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A MODIFICATION OF THE TWO-LEVEL ALGORITHM WITH OVERCORRECTION

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Summary. In this paper we analyse an algorithm which is a modification of the so-called two-level algorithm with overcorrection, published in [2].
We illustrate the efficiency of this algorithm by a model example.

Keywords: Two-level algorithm, overcorrection

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INTRODUCTION

A standard multigrid algorithm for solving systems of linear algebraic equations consists of two main parts: the correction of error on the "coarse level" and smoothing. The rate of convergence of the algorithm is strongly dependent on the properties of the vector which is the correction of the error obteined on the "coarse level". This correction usually aproximates the error of solution very well in the sense of its "progress", but not in the sense of its "size". Therefore, the algorithm can be accelerated when we multiply the correction by a suitable scalar factor. The main goal of this paper is to show how to get this scalar factor. The presented algorithm is the modification of that published in [2]. The convergence analysis is made for the two-level case.

1. Notation

Consider two finitedimensional real spaces H^1, H^2, where $n = \dim(H^1)$, $m = \dim(H^2)$, $m < n$. Let the space H^i be equipped with an inner product $\langle .,. \rangle_i$ and the associated norm $\|.\|_i = \langle .,. \rangle_i^{\frac{1}{2}}$, $i = 1, 2$. In most applications H^i, $i = 1, 2$ will be the Euclidean space.

We are interested in numerical solution $\hat{u} \in H^1$ of the problem

$$(1.1) \qquad \mathsf{A}u = f,$$

where $f \in H^1$, $u \in H^1$ and $A\colon H^1 \to H^1$ is a linear, symmetric and positive operator. The problem (1.1) has a unique solution for any $f \in H^1$. Let $\mathsf{p}: H^2 \to H^1$ be a linear injective operator called prolongation. Adjoint operators relative to inner products $\langle .,. \rangle_1$, $\langle .,. \rangle_2$ will be denoted by *.

Define the restriction operator $\mathsf{r}: H^1 \to H^2$ by

$$(1.2) \qquad \mathsf{r} = \mathsf{p}^*,$$

i.e.

$$(1.3) \qquad \langle x, \mathsf{p}y \rangle_1 = \langle \mathsf{r}x, y \rangle_2 \quad \text{for any } x \in H^1,\ y \in H^2.$$

For technical details of the construction of r, p see [3]. Put

$$(1.4) \qquad {}^2\mathsf{A} = \mathsf{rAp}.$$

It is easy to see that ${}^2\mathsf{A}$ is a linear, symmetric and positive operator. Hence we can define other inner products

$$(1.5) \qquad (.,.)_1 = \langle \mathsf{A}.,. \rangle_1,$$
$$(1.6) \qquad (.,.)_2 = \langle {}^2\mathsf{A}.,. \rangle_2$$

and the associated norms by

$$(1.7) \qquad |\!|\!|.|\!|\!|_1 = (.,.)_1^{\frac{1}{2}},$$
$$(1.8) \qquad |\!|\!|.|\!|\!|_2 = (.,.)_2^{\frac{1}{2}}.$$

2. Standard two-level algorithm

Let

$$\mathscr{S}(.): H^1 \to H^1 \tag{2.1}$$

be an iterative method for the solution of (1.1) satisfying the fix-point condition

$$\mathscr{S}(\hat{u}) = \hat{u}. \tag{2.2}$$

For any integer $\nu > 0$ we define

$$\mathscr{S}^{(\nu)}(.) = \mathscr{S}(\mathscr{S}^{(\nu-1)}(.)) \tag{2.3}$$

and for $\nu = 0$ we put

$$\mathscr{S}^{(0)}(.) = \mathsf{I}^1, \tag{2.4}$$

where I^1 is the identity operator on H^1. Further we shall suppose that $\mathscr{S}(x)$ can be written in the form

$$\mathscr{S}(x) = \mathsf{M}x + \mathsf{N}f, \tag{2.5}$$

where $\mathsf{M}, \mathsf{N}: H^1 \to H^1$ are linear operators satisfying the consistence condition

$$\mathsf{I}^1 = \mathsf{NA} + \mathsf{M}. \tag{2.6}$$

Note that (2.6) implies (2.2).

Let $u^i \in H^1$ be an arbitrary vector, $\nu_1 \geqslant 0$, $\nu_2 > 0$ given integers. One iteration $(u^i \to u^{i+1})$ of the standard two-level algorithm is defined as follows:

$$\tilde{u} := \mathscr{S}^{(\nu_1)}(u^i), \quad \tilde{u} \in H^1, \tag{2.7a}$$

$$v^2 := ({}^2\mathsf{A})^{-1}\mathsf{r}(\mathsf{A}\tilde{u} - f), \quad v^2 \in H^2, \tag{2.7b}$$

$$\tilde{\tilde{u}} := \tilde{u} - \mathsf{p}v^2, \quad \tilde{\tilde{u}} \in H^1, \tag{2.7c}$$

$$u^{i+1} := \mathscr{S}^{(\nu_2)}(\tilde{\tilde{u}}), \quad u^{i+1} \in H^1. \tag{2.7d}$$

This algorithm is analysed in [1], [3].

