|\int_{z_j-(h/2)}^{z_j+(h/2)} [\beta(z + h) - 2\beta(z) + \beta(z - h)]\,\mathrm{d}z\right|\ , \tag{3.6}$$
$$j = 1, 2, \dots, N - 1\ .$$

Using the Taylor's expansion

$$\beta(z \pm h) = \beta(z) \pm h\ \beta'(z) + \int_z^{z \pm h} \beta''(t)\,(z \pm h - t)\,\mathrm{d}t\ ,$$
$$z \in \left[\frac{h}{2},\ 1 - \frac{h}{2}\right]$$

we obtain

$$(3.7) \qquad |\beta(z+h) - 2\,\beta(z) + \beta(z-h)| \leqq$$
$$\leqq \int_z^{z+h} |\beta''(t)|\,(z+h-t)\,\mathrm{d}t + \int_{z-h}^{z} |\beta''(t)|\,(t-z+h)\,\mathrm{d}t \leqq C_2 h^2$$

since

$$|\beta''(t)| = |(Z_{\mu 1}\alpha)''(t)| \leqq (1-\mu)\,|\alpha''(t)| \leqq C_2\,,$$
$$t \in \left(-\frac{h}{2},\ 1+\frac{h}{2}\right), \quad \forall h < 2\varepsilon\,.$$

Note that in the proof of Lemma A.1 of [5] one can choose the constant parameter $k = 1$. In fact, $k = 1$ is compatible with both (A.11) and (A.18). The latter assertion follows from the fact that

$$S\bar{\gamma} = \int_0^1 \bar{\gamma}\alpha\,\mathrm{d}z > \int_0^1 \gamma\alpha\,\mathrm{d}z = \int_{I^+} \gamma\alpha\,\mathrm{d}z + \int_{I^-} \gamma\alpha\,\mathrm{d}z$$

which implies

$$-\int_{I^-} \gamma\alpha\,\mathrm{d}z > \int_{I^+} \gamma\alpha\,\mathrm{d}z - S\bar{\gamma}\,.$$

Therefore we have again

$$Z_{\mu k}\alpha \equiv Z_{\mu 1}\alpha \in C^{(1),1}([-\varepsilon,\ 1+\varepsilon]) \quad \text{for some} \quad \varepsilon > 0$$

and the second derivative $(Z_{\mu 1}\alpha)''$ exists almost everywhere in $[-\varepsilon,\ 1+\varepsilon]$.

Now inserting from (3.7) into (3.6), we arrive at

$$(3.8) \qquad |\delta_h^2\, \alpha_h(z_j)| \leqq C_2\,, \quad j = 1, 2, \ldots, N-1\,.$$

Q.E.D.

Lemma 3.2. *Let $\{\alpha_h\}$, $h \to 0$, be a sequence of $\alpha_h \in U_{\mathrm{ad}}^h$ such that $\alpha_h \to \alpha$ in $C([0, 1])$. Then $\alpha \in U_{\mathrm{ad}}$.*

Proof. Let us define function $\gamma_h \in C^{(0),1}([0, 1])$ by

$$\gamma_h(z) = \begin{cases} \alpha_h'\left(\dfrac{h}{2}\right), & z \leqq \dfrac{h}{2} \\ \alpha_h'\left(1-\dfrac{h}{2}\right), & z \geqq 1-\dfrac{h}{2} \\ \text{piecewise linear interpolate of points} & \\ \left(jh+\dfrac{h}{2},\ \alpha_h'\left(jh+\dfrac{h}{2}\right)\right), \quad j = 0, \ldots, N-1\,, & \dfrac{h}{2} < z < 1-\dfrac{h}{2}\,. \end{cases}$$

Now $\gamma_h \in E_{ad}\ \forall h$, where $E_{ad} = \{\beta \in C^{(0),1}([0, 1]) \mid -C_1 \leqq \beta \leqq C_1, |\beta'| \leqq C_2$ a.e. in $(0, 1)\}$. As E_{ad} is compact in $C([0, 1])$ we can find a subsequence $\{\gamma_{h_m}\}$ such that $\gamma_{h_m} \to \gamma$ in $C([0, 1])$ and $|\gamma(z_1) - \gamma(z_2)| \leqq C_2|z_1 - z_2|\ \forall z_1, z_2 \in [0, 1]$. From this and the inequality

$$\forall h\ \gamma_h(z) - \frac{C_2}{2} h \leqq \alpha_h'(z) \leqq \gamma_h(z) + \frac{C_2}{2} h \quad \text{a.e. in } (0, 1)$$

we obtain the convergence of the corresponding subsequence $\alpha'_{h_m} \to \gamma$ in $L^\infty(0, 1)$. For this subsequence we have

$$\int^z \gamma(t)\,\mathrm{d}t = \int^z \lim_{h_m \to 0} \alpha'_{h_m}(t)\,\mathrm{d}t = \lim_{h_m \to 0} \int^z \alpha'_{h_m}(t)\,\mathrm{d}t = \lim_{h_m \to 0} \alpha_{h_m}(z) = \alpha(z)\,.$$

Thus $\gamma = \alpha' \in C^{(0),1}([0, 1])$.

