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AFFINE-INVARIANT MONOTONE ITERATION METHODS WITH APPLICATION TO SYSTEMS OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

RUDOLF L. VOLLER

Summary. In this paper we present a new theorem for monotone including iteration methods. The conditions for the operators considered are affine-invariant and no topological properties neither of the linear spaces nor of the operators are used. Furthermore, no inverse-isotony is demanded. As examples we treat some systems of nonlinear ordinary differential equations with two-point boundary conditions.

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1. INTRODUCTION

Given the equation

$$F(u) = 0 \tag{1}$$

where F is an (in general nonlinear) operator acting from a subset D of a partially ordered space V into a real linear space W, we investigate Newton-like iteration schemes of the form

$$\begin{aligned} S_{1k}(v_{k+1} - v_k) &= -F(v_k) \\ S_{2k}(u_{k+1} - u_k) &= -F(u_k), \end{aligned}$$

to obtain sequences $\{u_k\}_{k\in\mathbb{N}_0}$, $\{v_k\}_{k\in\mathbb{N}_0}$ such that starting with two elements $u_0 \leqslant v_0$ we have for all $k \in \mathbb{N}_0$: $u_k \leqslant u_{k+1} \leqslant \cdots \leqslant v_{k+1} \leqslant v_k$, so that for all $k \in \mathbb{N}$ and every solution u^* of $F(u) = 0$ from $[u_0, v_0]$: $u_k \leqslant u^* \leqslant v_k$. Here the linearizations S_{ik}, $i = 1, 2$ depend on u_k and/or v_k. In earlier papers it was assumed (comp. [12], [13]) that the S_{ik} are inverse-isotone or have at least positive left subinverses B_{ik}. These assumptions are weakened in Theorem 2 for the methods from [12] and [13] similar as in [7], [8], so that any monotonicity assumptions are superfluous. Furthermore the new conditions are affine-invariant, so that they are as in [9] fulfilled for AF, iff they are fulfilled for F, where A is a linear, bijective mapping from W into V. Beside the results in [7] we can not only drop the inverse-isotony of the S_{ik} or the positivity of their left subinverses, but also the inverse-isotony of the mappings $B_{ik}S_{ik}$. Since the theorem will be proved for a partially ordered space V we also do not use any property depending on the topology neither of F nor the S_{ik} nor of V nor W.

In Section 2 we demonstrate how the results can be applied to certain systems of weakly nonlinear two-point boundary value problems. First a system is considered where the discretization by finite differences leads to a nonlinear system of equations whose "natural" linearization is not inverse-isotone, so that the theorems from [1], [10], [11], [12], [13] and [14] are not applicable. Secondly a predator-prey-model is considered.

2. Affine-invariant monotone iteration methods in partially ordered linear spaces

In this section we first present some definitions for operators acting on partially ordered spaces which are used in the following. For a detailed discussion see [4].

Definition 1. Let V be a linear space over $\mathbf{R}$. Then V is called a *partially ordered linear space* (POS), iff there exists a reflexive, antisymmetric and transitive relation $\leqslant$ on $V \times V$ with the following properties:

$$\forall x, y, z \in V: \; x \leqslant y \Rightarrow x + z \leqslant y + z$$
$$\forall x, y \in V \; \forall \lambda \in \mathbf{R}_+: \; x \leqslant y \Rightarrow \lambda x \leqslant \lambda y$$

Definition 1. $x \geqslant y: \Leftrightarrow y \leqslant x$, $x < y: \Leftrightarrow x \leqslant y$ and $x \neq y$ and $x > y: \Leftrightarrow y < x$.

Definition 1. Let V be a POS. Then the set $K_V := \{x \in V \mid 0 \leqslant x\}$ is called the *cone* of V with respect to $\leqslant$.

Definition 1. Let X be a real Banach space. Then X is called a *partially ordered Banach space* (POB), iff X is a POS and K_X is closed with respect to the norm topology.

Definition 1. Let V be a POS and X a POB.