3. Accelerated two-level algorithm

Let $\nu_1 \geqslant 0$, $\nu_2 > 0$, $u^i \in H^1$ be given. One step of the accelerated two-level algorithm is defined by

$$\tilde{u} := \mathscr{S}^{(\nu_1)}(u^i), \quad \tilde{u} \in H^1, \tag{3.1a}$$

$$v^2 := ({}^2\mathsf{A})^{-1}\mathsf{r}(\mathsf{A}\tilde{u} - f), \quad v^2 \in H^2, \tag{3.1b}$$

$$v := \mathsf{p}v^2, \quad v \in H^1, \tag{3.1c}$$

$$\tilde{\tilde{u}} := \tilde{u} - v, \quad \tilde{\tilde{u}} \in H^1, \tag{3.1d}$$

$$\bar{u} := \mathscr{S}^{(\nu_2)}(\tilde{\tilde{u}}), \quad \bar{u} \in H^1, \tag{3.1e}$$

$$\bar{v} := \mathsf{M}^{\nu_2} v, \quad \bar{v} \in H^1; \tag{3.1f}$$

$$\hat{t} := t(\bar{u}, \bar{v}), \quad \hat{t} \in R \text{ (see (3.3))}, \tag{3.1g}$$

$$u^{i+1} := \bar{u} - \hat{t}\bar{v}, \quad u^{i+1} \in H^1, \tag{3.1h}$$

where the number $\hat{t} \in R$ is defined by the condititon

$$|\!|\!|(\bar{u} - \hat{t}\bar{v}) - \hat{u}|\!|\!|_1 = \min_{t \in R}\{|\!|\!|(\bar{u} - t\bar{v}) - \hat{u}|\!|\!|_1\}. \tag{3.2}$$

It is easy to see that (3.2) holds for

$$\hat{t} = \frac{\langle \mathsf{A}\bar{u} - f, \bar{v}\rangle_1}{|\!|\!|\bar{v}|\!|\!|_1} \quad \text{if } \bar{v} \neq 0 \tag{3.3}$$

and for any $\hat{t} \in R$ if $\bar{v} = 0$.

Remark 3.1. If we put $\hat{t} := 0$ in (3.1g) instead of $\hat{t} := t(\bar{u}, \bar{v})$ defined by (3.2) then we have $u^{i+1} = \bar{u} = \mathscr{S}^{(\nu_2)}(\tilde{\tilde{u}})$, i.e. we obtain the same result as by using algorithm (2.7) (see (2.7d)).

4. Convergence analysis

Define subspaces S, T of H^1 by

$$S = \mathrm{Im}(\mathsf{p}) = \{x \in H^1 : x = \mathsf{p}y \text{ for some } y \in H^2\}, \tag{4.1}$$

$$T = S^{\perp} = \{x \in H^1 : (x, z)_1 = 0 \text{ for every } z \in S\}. \tag{4.2}$$

Lemma 4.1. $T = \mathrm{Ker}(\mathsf{rA})$, *where* $\mathrm{Ker}(\mathsf{rA}) = \{x \in H^1 : \mathsf{rA}x = 0\}$.

Proof. The proof consists in the verification of the equality

$$S^{\perp} = \mathrm{Ker}(\mathsf{rA}). \tag{4.3}$$

It is evident that $(x, \mathsf{p}z)_1 = \langle \mathsf{rA}x, z\rangle_2$ for any $x \in H^1$, $z \in H^2$.

Let $x \in S^{\perp}$. Then $(x, \mathsf{p}z)_1 = 0$ for any $z \in H^2$. By virtue of the previous identity we arrive at $\mathsf{rA}x = 0$, i.e. $x \in \mathrm{Ker}(\mathsf{rA})$.

Let $x \in \mathrm{Ker}(\mathsf{rA})$. Then $\langle \mathsf{rA}x, z\rangle_2 = 0$ for any $z \in H^2$ and hence we may write $x \in S^{\perp}$. □

Let $x \in H^1$. We define the error of x by

$$e(x) = x - \hat{u} \tag{4.4}$$

and the defect of x by

$$d(x) = \mathsf{A}x - f. \tag{4.5}$$

Note that

$$d(x) = \mathsf{A}e(x). \tag{4.6}$$

Lemma 4.2. *The following equalities are valid:*

$$e(\tilde{u}) = \mathsf{M}^{\nu_1} e(u^i), \tag{4.7}$$

$$v = \mathsf{p}(\mathsf{rAp})^{-1}\mathsf{rA}e(\tilde{u}), \tag{4.8}$$

$$e(\tilde{\tilde{u}}) = [\mathsf{I}^1 - \mathsf{p}(\mathsf{rAp})^{-1}\mathsf{rA}]e(\tilde{u}), \tag{4.9}$$

$$e(\tilde{u}) = \mathsf{M}^{\nu_2} e(\tilde{\tilde{u}}), \tag{4.10}$$

$$e(u^{i+1}) = \mathsf{M}^{\nu_2}[e(\tilde{\tilde{u}}) = \hat{t}v], \tag{4.11}$$

$$|||e(u^{i+1})|||_1 = |||\mathsf{M}^{\nu_2}[e(\tilde{\tilde{u}}) - \hat{t}v]|||_1 = \\ = \min_{t \in R}\{|||\mathsf{M}^{\nu_2}[e(\tilde{\tilde{u}}) - tv]|||_1\}. \tag{4.12}$$

P r o o f. The equalities follow immediately. □

R e m a r k 4.1. Let $u^i \in H^1$ be given. Using algorithm (2.7) we obtain the iteration u^{i+1} with an error $\mathsf{M}^{\nu_2} e(\tilde{\tilde{u}})$ (see Remark 3.1 and (4.11)), using algorithm (3.1) we obtain the iteration u^{i+1} with an error $\mathsf{M}^{\nu_2}[e(\tilde{\tilde{u}}) - \hat{t}v]$. According to (4.12) we have $|||\mathsf{M}^{\nu_2}[e(\tilde{\tilde{u}}) - \hat{t}v]|||_1 \leqslant |||\mathsf{M}^{\nu_2} e(\tilde{\tilde{u}})|||_1$.