Finally we have $|\alpha'(z_1) - \alpha'(z_2)| = |\gamma(z_1) - \gamma(z_2)| \leqq C_2|z_1 - z_2|\ \forall z_1, z_2 \in (0, 1)$. Consequently, $\alpha \in U_{ad}$ holds. Q.E.D.

For a fixed parameter h, we define the Approximate Domain Optimization Problems:

$$(\mathrm{P}_{hi}) \qquad \begin{cases} \text{Find} \quad \alpha_h^* \in U_{ad}^h \quad \text{such that} \\ j_i(\alpha_h^*, y_h(\alpha_h^*)) \leqq j_i(\alpha_h, y_h(\alpha_h)) \quad \forall \alpha_h \in U_{ad}^h\,, \end{cases}$$

where $i \in \{1, 2, 3\}$ and $y_h(\alpha_h)$ is the solution of the approximate state problem (3.1). The existence of at least one optimal solution of (P_{hi}) follows easily (cf. [5] – p. 238).

Theorem 2. *Let the assumptions* (3.3), (3.4) *hold. Let* $\{\alpha_h\}$, $h \to 0$, *be a sequence of solutions of the Approximation Domain Optimization Problem* (P_{hi}), $i \in \{1, 2, 3\}$.

Then a subsequence $\{\alpha_{h_m}\} \subset \{\alpha_h\}$ *exists such that*

$$(3.9) \qquad \alpha_{h_m} \to \alpha^* \quad in \quad C([0, 1])\,,$$

$$(3.10) \qquad Ey_{h_m}|_{D(\alpha^*)} \to y(\alpha^*) \quad (weakly) \ in \ W_{2,r}^1(D(\alpha^*))\,,$$

where α^* *is a solution of the Domain Optimization Problem* (P_i), Ey_{h_m} *are the solutions* $y_{h_m}(\alpha_{h_m})$, *extended according to the formula*

$$(3.11) \qquad Ey(r, z) = y(2\,\alpha(z) - r, z)\,,$$

$y(\alpha^*)$ *is the solution of the state problem* (1.3) *on* $D(\alpha^*)$.

The limit of any uniformly convergent subsequence of $\{\alpha_h\}$ *represents a solution of* (P_i) *and an analogue of* (3.10) *holds.*

Proof. Since U_{ad} is compact in $C([0, 1])$, a subsequence $\{\alpha_{h_m}\} \subset \{\alpha_h\}$ exists such that (3.9) holds. By Lemma 3.2 $\alpha^* \in U_{ad}$.

Let $\alpha \in U_{ad}$ be given. By virtue of Lemma 3.1 there exists a sequence $\{\beta_h\}$, $\beta_h \in U_{ad}^h$, such that $\beta_h \to \alpha$ in $C([0, 1])$.

We have

$$j_i(\alpha_{h_m}, y_{h_m}) \leqq j_i(\beta_{h_m}, y_{h_m}(\beta_{h_m})) \quad \forall h_m$$

by definition.

Passing to the limit with $h_m \to 0$ and using Proposition 3.1 on both sides, we obtain

$$j_i(\alpha^*, y(\alpha^*)) \leqq j_i(\alpha, y(\alpha)) .$$

Consequently, α^* is a solution of the problem (P_i).

The convergence (3.10) follows from Proposition 3 in [5]. The rest of the Theorem is obvious. Q.E.D.

4. NUMERICAL REALIZATION

4.1. Formulation of the equivalent mathematical programming problem. We now express our Domain Optimization Problem (P_{hi}), $i \in \{1, 2, 3\}$ for fixed $h > 0$ in matrix notation.

Let us denote

$$\boldsymbol{x} = (x_0, x_1, \ldots, x_N)^{\mathrm{T}} \equiv (\alpha_h(0), \alpha_h(h), \ldots, \alpha_h(Nh))^{\mathrm{T}} \in \mathbb{R}^{N+1}$$

the r-coordinate vector of the design nodes

$$(\alpha_h(jh), jh) \quad j = 0, 1, \ldots, N .$$

For fixed h the approximate solution $y_h(\alpha_h)$ is given by the vector of nodal values $\boldsymbol{q} \equiv \boldsymbol{q}(\boldsymbol{x})$, which is the solution of the linear system of equations

$$\boldsymbol{K}(\boldsymbol{x})\, \boldsymbol{q} = \boldsymbol{f}(\boldsymbol{x}) . \tag{4.1}$$

$\boldsymbol{K}(\boldsymbol{x})$ is the stiffness matrix of our problem and $\boldsymbol{f}(\boldsymbol{x})$ is the vector arising from the discretization of $L(\alpha; v)$ in (1.3).