(i) Let $x, y \in V$. The subset $\{z \in V \mid x \leqslant z \leqslant y\}$ of V is called an *order-interval* and is denoted by $[x, y]$.

(ii) $D \subset V$ is called *o-bounded*, iff there exists two elements $x, y \in V$, such that $\forall z \in D: \; x \leqslant z \leqslant y$ (analogously o-bounded from above or from below).

(iii) K_V is called (*strongly*) *minihedral*, iff every pair of elements of V (each subset of V, which is o-bounded from above) has a supremum.

(iv) K_X is called *normal*, iff $\exists C \in \mathbf{R}_+$: $\forall x, y \in K_X : \; x \leqslant y \Rightarrow \|x\| \leqslant C\|y\|$.

(v) K_X is called *regular*, iff each o-bounded, monotone sequence in X is convergent.

R e m a r k . Each regular cone is normal.

Definition 1. Let V, W be POS and $A\colon D \subset V \to W$ an operator. Then A is called

$$\begin{aligned} isotone &: \Leftrightarrow x, y \in D,\ x \leqslant y \Rightarrow A(x) \leqslant A(y) \\ inverse - isotone &: \Leftrightarrow x, y \in D,\ A(x) \leqslant A(y) \Rightarrow x \leqslant y \\ positive &: \Leftrightarrow x \in D \cap K_V \Rightarrow 0 \leqslant A(x). \end{aligned}$$

R e m a r k . For linear operators isotone and positive are equivalent properties. Each inverse-isotone operator is injective.

Furthermore we need the following definitions:

Definition 1. Let F be an operator from a POS V into a real linear space W. Then any operator $F_l^{-1}\colon W \to V$ having the property

$$\forall x \in V: \ F_l^{-1}F(x) \leqslant x$$

is called a *left subinverse* of F. Analogously a right subinverse, superinverse etc. are defined.

Definition 1. Let V, W be linear spaces and $F\colon D \subset V \to W$ an operator. A mapping $S\colon D \times D \to L(V, W)$ is called a *divided difference operator* (ddo) for F, iff

$$\forall x, y \in D: \ S(x, y)(x - y) = F(x) - F(y).$$

If V and W are POS and

$$\forall x, y \in D \text{ with } x \leqslant y \text{ or } y \leqslant x: \ S(x, y)(x - y) \leqslant F(x) - F(y) \tag{2}$$

holds, then S is called a *generalized divided difference operator* (gddo). If (2) is only true for $x \leqslant y$, then S is called a *quasi-divided-difference-operator* (qddo).

Definition 1. Let V, W be POS, D a convex subset of V. Then an operator $F\colon D \to W$ is called *convex*, iff

$$\forall x, y \in D,\ \forall t \in (0, 1): \ F(tx + (1 - t)y) \leqslant tF(x) + (1 - t)F(y)$$

and *o-convex*, iff this inequality holds only in the case $x \leqslant y$ or $y \leqslant x$.

If F has a ddo S and

$$\forall x, y, z \in D \text{ with } x \leqslant y \leqslant z: \ S(x, y) \leqslant S(x, z) \leqslant S(y, z),$$

then F is called *ddo-convex*.

For a further discussion see [12]. In the following it is important to know, that *ddo*-convex operators and operators with an isotone *gddo* are o-convex (comp. [12], 1.5 Lemma 3), while operators with a *qddo* are not necessarily o-convex.

Finally we need the following theorem about solvability of the fixed-point-equation and successive approximation for isotone operators

Theorem 1 ([4]). *Let V be a POS and $A\colon D \subset V \to V$ an operator. Furthermore we may have elements $x_0, y_0 \in D$ with $[x_0, y_0] \subset D$ and $x_0 \leqslant A(x_0)$, $A(y_0) \leqslant y_0$. Let $A|_{[x_0,y_0]}$ be isotone and let one of the following three conditions be true:*