R e m a r k 4.2. Let $x \in H^1$. Since S, T are A-orthogonal subspaces of H^1 there exist unique two vectors $x_S \in S$, $x_T \in T$ such that $x = x_S + x_T$. The following is true:

$$|||x|||_1^2 = |||x_S|||_1^2 + |||x_T|||_1^2, \quad |||x_S|||_1 \leqslant |||x|||_1, \quad |||x_T|||_1 \leqslant |||x|||_1.$$

Lemma 4.3. *Let* $\mathsf{Q} = \mathsf{I}^1 - \mathsf{p}(\mathsf{rAp})^{-1}\mathsf{rA}$. *Then*

$$\mathsf{Q}x = x_T. \tag{4.13}$$

P r o o f. The proof consists in the verification of the following equalities:

$$\mathsf{Q}x_T = x_T, \tag{4.14}$$

$$\mathsf{Q}x_S = 0. \tag{4.15}$$

The equality (4.14) follows immediately from Lemma 4.1.
The vector $x_S \in S$ can be written in the form $x_S = \mathsf{p}w$ for some $w \in H^2$. Therefore $\mathsf{Q}x_S = \mathsf{Qp}w - \mathsf{p}(\mathsf{rAp})^{-1}\mathsf{rAp}w = 0$, which is nothing but (4.15). □

Note that

$$\text{(4.16)}\qquad \|Qx\|_1 \leqslant \|x\|_1 \quad \text{for all } x \in H^1,$$

$$\text{(4.17)}\qquad 0 < \|Qx\|_1 < \|x\|_1 \quad \text{for all } x \in H^1\backslash(T \cup S).$$

Define

$$\text{(4.18)}\qquad k(x) = \frac{\|Qx\|_1}{\|x\|_1}, \quad x \neq 0.$$

We have $k(x) \in \langle 0, 1\rangle$ for all $x \in H^1$, $x \neq 0$ and $k(x) \in (0,1)$ for all $x \in H^1\backslash(T \cup S)$.

Lemma 4.4. *The equality*

$$\text{(4.19)}\qquad \frac{\|e(u^{i+1})\|_1}{\|e(u^i)\|_1} = \min_{t\in R} \frac{\|M^{\nu_2}[e(\tilde{\tilde{u}}) - tv]\|_1}{\|e(\tilde{\tilde{u}})\|_1} \cdot k(e(\tilde{u})) \cdot \frac{\|M^{\nu_1} e(u^i)\|_1}{\|e(u^i)\|_1},$$

where $e(\tilde{\tilde{u}}) \in T$, $v \in S$, *is valid.*

Proof. As $e(\tilde{\tilde{u}}) = Qe(\tilde{u})$ and $v = p(rAp)^{-1}rAe(\tilde{u})$, we have $e(\tilde{\tilde{u}}) \in T, v \in S$. We can write $\frac{\|e(u^{i+1})\|_1}{\|e(u^i)\|_1}$ in the form

$$\text{(4.20)}\qquad \frac{\|e(u^{i+1})\|_1}{\|e(u^i)\|_1} = \frac{\|e(u^{i+1})\|_1}{\|e(\tilde{\tilde{u}})\|_1} \cdot \frac{\|e(\tilde{\tilde{u}})\|_1}{\|e(\tilde{u})\|_1} \cdot \frac{\|e(\tilde{u})\|_1}{\|e(u^i)\|_1}.$$

Since $e(\tilde{\tilde{u}}) = Qe(\tilde{u})$ we may write

$$\text{(4.21)}\qquad k(e(\tilde{u})) = \frac{\|e(\tilde{\tilde{u}})\|_1}{\|e(\tilde{u})\|_1}.$$

From the equality (4.7) we obtain

$$\text{(4.22)}\qquad \frac{\|e(\tilde{u})\|_1}{\|e(u^i)\|_1} = \frac{\|M^{\nu_1} e(u^i)\|_1}{\|e(u^i)\|_1}$$

and using (4.12) we have

$$\text{(4.23)}\qquad \frac{\|e(u^{i+1})\|_1}{\|e(\tilde{\tilde{u}})\|_1} = \min_{t\in R} \frac{\|M^{\nu_2}[e(\tilde{\tilde{u}}) - tv]\|_1}{\|e(\tilde{\tilde{u}})\|_1}.$$

Substituting (4.21), (4.22), (4.23) in formula (4.20) we obtain (4.19). □

Theorem 1. *Let $\nu_1 \geqslant 0$, $\nu_2 > 0$ be given integers. We shall suppose that there exists a real number $q \in (0,1)$ such that*

$$\|\mathsf{M}x\|_1 \leqslant q\|x\|_1 \quad \text{for any } x \in H^1. \tag{4.24}$$

Then the iterative process given by algorithm (3.1) converges for every $u^0 \in H^1$ and the following inequality holds for any $u^i \in H^1$:

$$\|e(u^{i+1})\|_1 \leqslant q^{\nu_1+\nu_2}\|e(u^i)\|_1. \tag{4.25}$$

P r o o f. The proof follows immediately from Lemma 4.4. $\square$

R e m a r k 4.3. Since $e(\tilde{\tilde{u}}) = e(\tilde{u}) - v$, $e(\tilde{\tilde{u}}) \in T$, $v \in S$ we have

$$\|v\|_1^2 = [1 - k^2(e(\tilde{u}))] \cdot \|e(\tilde{u})\|_1^2. \tag{4.26}$$

In what follows we shall suppose that all assumptions of Theorem 1 are satisfied.