Consequently, the problem (P_{hi}) expressed in algebraic form is equivalent to the following mathematical programming problem

$$\begin{cases} \text{Find} \quad \boldsymbol{x}^* \in U \quad \text{such that} \\ J_i(\boldsymbol{x}^*, \boldsymbol{q}(\boldsymbol{x}^*)) \leqq J_i(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})) \quad \forall \boldsymbol{x} \in U , \end{cases} \tag{4.2i}$$

where $\boldsymbol{q}(\boldsymbol{x})$ solves (4.1) and U denotes the set of admissible design vectors given by

$$U = \Big\{ \boldsymbol{x} \in \mathbb{R}^{N+1} \,|\, \alpha_{\min} \leqq x_j \leqq \alpha_{\max} , \quad j = 0, 1, \ldots, N ;$$

$$-C_1 h \leqq x_{j+1} - x_j \leqq C_1 h , \quad j = 0, 1, \ldots, N - 1 ;$$

$$-C_2 h^2 \leqq x_{j+2} - 2x_{j+1} + x_j \leqq C_2 h^2 , \quad j = 0, 1, \ldots, N - 2 ;$$

$$\frac{h}{2}(x_0^2 + x_N^2) + h \sum_{j=1}^{N-1} x_j^2 = C_3 \Big\}$$

Remark 4.1. In the above definition of U the integral constraint $\int_0^1 \alpha_h^2 \, \mathrm{d}z$ is approximated by means of the trapezoidal rule. As α_h^2 is piecewise quadratic, an exact evaluation of the integral is also possible.

4.2. Sensitivity analysis. In order to utilize efficient numerical optimization algorithms in solving (4.2i), we need an efficient method for evaluating the gradient $\nabla_x J_i(\mathbf{x})$.

Lemma 4.1. *Let* $\boldsymbol{p}$ *denote the solution of the adjoint state problem*

$$\mathbf{K}(\mathbf{x})\, \boldsymbol{p} = \nabla_q J_i(\mathbf{x}, \boldsymbol{q}) \,. \tag{4.3}$$

Then the gradient of the cost functional with respect to $\mathbf{x}$ *is given by*

$$\nabla_x J_i(\mathbf{x}, \boldsymbol{q}(\mathbf{x})) = \nabla_x J_i(\mathbf{x}, \boldsymbol{q}) + \boldsymbol{p}^{\mathrm{T}}(\nabla_x \boldsymbol{f}(\mathbf{x}) - \nabla_x \mathbf{K}(\mathbf{x})\, \boldsymbol{q}) \,. \tag{4.4}$$

Proof. Differentiating $J_i(\mathbf{x}, \boldsymbol{q})$ with respect to x_j we obtain

$$\frac{\partial J_i(\mathbf{x}, \boldsymbol{q}(\mathbf{x}))}{\partial x_j} = \frac{\partial J_i(\mathbf{x}, \boldsymbol{q})}{\partial x_j} + (\nabla_q J_i(\mathbf{x}, \boldsymbol{q}))^{\mathrm{T}} \frac{\partial \boldsymbol{q}}{\partial x_j} \,. \tag{4.5}$$

As $\mathbf{K}(\mathbf{x})$ and $\boldsymbol{f}(\mathbf{x})$ are smooth functions of $\mathbf{x}$ we may differentiate (4.1) implicitly to obtain

$$\frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_j} \boldsymbol{q} + \mathbf{K}(\mathbf{x}) \frac{\partial \boldsymbol{q}}{\partial x_j} = \frac{\partial \boldsymbol{f}(\mathbf{x})}{\partial x_j} \,, \quad j = 0, 1, \ldots, N \,. \tag{4.6}$$

Using (4.3) and (4.6) we may write

$$\begin{aligned} (\nabla_q J_i(\mathbf{x}, \boldsymbol{q}))^{\mathrm{T}} \frac{\partial \boldsymbol{q}}{\partial x_j} &= (\mathbf{K}(\mathbf{x})\, \boldsymbol{p})^{\mathrm{T}} \frac{\partial \boldsymbol{q}}{\partial x_j} = \boldsymbol{p}^{\mathrm{T}} \mathbf{K}(\mathbf{x}) \frac{\partial \boldsymbol{q}}{\partial x_j} = \\ &= \boldsymbol{p}^{\mathrm{T}} \left(\frac{\partial \boldsymbol{f}(\mathbf{x})}{\partial x_j} - \frac{\partial \mathbf{K}(\mathbf{x})}{\partial x_j} \boldsymbol{q} \right) . \end{aligned} \tag{4.7}$$

Substituting from (4.7) into (4.5), we arrive at (4.4). Q.E.D.

The components of the stiffness matrix and the force vector are given by

$$k_{ij} = \int_{D(\alpha)} \left(a_r \frac{\partial \varphi_i}{\partial r} \frac{\partial \varphi_j}{\partial r} + a_z \frac{\partial \varphi_i}{\partial z} \frac{\partial \varphi_j}{\partial z} \right) r \, \mathrm{d}r \, \mathrm{d}z$$

and

$$f_j = \int_{D(\alpha)} f \varphi_j r \, \mathrm{d}r \, \mathrm{d}z + \int_{\Gamma_1(\alpha)} g \varphi_j r \, \mathrm{d}r \, \mathrm{d}z \,,$$

where φ_j, $j = 1, \ldots, M \equiv \dim V_h$, are the basis functions of V_h.

To compute $\partial \boldsymbol{f}(\mathbf{x})/\partial x_j$ and $\partial \mathbf{K}(\mathbf{x})/\partial x_j$ we utilize the isoparametric element technique.