(i) *The cone of V is strongly minihedral.*

(ii) *V is a POB with a regular cone, and A is continuous.*

(iii) *V is a POB with a normal cone, and A is completely continuous.*

Then A has at least one fixed point $x^* \in [x_0, y_0]$ and for the sequences $\{x_n\}_{n\in\mathbb{N}_0}$, $\{y_n\}_{n\in\mathbb{N}_0}$ determined by

$$x_{n+1} = A(x_n), \quad y_{n+1} = A(y_n), \quad n \in \mathbb{N}_0$$

we have

$$x_n \leqslant x_{n+1} \leqslant \cdots \leqslant x^* \leqslant \cdots \leqslant y_{n+1} \leqslant y_n.$$

If (ii) *or* (iii) *holds, both sequences converge against fixed points of A.*

Now we are able to prove the main theorem for monotone including iteration-methods, which we obtain by linearizing the operator equations considered, and thus we call them "Newton-like", which also means that they converge faster than the iteration of Theorem 1.

Theorem 2. *Given an equation* (1) *let F be a (in general nonlinear) operator acting from a subset D of a partially ordered space V into a real linear space W, and let two elements $u_0 \leqslant v_0$ be given with $[u_0, v_0] \subset D$. The cone of V is supposed to be strongly minihedral. Furthermore let linearizations $S_i\colon [u_0, v_0] \times [u_0, v_0] \to L(V, W)$ be given, such that for all $u, v \in [u_0, v_0]$ with $u \leqslant v$ $S_i(u, v)$ has a left subinverse $B_i(u, v) \in L(W, V)$ with the property $B_i(u, v)w = 0 \Rightarrow w = 0$, $i = 1, 2$.*

Let at last one of the following groups of inequalities I–IV *hold:*

$$\begin{aligned}
\text{I. } & B_1(v_0, v_0)F(u_0) \leqslant 0 \leqslant B_1(v_0, v_0)F(v_0) \\
& B_2(u_0, v_0)F(u_0) \leqslant 0 \leqslant B_2(u_0, v_0)F(v_0) \\
& \forall u, v \in [u_0, v_0],\ u \leqslant v,\ z \in \{u, v\}: \\
& \qquad 0 \leqslant B_1(z, z)[F(u) - F(v) - S_1(v, v)(u - v)] \\
& \forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v: \\
& \qquad 0 \leqslant B_1(z, z)[F(u) - F(w) - S_2(u, v)(u - w)] \\
& \forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v,\ z \in \{w, v\}:
\end{aligned}$$

$$0 \leqslant B_2(u,v)[F(u) - F(z) - S_2(u,v)(u-w)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_2(w,z)[F(u) - F(w) - S_2(u,v)(u-w)]$$

$\forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v:$

$$0 \leqslant B_2(u,z)[F(z) - F(v) - S_1(v,v)(z-v)]$$

II. $B_1(u_0, v_0)F(u_0) \leqslant 0 \leqslant B_1(u_0, v_0)F(v_0)$

$B_2(u_0, u_0)F(u_0) \leqslant 0 \leqslant B_2(u_0, u_0)F(v_0)$

$\forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v,\ w \in \{u, z\}:$

$$0 \leqslant B_1(u,v)[F(w) - F(v) - S_1(u,v)(z-v)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_1(w,z)[F(z) - F(v) - S_1(u,v)(z-v)]$$

$\forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v:$

$$0 \leqslant B_1(z,v)[F(u) - F(z) - S_2(u,u)(u-z)]$$

$\forall u, v \in [u_0, v_0],\ u \leqslant v,\ z \in \{u, v\}:$

$$0 \leqslant B_2(z,z)[F(u) - F(v) - S_2(u,u)(u-v)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_2(w,w)[F(z) - F(v) - S_1(u,v)(z-v)]$$

III. $B_i(u_0, v_0)F(u_0) \leqslant 0 \leqslant B_i(u_0, v_0)F(v_0), i = 1, 2$

$\forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v:$

$$0 \leqslant B_1(u,v)[F(w) - F(v) - S_1(u,v)(w-v)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_1(w,z)[F(z) - F(v) - S_1(u,v)(z-v)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_1(w,z)[F(u) - F(w) - S_2(u,v)(u-w)]$$