Lemma 4.5. *Let $e(\tilde{u}) \in H^1 \backslash (S \cup T)$. Denote*

$$\varphi(x) = \frac{\|\mathsf{M}^{\nu_2}x\|_1}{\|x\|_1}, \quad x \in H^1 \backslash \{0\}. \tag{4.27}$$

Put

$$q_T = \varphi(e(\tilde{\tilde{u}})), \tag{4.28}$$
$$q_S = \varphi(v), \tag{4.29}$$
$$r = \varphi(e(\tilde{u})), \tag{4.30}$$
$$k = k(e(\tilde{u})). \tag{4.31}$$

Then the following equality is valid:

$$\|e(u^{i+1})\|_1^2 = \left\{ q_T^2 k^2 - \frac{\left[r^2 - q_T^2 k^2 - q_S^2(1-k^2)\right]^2}{4q_S^2(1-k^2)} \right\} \|e(\tilde{u})\|_1{}^2. \tag{4.32}$$

P r o o f. Since $e(\tilde{u}) \notin S \cup T$, we have $v \neq 0$, $e(\tilde{\tilde{u}}) \neq 0$, $k \in (0,1)$. Therefore q_T, q_S and r are well-defined. From Lemma 4.2 and (3.3) we have

$$\|e(u^{i+1})\|_1 = \|\mathsf{M}^{\nu_2}[e(\tilde{\tilde{u}}) - \hat{t}v]\|_1, \tag{4.33}$$

where

$$\hat{t} = \frac{\left(\mathsf{M}^{\nu_2}v, \mathsf{M}^{\nu_2}e(\tilde{\tilde{u}})\right)_1}{\|\mathsf{M}^{\nu_2}v\|_1^2}.$$

Using elementary calculations we obtain

$$\|e(u^{i+1})\|_1^2 = \|\mathrm{M}^{\nu_2} e(\tilde{\tilde{u}})\|_1^2 - \frac{(\mathrm{M}^{\nu_2} v, \mathrm{M}^{\nu_2} e(\tilde{\tilde{u}}))_1^2}{\|\mathrm{M}^{\nu_2} v\|_1^2}. \tag{4.34}$$

Because

$$\begin{aligned}\|\mathrm{M}^{\nu_2} e(\tilde{u})\|_1^2 &= \|\mathrm{M}^{\nu_2}[e(\tilde{\tilde{u}}) + v]\|_1^2 = \\ &= \|\mathrm{M}^{\nu_2} e(\tilde{\tilde{u}})\|_1^2 + 2(\mathrm{M}^{\nu_2} e(\tilde{\tilde{u}}), \mathrm{M}^{\nu_2} v)_1 + \|\mathrm{M}^{\nu_2} v\|_1^2 = \\ &= q_T^2 \|e(\tilde{\tilde{u}})\|_1^2 + 2(\mathrm{M}^{\nu_2} e(\tilde{\tilde{u}}), \mathrm{M}^{\nu_2} v)_1 + q_S^2 \|v\|_1^2 = \\ &= q_T^2 k^2 \|e(\tilde{u})\|_1^2 + 2(\mathrm{M}^{\nu_2} e(\tilde{\tilde{u}}), \mathrm{M}^{\nu_2} v)_1 + q_S^2(1-k^2)\|e(\tilde{u})\|_1^2\end{aligned}$$

and

$$\|\mathrm{M}^{\nu_2} e(\tilde{u})\|_1^2 = r^2 \|e(\tilde{u})\|_1^2$$

we arrive at

$$(\mathrm{M}^{\nu_2} e(\tilde{\tilde{u}}), \mathrm{M}^{\nu_2} v)_1 = \frac{1}{2}[r^2 - q_T^2 k^2 - q_S^2(1-k^2)]. \tag{4.35}$$

Substituting (4.35) in formula (4.34) we obtain (4.32). □

We denote

$$\bar{q}_T = \sup_{x \in T \setminus \{o\}} \varphi(x), \tag{4.36}$$

$$\bar{q}_S = \sup_{x \in S \setminus \{o\}} \varphi(x). \tag{4.37}$$

Since $e(\tilde{\tilde{u}}) \in T$, $v \in S$ we have

$$q_T \leqslant \bar{q}_T, \tag{4.38}$$

$$q_S \leqslant \bar{q}_S. \tag{4.39}$$

Hence $\bar{q}_T \leqslant q^{\nu_2}$, $\bar{q}_S \leqslant q^{\nu_2}$, $q < 1$ (see (4.24)).

Consider a real number $\bar{r} \in (\max\{\bar{q}_T, \bar{q}_S\}, 1)$ and define the set

$$\mathscr{A}(\bar{r}) = \{x \in H^1 \setminus \{0\} : \varphi(e(x)) \geqslant \bar{r}\}. \tag{4.40}$$

Note that $\tilde{u} \in \mathscr{A}(\bar{r})$ if and only if $r \geqslant \bar{r}$.

Theorem 2. *Let us suppose that all assumptions of Theorem 1 are valid and let a real number $\bar{r} \in (\max\{\bar{q}_T, \bar{q}_S\}, 1)$ be given. Then for any $u^i \in H^1$ such that*

$$\mathscr{S}^{(\nu_1)}(u^i) \in \mathscr{A}(\bar{r}) \tag{4.41}$$

the following error estimate holds:

$$\frac{\|e(u^{i+1})\|_1^2}{\|e(u^i)\|_1^2} \leqslant \sup_{h\in(0,1)} \left\{ \bar{q}_T^2 h^2 - \frac{[\bar{r}^2 - \bar{q}_T^2 h^2 - \bar{q}_S^2(1-h^2)]^2}{4\bar{q}_S^2(1-h^2)} \right\}. \tag{4.42}$$

P r o o f. Since $\tilde{u} \in \mathscr{A}(\bar{r})$ and $\bar{q}_T < \bar{r}$, $\bar{q}_S < \bar{r}$ we have $q_T < r$, $q_S < r$. Therefore $e(\tilde{u}) \notin T \cup S$, i.e. the assumptions of Lemma 4.5 hold and we obtain

$$\|e(u^{i+1})\|_1^2 = \left\{ q_T^2 k^2 - \frac{[r^2 - q_T^2 k^2 - q_S^2(1-k^2)]^2}{4q_S^2(1-k^2)} \right\} \|e(\tilde{u})\|_1^2$$

for some $k \in (0,1)$. Now, it is easy to see that (4.42) is true. □

We shall consider a special iterative method $\mathscr{S}(.)$ defined by

$$\mathscr{S}(x) = (\mathsf{I}^1 - \omega\mathsf{A})x + \omega f, \quad x \in H^1, \ \omega \in (0, 2/\rho(\mathsf{A})), \tag{4.43}$$

where $\rho(\mathsf{A})$ is the spectral radius of A.