We assume for simplicity that $g = 0$ and a_r, a_z are constants. Let

$$\boldsymbol{X}_T = \begin{pmatrix} r^{(1)} & z^{(1)} \\ r^{(2)} & z^{(2)} \\ r^{(3)} & z^{(3)} \end{pmatrix}$$

denote the matrix of nodal coordinates of an element T. Then the element stiffness matrix and force vector corresponding with T are given by

$$K_T = \int_{\hat{T}} \mathbf{B}^{\mathrm{T}} \boldsymbol{A} \mathbf{B} r |\boldsymbol{J}| \, \mathrm{d}\varrho \, \mathrm{d}\zeta$$

and

$$\boldsymbol{f}_T = \int_{\hat{T}} f \mathbf{N} r |\boldsymbol{J}| \, \mathrm{d}\varrho \, \mathrm{d}\zeta \, .$$

Here $\hat{T}$ denotes the reference element, $\boldsymbol{N}$ and $\boldsymbol{B}$ contain the values of the elements basis functions and their derivatives respectively,

$$\boldsymbol{A} = \begin{pmatrix} a_r & 0 \\ 0 & a_z \end{pmatrix},$$

and $|\boldsymbol{J}|$ is the Jacobian determinant of the isoparametric mapping $\hat{T} \to T$: $(\varrho, \zeta) \mapsto$ $\mapsto (r(\varrho, \zeta), z(\varrho, \zeta))$ which is defined by the element basis functions

$$r(\varrho, \zeta) = \sum_{j=1}^{3} \varphi_j(\varrho, \zeta) \, r^{(j)} \, ,$$

$$z(\varrho, \zeta) = \sum_{j=1}^{3} \varphi_j(\varrho, \zeta) \, z^{(j)} \, .$$

Employing quadrature formula with quadrature points $(\varrho_k, \zeta_k) \in \hat{T}$ and weights W_k, $k = 1, \ldots, Q$ we have

$$\boldsymbol{K}_T = \sum_{k=1}^{Q} W_k \boldsymbol{B}_k^T \boldsymbol{A}_k \boldsymbol{B}_k r_k |\boldsymbol{J}_k|$$

and

$$\boldsymbol{f}_T = \sum_{k=1}^{Q} W_k f(r_k, z_k) \, \boldsymbol{N}_k r_k |\boldsymbol{J}_k| \, .$$

Here the subscript k denotes the evaluation at the k:th integration point. In what follows we denote $(\,)' = \partial/\partial x_j \,(\,)$.

Lemma 4.2. *We have*

$$\boldsymbol{B}_k' = -\boldsymbol{B}_k \boldsymbol{X}_T' \boldsymbol{B}_k$$

$$\begin{pmatrix} r_k' \\ z_k' \end{pmatrix} = (\boldsymbol{X}_T')^{\mathrm{T}} \, \boldsymbol{N}_k$$

$$|\boldsymbol{J}_k|' = |\boldsymbol{J}_k| \sum_{j=1}^{3} (\nabla \varphi_j(r_k, z_k))^{\mathrm{T}} \begin{pmatrix} r^{(j)} \\ z^{(j)} \end{pmatrix}' .$$

Proof. See [2] and [8].

The following Lemma completes our sensitivity analysis:

Lemma 4.3. *The derivatives of the element stiffness matrix and the force vector are given by*

$$\boldsymbol{K}_T' = \sum_k W_k[(\boldsymbol{B}_k')^{\mathrm{T}} \boldsymbol{A}_k \boldsymbol{B}_k r_k |\boldsymbol{J}_k| + \boldsymbol{B}_k^{\mathrm{T}} \boldsymbol{A}_k \boldsymbol{B}_k' r_k |\boldsymbol{J}_k| + $$
$$+ \boldsymbol{B}_k^{\mathrm{T}} \boldsymbol{A}_k \boldsymbol{B}_k r_k' |\boldsymbol{J}_k| + \boldsymbol{B}_k^{\mathrm{T}} \boldsymbol{A}_k \boldsymbol{B}_k r_k |\boldsymbol{J}_k|']$$

and

$$\boldsymbol{f}_T' = \sum_k W_k \left[\left(\frac{\partial f_k}{\partial r} r_k' + \frac{\partial f_k}{\partial z} z_k' \right) \boldsymbol{N}_k r_k |\boldsymbol{J}_k| + f_k \boldsymbol{N}_k r_k' |\boldsymbol{J}_k| + f_k \boldsymbol{N}_k r_k |\boldsymbol{J}_k|' \right].$$

Remark 4.2. The sensitivity analysis above is in fact valid for all isoparametric Lagrangian finite elements (with obvious modifications). As in usual the stiffness matrix and force vector are formed by summing element contributions, it is natural to use the same element by element technique when computing (4.4).