$\forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v,\ z \in \{w, v\}:$

$$0 \leqslant B_2(u,v)[F(u) - F(z) - S_2(u,v)(u-w)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_2(w,z)[F(u) - F(w) - S_2(u,v)(u-w)]$$

$\forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v:$

$$0 \leqslant B_2(w,z)[F(z) - F(v) - S_1(u,v)(z-v)]$$

IV. $B_i(z_i, z_i)F(u_0) \leqslant 0 \leqslant B_i(z_i, z_i)F(v_0), i = 1, 2,\ z_1 = v_0, z_2 = u_0$

$\forall u, v \in [u_0, v_0],\ u \leqslant v,\ z \in \{u, v\}:$

$$0 \leqslant B_1(z,z)[F(u) - F(v) - S_1(v,v)(u-v)]$$

$\forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v:$

$$
\begin{aligned}
&\quad 0 \leqslant B_1(v,v)[F(u) - F(z) - S_2(u,u)(u-z)] \\
&\forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v,\ z \in \{u, v\}: \\
&\quad 0 \leqslant B_2(z,z)[F(u) - F(v) - S_2(u,u)(u-w)] \\
&\forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v: \\
&\quad 0 \leqslant B_2(u,u)[F(z) - F(v) - S_1(v,v)(z-v)]
\end{aligned}
$$

Then the iteration processes

$$S_{1k}(v_{k+1} - v_k) = -F(v_k) \tag{3}$$

$$S_{2k}(u_{k+1} - u_k) = -F(u_k) \tag{4}$$

where corresponding to our assumptions we have, respectively

$$
\begin{aligned}
\text{I. } S_{1k} &:= S_1(v_k, v_k),\ S_{2k} := S_2(u_k, v_k) \\
\text{II. } S_{1k} &:= S_1(u_k, v_k),\ S_{2k} := S_2(u_k, u_k) \\
\text{III. } S_{1k} &:= S_1(u_k, v_k),\ S_{2k} := S_2(u_k, v_k) \\
\text{IV. } S_{1k} &:= S_1(v_k, v_k),\ S_{2k} := S_2(u_k, u_k)
\end{aligned}
$$

starting with u_0 and v_0 give us monotone including sequences $\{u_k\}_{k\in\mathbf{N}_0}$, $\{v_k\}_{k\in\mathbf{N}_0}$, such that for all $k \in \mathbf{N}$: $u_k \leqslant u_{k+1} \leqslant \cdots \leqslant v_{k+1} \leqslant v_k$ and for all $k \in \mathbf{N}$ and every solution u^* of $F(u) = 0$ from $[u_0, v_0]$: $u_k \leqslant u^* \leqslant v_k$.

Proof. We prove our assertions for the case I., all other cases can be shown analogously, where in case II. first equation (4) and then (3) has to be solved.

The proof is done by induction. Since the first step is proved exactly as the step from k to $k+1$, we restrict ourselves to the latter one.

First we have to define operators $T_{ik}: V \to V$, $i = 1, 2$ by

$$
\begin{aligned}
T_{1k}h &:= h + B_{1k}[F(v_k) - S_{1k}h] \\
T_{2k}h &:= h - B_{2k}[F(u_k) + S_{2k}h]
\end{aligned}
$$

where B_{ik} are the left subinverses of the S_{ik}, $i = 1, 2$. Then we show by Theorem 1 that they have at least one fixed-point.