R e m a r k 4.4. Put

$$\mathsf{M} = \mathsf{I}^1 - \omega\mathsf{A}, \tag{4.44}$$

$$\mathsf{N} = \omega\mathsf{I}^1. \tag{4.45}$$

Then the iterative method (4.43) can be written in the form (2.5). The condition (2.6) follows immediately from (4.44), (4.45). Moreover, it is easy to see that the iterative method (4.43) satisfies the condition (4.24) with the constant $q = \max\{|1-\omega\lambda_{\min}(\mathsf{A})|, |1-\omega\rho(\mathsf{A})|\} < 1$, where $\lambda_{\min}(\mathsf{A})$ is the least eigenvalue of A. Of course, all eigenvalues are real and positive numbers, because A is a symmetric and positive operator.

Lemma 4.6. *Let the iterative method $\mathscr{S}(.)$ be given by (4.43). Then the inequality*

$$\|\mathsf{M}^{\xi_1+\xi_2}x\|_1 \, \|x\|_1 \geqslant \|\mathsf{M}^{\xi_2}x\|_1 \, \|\mathsf{M}^{\xi_1}x\|_1 \tag{4.46}$$

holds for any integers $\xi_1 \geqslant 0$, $\xi_2 \geqslant 0$ and $x \in H^1$.

P r o o f. Let $\{\lambda_i\}_{i=1}^n (n = \dim(H^1))$ be the spectrum of A and $\{v_i\}_{i=1}^n$ the sequence of the corresponding eigenvectors such that $\|v_i\|_1 = \text{const}$, 1, 2, ..., n. The operator M (see (4.44)) has the same eigenvectors and the eigenvalues

$$\{1 - \omega\lambda_i\}_{i=1}^n. \tag{4.47}$$

The vector $x \in H^1$ can be decomposed as follows:

$$x = \sum_{i=1}^n \alpha_i v_i, \quad \alpha_i \in R, \ i = 1, 2 \ldots, n. \tag{4.48}$$

Then the inequality (4.46) can be written in the form

$$(4.49)\qquad \{\sum_{i=1}^{n}\lambda_i[(1-\omega\lambda_i)^{\xi_1+\xi_2}]^2\alpha_i^2\}\{\sum_{i=1}^{n}\lambda_i\alpha_i^2\} \geqslant$$
$$\geqslant \{\sum_{i=1}^{n}\lambda_i[(1-\omega\lambda_i)^{\xi_2}]^2\alpha_i^2\}\cdot\{\sum_{i=1}^{n}\lambda_i[(1-\omega\lambda_i)^{\xi_1}]^2\alpha_i^2\}.$$

Put

$$(4.50)\qquad b_i = \lambda_i\alpha_i^2,\quad i=1,2,\dots,n,$$
$$(4.51)\qquad a_i = 1-\omega\lambda_i,\quad i=1,2,\dots,n.$$

Of course, $b_i \geqslant 0$ for $i = 1, 2, \dots, n$. We shall prove that the inequality

$$(4.52)\qquad \Big(\sum_{i=1}^{n} b_i\Big)\Big(\sum_{i=1}^{n} b_i a_i^{2(\xi_1+\xi_2)}\Big) \geqslant \Big(\sum_{i=1}^{n} b_i a_i^{2\xi_2}\Big)\Big(\sum_{i=1}^{n} b_i a_i^{2\xi_1}\Big)$$

is valid for any sequence $\{b_i\}_{i=1}^{n}$, $b_i \geqslant 0$. Using elementary calculations we obtain

$$(4.53)\qquad \sum_{i,j=1}^{n} b_ib_ja_i^{2(\xi_1+\xi_2)} \geqslant \sum_{i,j=1}^{n} b_ib_ja_i^{2\xi_2}a_j^{2\xi_1}.$$

Since

$$\sum_{i,j=1}^{n} b_ib_ja_i^{2(\xi_1+\xi_2)} - \sum_{i=1}^{n} b_i^2a_i^{2(\xi_1+\xi_2)} = \sum_{i=1,j<i}^{n} b_ib_j(a_i^{2(\xi_1+\xi_2)} + a_j^{2(\xi_1+\xi_2)})$$

and

$$\sum_{i,j=1}^{n} b_ib_ja_i^{2\xi_2}a_j^{2\xi_1} - \sum_{i=1}^{n} b_i^2a_i^{2(\xi_1+\xi_2)} = \sum_{i=1,j<i}^{n} b_ib_j(a_i^{2\xi_1}a_j^{2\xi_2} + a_i^{2\xi_2}a_j^{2\xi_1})$$

we arrive at

$$(4.54)\qquad \sum_{i=1,j<i} b_ib_j(a_i^{2(\xi_1+\xi_2)} + a_j^{2(\xi_1+\xi_2)}) \geqslant \sum_{i=1,j<i}^{n} b_ib_j(a_i^{2\xi_1}a_j^{2\xi_2} + a_i^{2\xi_2}a_j^{2\xi_1}).$$

The inequality (4.54) holds for any $\{b_i\}_{i=1}^{n}$, $b_i \geqslant 0$, because we have

$$a_i^{2(\xi_1+\xi_2)} + a_j^{2(\xi_1+\xi_2)} - a_i^{2\xi_1}a_j^{2\xi_2} - a_i^{2\xi_2}a_j^{2\xi_1} = (a_i^{2\xi_2} - a_j^{2\xi_2})\cdot(a_i^{2\xi_1} - a_j^{2\xi_1}) \geqslant 0$$

for any sequence $\{a_i\}_{i=1}^{n}$.