Example 4.1. In the case $i = 1$ we have

$$J_1(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})) = (\boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{q}_0)^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{x}) (\boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{q}_0),$$

where $\boldsymbol{M}(\boldsymbol{x})$ is the "mass" matrix and $\boldsymbol{q}_0$ is a vector of nodal values of the function y_d. The gradient $\nabla_x J_1(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x}))$ is given by

$$\nabla_x J_1(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})) = (\boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{q}_0)^{\mathrm{T}} \nabla_x \boldsymbol{M}(\boldsymbol{x}) (\boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{q}_0) +$$
$$+ \boldsymbol{p}^{\mathrm{T}}(\nabla_x \boldsymbol{f}(\boldsymbol{x}) - \nabla_x \boldsymbol{K}(\boldsymbol{x}) \boldsymbol{q}(\boldsymbol{x})),$$

where $\boldsymbol{p}$ solves

$$\boldsymbol{K}(\boldsymbol{x}) \boldsymbol{p} = 2 \boldsymbol{M}(\boldsymbol{x}) (\boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{q}_0).$$

Example 4.2. In the case $i = 3$ we have

$$J_3(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})) = \boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{K}(\boldsymbol{x}) \boldsymbol{q}(\boldsymbol{x}).$$

The gradient $\nabla_x J_3(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x}))$ is given by

$$\nabla_x J_3(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})) = 2\, \boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}} \nabla_x \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{q}(\boldsymbol{x})^{\mathrm{T}} \nabla_x \boldsymbol{K}(\boldsymbol{x}) \boldsymbol{q}(\boldsymbol{x}).$$

In this case no adjoint state problem needs to be solved numerically, since

$$\nabla_q J_3(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x}) = 2\, \boldsymbol{K}(\boldsymbol{x}) \boldsymbol{q}(\boldsymbol{x})$$

resulting $\boldsymbol{p} = 2\, \boldsymbol{q}(\boldsymbol{x})$.

5. NUMERICAL EXAMPLES

In this section we present numerical results of several test cases. In optimization we have used Sequential Quadratic Programming algorithm E04VCF from NAG-library. E04VCF is essentially the code NPSOL due to Gill et al. (see [3]). The state problem (4.1) and the adjoint problem (4.3) were solved iteratively using preconditioned conjugate gradient method. All computations were done in double precision using VAX 8650 computer.

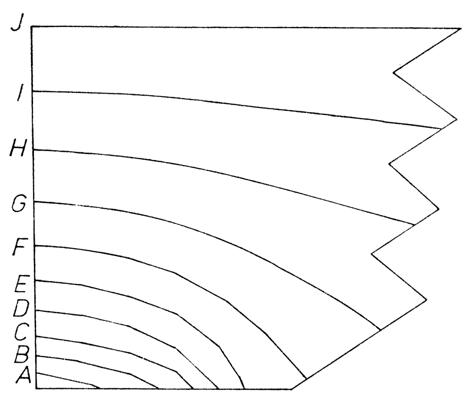
Example 5.1. In this example we study the convergence of the approximate solution α_h as $h \to 0$. We relax the cpnstraint for the second difference, so the situation here corresponds with the theory presented in [5]. Here $f = 0$, $a_r = a_z = 1$ and

$$g = \begin{cases} 1, & \text{for } 0 < r < 1/2 \\ 0, & \text{for } 1/2 < r < \delta. \end{cases}$$

We choose $\alpha_{\min} = 0{\cdot}6$, $\alpha_{\max} = 1{\cdot}2$, $C_1 = 1{\cdot}5$ and $C_3 = 1$. The results for $h = 1/8$, $h = 1/16$ and $h = 1/32$ are presented in Table 5.1 and Figures 5.1 – 5.3 respectively. In all cases the initial value of the unknown boundary was chosen to be $\alpha_h^{(0)} = 1$.

Table 5.1

h	SQP-iter.	$j_3(\alpha_h^{(0)})$	$j_3(\alpha_h^{*})$	CPU-sec.
1/8	17	$2{\cdot}499 \times 10^{-2}$	$2{\cdot}439 \times 10^{-2}$	8·8
1/16	21	$2{\cdot}543 \times 10^{-2}$	$2{\cdot}493 \times 10^{-2}$	52
1/32	11	$2{\cdot}555 \times 10^{-2}$	$2{\cdot}508 \times 10^{-2}$	105



A — 0·450E + 00, *B* — 0·400E + 00, *C* — 0·350E + 00, *D* — 0·300E + 00, *E* — 0·250E + 00, *F* — 0·200E + 00, *G* — 0·150E + 00, *H* — 0·100E + 00, *I* — 0·500E — 01, *J* — 0·000E + 00

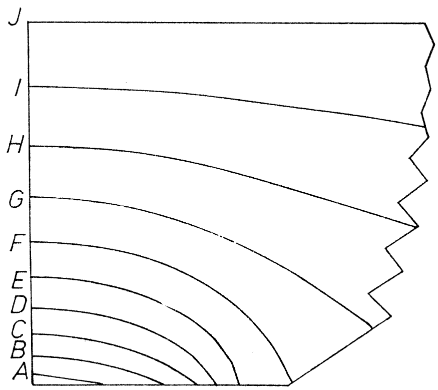
Figure 5.1 ($h = 1/8$)

Example 5.2. In this example we study the effects of regularizing the approximate boundary in the case $i = 3$.