Therefore inequalities (4.53), (4.52), (4.49) and (4.46) hold too. □

Theorem 3. *Let $\mathscr{S}(.)$ be given by (4.43), $\nu_1 \geqslant 0$, $\nu_2 = 1$, $\bar{r} \in (\max\{\bar{q}_T, \bar{q}_S\}, 1)$. Put*

$$G(\bar{r}, \bar{q}_T, \bar{q}_S) = \sup_{h \in (0,1)} \left\{ \bar{q}_T^2 h^2 - \frac{[\bar{r}^2 - \bar{q}_T^2 h^2 - \bar{q}_S^2(1-h^2)]^2}{4\bar{q}_S^2(1-h^2)} \right\}. \tag{4.55}$$

Then the error estimate

$$|||e(u^{i+1})|||_1^2 \leqslant \max\{q_T^2 \bar{r}^{2\nu_1}, G(\bar{r}, \bar{q}_T, \bar{q}_S)\}\ |||e(u^i)|||_1^2 \tag{4.56}$$

holds.

Proof. We shall consider the following cases:
(i) $\tilde{u} \in \mathscr{A}(\bar{r})$,
(ii) $e(\tilde{u}) = 0$,
(iii) $e(\tilde{u}) \neq 0$, $\tilde{u} \notin \mathscr{A}(\bar{r})$.

ad (i). According to Remark 4.4 the assumptions of Theorem 1 are satisfied. Since $\tilde{u} = \mathscr{S}^{(\nu_1)}(u^i) \in \mathscr{A}(\bar{r})$, the assumptions of Theorem 2 are fulfilled, too, and we immediately obtain

$$|||e(u^{i+1})|||_1^2 \leqslant G(\bar{r}, \bar{q}_T, \bar{q}_S)\ |||e(u^i)|||_1^2. \tag{4.57}$$

ad (ii). Since $e(\tilde{u}) = 0$, we have $e(\tilde{\tilde{u}}) = \mathsf{Q}e(\tilde{u}) = 0$. From (4.11) we arrive at $e(u^{i+1}) = 0$. Then (4.56) is trivially satisfied.

ad (iii). Since $\tilde{u} \notin \mathscr{A}(\bar{r})$ and $e(\tilde{u}) \neq 0$, we may write $\varphi(e(\tilde{u})) < r$. Put $x = \mathsf{M}^\xi e(u^i)$, $\xi_2 = 1$, $\xi_1 = \nu_1 - \xi$, where ξ is an integer, $0 \leqslant \xi \leqslant \nu_1$. On the basis of (4.46) we obtain

$$|||\mathsf{M}[\mathsf{M}^\xi e(u^i)]|||_1 \cdot |||\mathsf{M}^{\nu_1} e(u^i)|||_1 \leqslant |||\mathsf{M}[\mathsf{M}^{\nu_1} e(u^i)]|||_1 \cdot |||\mathsf{M}^\xi e(u^i)|||_1. \tag{4.58}$$

However, $e(\tilde{u}) = \mathsf{M}^{\nu_1} e(u^i) \neq 0$. Therefore $M^\xi e(u^i) \neq 0$ for any $\xi = 0, 1, \ldots, \nu_1$ and we may write

$$\frac{|||\mathsf{M}[\mathsf{M}^\xi e(u^i)]|||_1}{|||\mathsf{M}^\xi e(u^i)|||_1} \leqslant \frac{|||\mathsf{M}[\mathsf{M}^{\nu_1} e(u^i)]|||_1}{|||\mathsf{M}^{\nu_1} e(u^i)|||_1}, \tag{4.59}$$

$\xi = 0, 1, \ldots, \nu_1$. Since $\nu_2 = 1$, we have

$$\varphi(e(\tilde{u})) = \frac{|||\mathsf{M}[\mathsf{M}^{\nu_1} e(u^i)]|||_1}{|||\mathsf{M}^{\nu_1} e(u^i)|||_1} < \bar{r}. \tag{4.60}$$

We will write $|||\mathsf{M}^{\nu_1} e(u^i)|||_1$ in the form

$$|||\mathsf{M}^{\nu_1} e(u^i)|||_1 = \frac{|||\mathsf{M}[\mathsf{M}^{\nu_1 - 1} e(u^i)]|||_1}{|||\mathsf{M}^{\nu_1 - 1} e(u^i)|||_1} \cdot \frac{|||\mathsf{M}[\mathsf{M}^{\nu_1 - 2} e(u^i)]|||_1}{|||\mathsf{M}^{\nu_1 - 2} e(u^i)|||_1} \times \\ \times \ldots \times \frac{|||\mathsf{M}[\mathsf{M}^0 e(u^i)]|||_1}{|||\mathsf{M}^0 e(u^i)|||_1}\ |||e(u^i)|||_1. \tag{4.61}$$

From (4.59), (4.60), (4.61), and (4.7) we obtain

$$(4.62) \qquad \| e(\tilde{u}) \|_1 < \bar{r}^{\nu_1} \| e(u^i) \|_1.$$

By virtue of (4.12) we have

$$(4.63) \qquad \| e(u^{i+1}) \|_1 \leqslant \| \mathsf{M}^{\nu_2} e(\tilde{\tilde{u}}) \|_1.$$

Further

$$(4.64) \qquad \| e(\tilde{\tilde{u}}) \|_1 \leqslant \| e(\tilde{u}) \|_1,$$

$$(4.65) \qquad \| \mathsf{M} e(\tilde{\tilde{u}}) \|_1 \leqslant \bar{q}_T \| e(\tilde{\tilde{u}}) \|_1.$$

From (4.62), (4.63), (4.64), (4.65) we immediately obtain

$$(4.66) \qquad \| e(u^{i+1}) \|_1 < \bar{q}_T \bar{r}^{\nu_1} \| e(u^i) \|_1.$$

Therefore (4.55) holds. □

5. A MODEL EXAMPLE

Let $\bar{m}$ be an integer. Put $\bar{n} = 3\bar{m}$, $m = \bar{m} - 1$, $n = \bar{n} - 1$,

$$(5.1) \qquad H^1 = E^n,$$

$$(5.2) \qquad H^2 = E^m,$$

where E^k is the k-dimensional Euclidean space.

We consider the problem (1.1), where A is the $n \times n$ matrix defined as follows

$$(5.3) \qquad \mathsf{A} = \begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

Define the prolongation

$$(5.4) \qquad \mathsf{p} = \frac{1}{3} \begin{pmatrix} 0 & & & \\ 1 & & & \\ 1 & & & \\ 1 & & & \\ & 1 & & \\ & 1 & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \\ & & & 1 \\ & & & 1 \\ & & & 0 \end{pmatrix} \qquad \text{(of type } n \times m\text{)}.$$

According to (1.3) we have

$$r = \frac{1}{3}\begin{pmatrix} 0 & 1 & 1 & 1 & & & & & & & \\ & & & & 1 & 1 & 1 & & & & \\ & & & & & & & \cdots & & & \\ & & & & & & & & 1 & 1 & 1 & 0 \end{pmatrix}$$

(of type $m \times n$). Let $\mathscr{S}(.)$ be given by (4.43), $\omega = \frac{1}{3}$, i.e.

$$\mathscr{S}(x) = (\mathsf{I}^1 - \frac{1}{3}\mathsf{A})x + \frac{1}{3}f, \quad x \in E^n, \tag{5.6}$$

where I^1 is the identity $n \times n$ matrix. From (1.4) we obtain

$$^2\mathsf{A} = \frac{1}{9}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \quad \text{(of type } m \times m\text{)}. \tag{5.7}$$

The proof of the following identity may be found in the paper [1]:

$$\sup_{\bar{m}} |||\mathsf{MQ}|||_1^2 = \frac{2}{3}. \tag{5.8}$$

(see[1]: (5.6a), (5.1), (5.2), (2.10a), (2.8), (2.1). According to Lemma 4.3 and (4.36) we have

$$q_T^2 \leqslant \frac{2}{3} \text{ for any integer } \bar{m} \tag{5.9}$$

It is well-known that the eigenvectors of A are the vectors with the entries

$$(v_i)_j = \sin\left(\frac{ij\pi}{\bar{n}}\right), \quad i,j = 1,2,\ldots,n \tag{5.10}$$

and the corresponding eigenvalues are

$$\lambda_i = 4\sin^2\left(\frac{i\pi}{2\bar{n}}\right), \quad i = 1,2,\ldots,n. \tag{5.11}$$

the eigenvectors v_i^2 of $^2\mathsf{A}$ are the vectors with the entries

$$(v_i^2)_j = \sin\left(\frac{ij\pi}{\bar{m}}\right), \quad i,j = 1,2,\ldots,m \tag{5.12}$$

and the spectrum of $^2\mathsf{A}$ is

$$\lambda_i = \frac{4}{9}\sin^2\left(\frac{i\pi}{2\bar{m}}\right), \quad i = 1, 2, \ldots, m. \tag{5.13}$$

Lemma 5.1. *The following equality is valid:*

$$\mathsf{p}v_i^2 = \frac{1}{9}(c_i v_i - c_{2\bar{m}-i} v_{2\bar{m}-i} + c_{2\bar{m}+i} v_{2\bar{m}+i}), \quad i = 1, 2, \ldots, m, \tag{5.14}$$

where

$$c_k = 1 + 2\cos\left(\frac{k\pi}{\bar{n}}\right), \quad k = 1, 2, \ldots, n. \tag{5.15}$$

For the proof see [1].

Lemma 5.2. *The vectors* $\mathsf{M}\mathsf{p}v_i^2$, $\mathsf{M}\mathsf{p}v_j^2$, $i, j = 1, 2, \ldots, m$, $i \neq j$ *are* A-*orthogonal.*

Proof. Since A, M have the same eigenvectors, the proof follows from Lemma 5.1 immediately. □

Lemma 5.3. *The following equality holds:*

$$\bar{q}_S = \max_{i=1,\ldots,m} \frac{|||\mathsf{M}\mathsf{p}v_i^2|||_1}{|||\mathsf{p}v_i^2|||_1}. \tag{5.16}$$

Proof. An arbitrary vector $x \in S$ can be written as

$$x = \sum_{i=1}^{m} a_i \mathsf{p}v_i^2, \quad a_i \in R, \; i = 1, 2, \ldots, m. \tag{5.17}$$

According to Lemma 5.2 we arrive at

$$|||\mathsf{M}x|||_1^2 = \sum_{i=1}^{m} a_i^2 |||\mathsf{M}\mathsf{p}v_i^2|||_1^2. \tag{5.18}$$

Therefore we have

$$|||\mathsf{M}x|||_1^2 \leqslant \left(\max_{i=1,\ldots,m} \frac{|||\mathsf{M}\mathsf{p}v_i^2|||_1}{|||\mathsf{p}v_i^2|||_1^2}\right) \cdot \sum_{i=1}^{m} a_i^2 |||\mathsf{p}v_i^2|||_1^2. \tag{5.19}$$

Since

$$|||x|||_1^2 = \sum_{i=1}^{m} a_i^2 |||\mathsf{p}v_i^2|||_1^2 \tag{5.20}$$

we obtain

$$\frac{|||\mathsf{M}x|||_1^2}{|||x|||_1^2} \leqslant \max_{i=1,\ldots,m} \frac{|||\mathsf{M}\mathsf{p}v_i^2|||_1^2}{\mathsf{p}v_i^2|||_1}. \tag{5.21}$$

Consider $j \in \{1, 2, \ldots, m\}$ such that

$$\frac{|||\mathsf{M}\mathsf{p}v_j^2|||_1^2}{|||\mathsf{p}v_j^2|||_1^2} = \max_{i=1,\ldots,m} \frac{|||\mathsf{M}\mathsf{p}v_i^2|||_1^2}{|||\mathsf{p}v_i^2|||_1^2} \tag{5.22}$$

and put $x = \mathsf{p}v_j^2$. Then (5.21), (5.22) imply (5.16). □

Lemma 5.4. *The equality*

$$\bar{q}_S^2 = \frac{1}{3} \tag{5.23}$$

holds for any integer $\bar{m}$.