We choose $h = 1/16$, $\alpha_{\min} = 0{\cdot}6$, $\alpha_{\max} = 1{\cdot}2$, $C_1 = 2$, $C_3 = 1$ and $\alpha_h^{(0)} = 1$ as the initial guess. The initial cost equals to $2{\cdot}54 \times 10^{-2}$. In Table 5.2 $C_2 = +\infty$ means that no restrictions were set on the second difference, $C_2 = 0$ means that an additional constraint

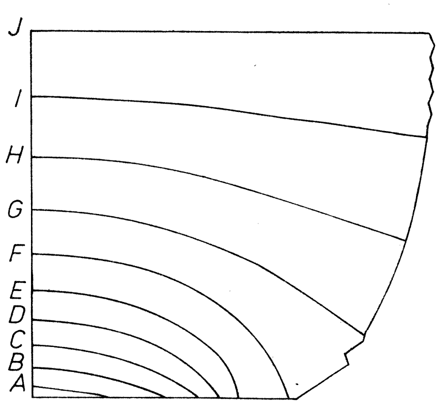
$$x_j = \tfrac{1}{2}(x_{j-1} + x_{j-1}), \quad j = 1, 3, 5, N - 1$$

was posed. The latter has the meaning of reducing the number of degrees of freedom of α_h to maintain stability. Note that this is also consistent with the theory in [5] as α_h is still piecewise linear.



A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E — 01, J — 0·000E + 00

Figure 5.2 ($h = 1/16$)



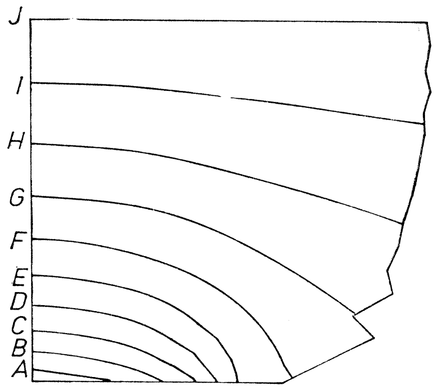
A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E — 01, J — 0·000E + 00

Figure 5.3 ($h = 1/32$)

The results are plotted in Figures 5.4–5.6.

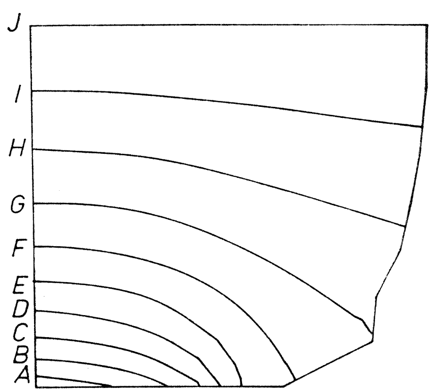
Table 5.2

C_2	SQP-Iter.	$j_3(\alpha_h^*)$	CPU-sec.
$+\infty$	11	$2{\cdot}489 \times 10^{-2}$	30
0	6	$2{\cdot}492 \times 10^{-2}$	22
5	5	$2{\cdot}495 \times 10^{-2}$	16



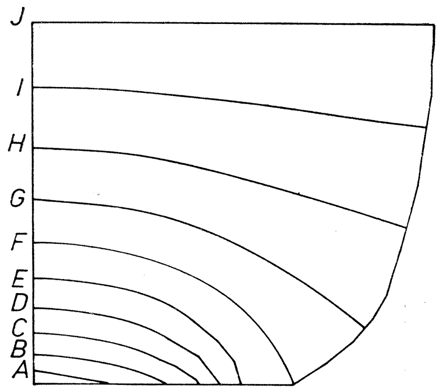
A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E — 01, J — 0·000E + 00

Figure 5.4 ($C_2 = +\infty$)



A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E — 01, J — 0·000E + 00

Figure 5.5 ($C_2 = 0$)



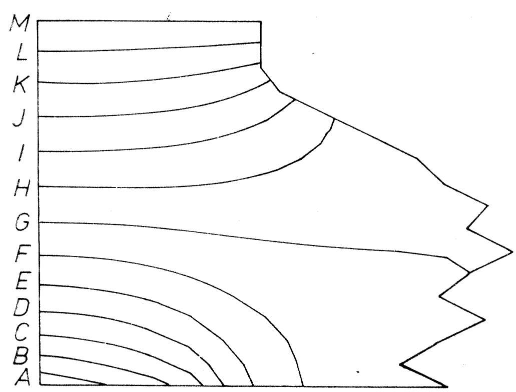
$A - 0{\cdot}450E + 00$, $B - 0{\cdot}400E + 00$, $C - 0{\cdot}350E + 00$, $D - 0{\cdot}300E + 00$, $E - 0{\cdot}250E + 00$, $F - 0{\cdot}200E + 00$, $G - 0{\cdot}150E + 00$, $H - 0{\cdot}100E + 00$, $I - 0{\cdot}500E - 01$, $J - 0{\cdot}000E + 00$

Figure 5.6 ($C_2 = 5$)

Example 5.3. In this example we have $i = 1$ with $y_d \equiv 0{\cdot}3$, $\alpha_{\min} = 0{\cdot}5$, $\alpha_{\max} = 1{\cdot}5$, $C_1 = 2$, $C_3 = 1$, $h = 1/16$ and $\alpha_h^{(0)} = 1$. The initial cost is $j_1(\alpha_h^{(0)}) = 9{\cdot}317 \times 10^{-3}$. In Table 5.3 and Figures 5.7–5.9 we see the results of the runs with different regularization parameters C_2.

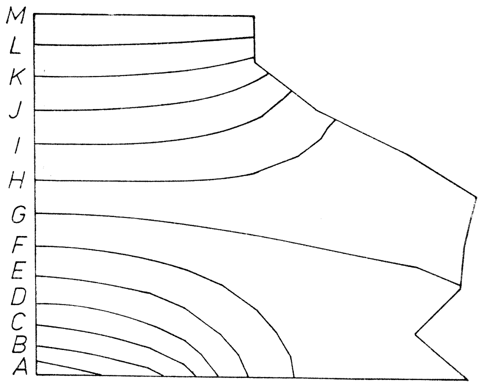
Table 5.3

C_2	SQP-Iter.	$j_3(\alpha_h^*)$	CPU-sec.
$+\infty$	25	$1{\cdot}948 \times 10^{-3}$	67
0	19	$1{\cdot}960 \times 10^{-3}$	54
5	16	$2{\cdot}007 \times 10^{-3}$	44



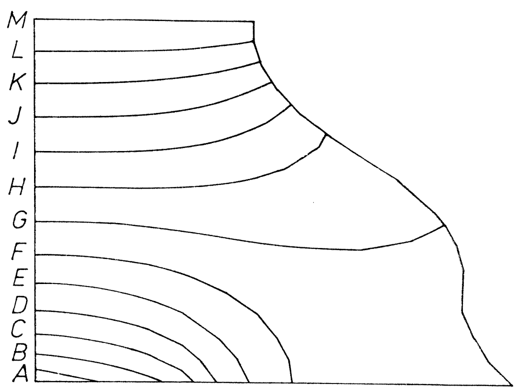
$A - 0{\cdot}600E + 00$, $B - 0{\cdot}550E + 00$, $C - 0{\cdot}500E + 00$, $D - 0{\cdot}450E + 00$, $E - 0{\cdot}400E + 00$, $F - 0{\cdot}350E + 00$, $G - 0{\cdot}300E + 00$, $H - 0{\cdot}250E + 00$, $I - 0{\cdot}200E + 00$, $J - 0{\cdot}150E + 00$, $K - 0{\cdot}100E + 00$, $L - 0{\cdot}500E - 01$, $M - 0{\cdot}000E + 00$

Figure 5.7 ($C_2 = +\infty$)



A — 0·600E + 00, B — 0·550E + 00, C — 0·500E + 00, D — 0·450E + 00, E — 0·400E + 00, F — 0·350E + 00, G — 0·300E + 00, H — 0·250E + 00, I — 0·200E + 00, J — 0·150E + 00, K — 0·100E + 00, L — 0·500E — 01, M — 0·000E + 00

Figure 5.8 ($C_2 = 0$)



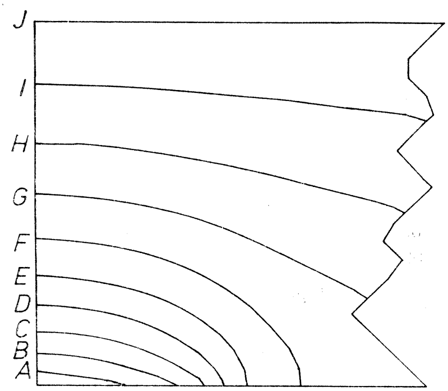
A — 0·600E + 00, B — 0·550E + 00, C — 0·500E + 00, D — 0·450E + 00, E — 0·400E + 00, F — 0·350E + 00, G — 0·300E + 00, H — 0·250E + 00, I — 0·200E + 00, J — 0·150E + 00, K — 0·100E + 00, L — 0·500E — 01, M — 0·000E + 00

Figure 5.9 ($C_2 = 5$)

Example 5.4. In this example $i = 2$ with $y_\gamma(z) = 0{\cdot}2(1 - z)$. The following parameter values are fixed: $\alpha_{\min} = 0{\cdot}6$, $\alpha_{\max} = 1{\cdot}2$, $C_1 = 1$, $C_3 = 1$. We vary parameters h and C_2. In Table 5.4 and Figures 5.10–5.13 we see the results of computations with $\alpha_h^{(0)} = 1$.

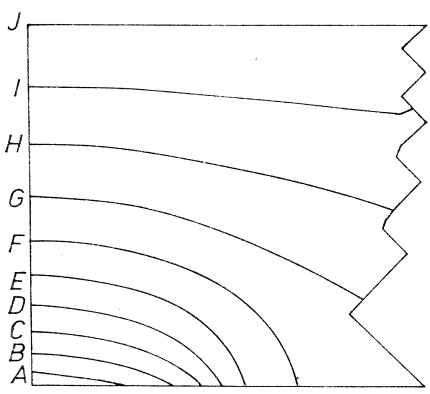
Table 5.4

h	C_2	SQP-Iter.	$j_2(\alpha_h^{(0)})$	$j_2(\alpha_h^*)$	CPU-sec.
1/20	$+\infty$	17	$1{\cdot}339 \times 10^{-3}$	$3{\cdot}587 \times 10^{-4}$	130
1/30	$+\infty$	15	$1{\cdot}879 \times 10^{-3}$	$4{\cdot}524 \times 10^{-4}$	290
1/10	8	14	$7{\cdot}986 \times 10^{-4}$	$3{\cdot}242 \times 10^{-4}$	22
1/20	8	20	$1{\cdot}339 \times 10^{-3}$	$4{\cdot}503 \times 10^{-4}$	137