Proof. It is easy to prove that

$$|||\mathrm{p}v_i^2|||_1 = |||v_i^2|||_2, \quad i = 1, 2, \dots, m, \tag{5.24}$$

$$||v_i||_1^2 = 2\bar{n}, \quad i = 1, 2, \dots, n, \tag{5.25}$$

$$||v_i^2||_2^2 = 2\bar{m}, \quad i = 1, 2, \dots, m, \tag{5.26}$$

$$\frac{c_i^2\lambda_i}{9\lambda_i^2} = \frac{c_{2\bar{m}-i}^2\lambda_{2\bar{m}-i}}{9\lambda_i^2} = \frac{c_{2\bar{m}+i}^2\lambda_{2\bar{m}+i}}{9\lambda_i^2} = 1 \tag{5.27}$$

$$(1 - \frac{1}{3}\lambda_i)^2 + (1 - \frac{1}{3}\lambda_{2\bar{m}-i})^2 + (1 - \frac{1}{3}\lambda_{2\bar{m}+i})^2 = 1. \tag{5.28}$$

According to Lemma 5.1, (5.24), (5.25) and (5.26) we arrive at

$$\frac{|||\mathrm{Mp}v_i^2|||_1^2}{|||\mathrm{p}v_i^2|||_1^2} = \frac{1}{27\lambda_i^2}[c_i^2(1 - \frac{1}{3}\lambda_i)^2\lambda_i + \\ + c_{2\bar{m}-i}^2(1 - \frac{1}{3}\lambda_{2\bar{m}-i})^2\lambda_{2\bar{m}-i} + c_{2\bar{m}+i}^2(1 - \frac{1}{3}\lambda_{2\bar{m}+i})^2\lambda_{2\bar{m}+i}]. \tag{5.29}$$

Using (5.27), (5.28) we obtain

$$\frac{|||\mathrm{Mp}v_i^2|||_1^2}{|||\mathrm{p}v_i^2|||_1^2} = \frac{1}{3}, \quad i = 1, 2, \dots, m. \tag{5.30}$$

From (5.16) and (5.30) we conclude (5.23). □

Since we know that $\bar{q}_T^2 \leqslant \frac{2}{3}$ and $\bar{q}_S^2 = \frac{1}{3}$ for any integer $\bar{m}$, we can use Theorem 2 and Theorem 3 for the error estimate of the iterative process given by algorithm (3.1). In both cases we need to know the value $G(\bar{r}, \bar{q}_T, \bar{q}_S)$ (see (4.55), (4.56) and (4.42)).

TABLE 1. (Theoretical results, independent of $\bar{m}$.)

$\bar{r}^2$	$G(\bar{r}, \bar{q}_T, \bar{q}_S)$
0.85	0.300
0.90	0.200
0.95	0.100
0.98	0.040
0.99	0.020

TABLE 2. (Numerical results, $\bar{n} = 900$, $\bar{m} = 300$.)

		Algorithm(2.7)	Algorithm (3.1)	
	i	$\lVert e(u^i)\rVert_1^2$	$\lVert e(u^i)\rVert_1^2$	$\hat{t}$
	0	0.547631×10^{-2}	0.547631×10^{-2}	———
$\nu_1 = 3$	1	0.251916×10^{-2}	0.138005×10^{-4}	1.97929
	2	0.110962×10^{-2}	0.170185×10^{-6}	1.02789
$\nu_2 = 1$	3	0.488981×10^{-2}	0.194823×10^{-8}	1.85861
	4	0.215570×10^{-3}	0.299069×10^{-10}	1.01789
	0	0.547631×10^{-2}	0.547631×10^{-2}	———
$\nu_1 = 3$	1	0.242473×10^{-2}	0.138005×10^{-4}	1.97929
	2	0.107420×10^{-2}	0.170185×10^{-6}	1.02789
$\nu_2 = 3$	3	0.476081×10^{-3}	0.194823×10^{-8}	1.85861
	4	0.211079×10^{-3}	0.299069×10^{-10}	1.01789
	0	0.547631×10^{-2}	0.547631×10^{-2}	———
$\nu_1 = 5$	1	0.242466×10^{-2}	0.138001×10^{-4}	1.97929
	2	0.107398×10^{-2}	0.141713×10^{-6}	1.12114
$\nu_2 = 3$	3	0.475879×10^{-3}	0.512204×10^{-9}	1.97739
	4	0.210943×10^{-4}	0.396811×10^{-11}	1.07900

References

[1] *R. Blaheta*: A multilevel method with correction by aggregation for solving discrete elliptic problems, Apl. Mat. *31* (1986), 365-378.

[2] *R. Blaheta*: Iterative methods for numerical solving of the boundary value problems of elasticity, Thesis, Ostrava, 1987.

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Souhrn

MODIFIKACE DVOJÚROVŇOVÉHO ALGORITMU SE SUPERKOREKCÍ

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V práci je navržen a analyzován algoritmus, který je modifikací tzv. dvojúrovňového algoritmu se superkorekcí, který byl publikován v práci [2]. Účinnost je ilustrována na modelovém příkladě.

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