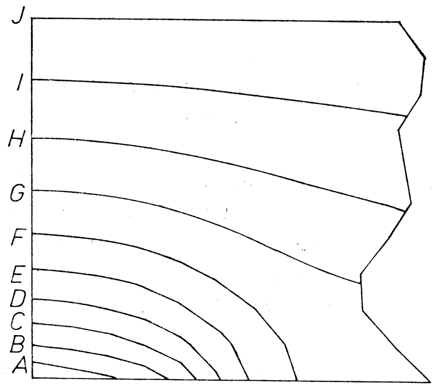
A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E − 01, J — 0·000E + 00

Figure 5.10 ($h = 1/20$, $C_2 = +\infty$)



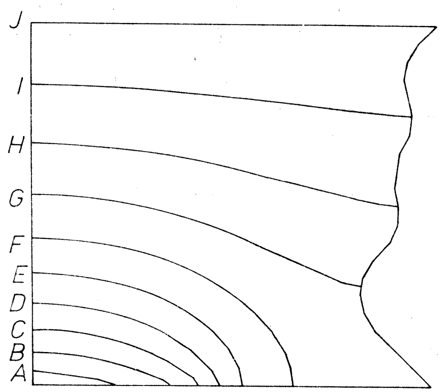
A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E − 01, J — 0·000E + 00

Figure 5.11 ($h = 1/30$, $C_2 = +\infty$)



A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·300E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·500E — 01, J — 0·000E + 00

Figure 5.12 ($h = 1/10$, $C_2 = 8$)



A — 0·450E + 00, B — 0·400E + 00, C — 0·350E + 00, D — 0·3C0E + 00, E — 0·250E + 00, F — 0·200E + 00, G — 0·150E + 00, H — 0·100E + 00, I — 0·5C0E — 01, J — 0·0C0E + 00

Figure 5.13 ($h = 1/20$, $C_2 = 8$)

6. CONCLUSIONS

The numerical tests confirm the theory in [5] as we obtain a converging sequence of solutions $\{\alpha_h^*\}$, $h \to 0$. However, from a designer's wiewpoint the shapes for large h are not acceptable. Moreover, in practice the use of fine meshes only to regularize oscillating boundaries is not desired due to larger computing costs.

The regularization procedures proposed in this paper have rigorous mathematical basis and proved to be efficient in the numerical tests. In almost all cases regularization reduced considerably the number of iterations in the optimization routine. The changes in the optimal cost were not substantial.

It is still recommended to have some flexibility of constraints in the domain optimization code. Because it is of great interest to find a least possible value of the cost, although the corresponding design would be for some reason unacceptable. One can then continue the optimization by adding more constraints on the regularity of the boundary and comparing the effect of regularization into the optimal cost.

It is also worth of noting that the optimization problem is generally non-convex. Thus the computed "minimum" may be only a local minimum. Therefore in practice, it is advised to perform several runs with different initial guesses for the possibility to find a better solution.

In the case of optimization of an axisymmetric elastic body, the same regularization procedure has proved to be efficient. For the theoretical study of this problem, see [6].

References

[1] *D. Begis, R. Glowinski:* Application de la méthode des éléments finis à l'approximation d'un problème de domaine optimal. Appl. Math. & Optim. 2 (1975), 130—169.

[2] *R. A. Brockman:* Geometric Sensitivity Analysis with Isoparametric Finite Elements, Commun. appl. numer. methods, 3 (1987), 495—499.

[3] *P. E. Gill, W. Murray, M. A. Saunders, M. H. Wright:* User's Guide for NPSOL. Technical Report SOL 84—7, Stanford University (1984).

[4] *I. Hlaváček:* Optimization of the Shape of Axisymmetric Shells. Apl. Mat. 28 (1983), 269—294.

[5] *I. Hlaváček:* Domain Optimization in Axisymmetric Elliptic Boundary Value Problems by Finite Elements. Apl. Mat. 33 (1988), 213—244.

[6] *I. Hlaváček:* Shape Optimization of Elastic Axisymmetric Bodies. Apl. Mat. 34 (1989), 225—245.

[7] *I. Hlaváček:* Domain Optimization in 3*D*-axisymmetric Elliptic Problems by Dual Finite Element Method. Apl. Mat. 35 (1990), 225—236.

[8] *R. Mäkinen:* Finite Element Design Sensitivity Analysis for Nonlinear Potential Problems. Submitted for publication in Commun. appl. numer. methods.

Souhrn

NUMERICKÉ ŘEŠENÍ OPTIMALIZACE TVARU OBLASTI V OSOVĚ SYMETRICKÝCH ELIPTICKÝCH ÚLOHÁCH

IVAN HLAVÁČEK, RAINO MÄKINEN

Uvažuje se osově symetrická eliptická úloha druhého řádu s kombinovanými okrajovými podmínkami. Je třeba nalézt část hranice oblasti tak, aby minimizovala jeden ze čtyř typů účelového funkcionálu. Je uvedena numerická realizace metody konečných prvků a důkaz konvergence po částech lineárních aproximací řešení.

Authors' addresses: Ing. *Ivan Hlaváček,* DrSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1, Czechoslovakia, Dr. *Raino Mäkinen,* University of Jyväskylä, Department of Mathematics, Seminaarinkatu 15, SF-40100 Jyväskylä, Finland.