Because of I. we have $T_{1k}0 = B_{1k}[F(v_k)] \geqslant 0$ and $T_{1k}(v_k - u_k) = v_k - u_k + B_{1k}[F(v_k) - S_{1k}(v_k - u_k)] = v_k - u_k - B_{1k}[F(u_k) - F(v_k) - S_{1k}(u_k - v_k)] + B_{1k}F(u_k) \leqslant v_k - u_k$ and furthermore $T_{1k}(v + dv) = T_{1k}v + dv - B_{1k}S_{1k}dv \geqslant TK_{1k}v$ for $dv \geqslant 0$. Thus T_{1k} fulfils the conditions of Theorem 1 and therefore has at least one fixed-point in $[0, v_k - u_k]$. Since the cone of V is strongly minihedral, the set $\{z \in [0, v_k - u_k] \mid T_{1k}z \leqslant z\}$, which is nonempty, since it contains the fixed-point, has an infimum $z_{1k} \geqslant 0$. Because $T_{1k}z_{1k} \leqslant T_{1k}z \leqslant z$ for all z from this set, $T_{1k}z_{1k}$ is also a lower bound of this set, which means $T_{1k}z_{1k} \leqslant z_{1k}$, so that z_{1k} lies in this set, but then also

$T_{1k}z_{1k}$, because $T_{1k}(T_{1k}z_{1k}) \leqslant T_{1k}z_{1k}$ and T_{1k} is isotone. So we have $z_{1k} \leqslant T_{1k}z_{1k}$, so that z_{1k} is the lowest fixed-point of T_{1k} in $[0, v_k - u_k]$.

By the definiton of T_{1k} it follows that $B_{1k}[F(v_k) + S_{1k}z_{1k}] = 0$ and from the assumption about the kernel of B_{1k}: $S_{1k}z_{1k} = -F(v_k)$. Thus we define $v_{k+1} := v_k - z_{1k}$ and obtain by $0 \leqslant z_{1k} \leqslant v_k - u_k$:

$$u_k \leqslant v_{k+1} \leqslant v_k.$$

Furthermore, because of I. we have $T_{2k}0 = -B_{2k}[F(u_k)] \geqslant 0$ and $T_{2k}(v_{k+1} - u_k) = v_{k+1} - u_k - B_{2k}[F(u_k) + S_{2k}(v_{k+1} - u_k)] = v_{k+1} - u_k - B_{2k}[F(u_k) - F(v_k) - S_{2k}(u_k - v_{k+1})] - B_{2k}F(v_k) \leqslant v_{k+1} - u_k$ as well as $T_{2k}(v + dv) = T_{2k}v + dv - B_{2k}S_{2k}dv \geqslant T_{2k}v$ for $dv \geqslant 0$. Then also T_{2k} fulfils the conditions of Theorem 1 and has a fixed-point in $[0, v_{k+1} - u_k]$. Like T_{1k} also T_{2k} has a lowest fixed-point z_{2k} in $[0, v_{k+1} - u_k]$ (which can be proved analogously).

From the definiton of T_{2k} it follows that $B_{2k}[F(u_k) - S_{2k}z_{2k}] = 0$ and because of the assumption about the kernel of B_{2k} : $S_{2k}z_{2k} = F(u_k)$. We define $u_{k+1} := u_k + z_{2k}$ and obtain because of $0 \leqslant z_{2k} \leqslant v_{k+1} - u_k$:

$$u_k \leqslant u_{k+1} \leqslant v_{k+1} \leqslant v_k. \tag{5}$$

From

$$B_{1k+1}F(u_{k+1}) = -B_{1k+1}[F(u_k) - F(u_{k+1}) - S_{2k}(u_k - u_{k+1})] \leqslant 0$$

and

$$B_{1k+1}F(v_{k+1}) = B_{1k+1}[F(v_{k+1}) - F(v_k) - S_{1k}(v_{k+1} - v_k)] \geqslant 0$$

as well as

$$B_{2k+1}F(u_{k+1}) = -B_{2k+1}[F(u_k) - F(u_{k+1}) - S_{2k}(u_k - u_{k+1})] \leqslant 0$$

and

$$B_{2k+1}F(v_{k+1}) = B_{2k+1}[F(v_{k+1}) - F(v_k) - S_{1k}(v_{k+1} - v_k)] \geqslant 0$$

it follows finally that

$$B_{1k+1}F(u_{k+1}) \leqslant 0 \leqslant B_{1k+1}F(v_{k+1})$$

and

$$B_{2k+1}F(u_{k+1}) \leqslant 0 \leqslant B_{2k+1}F(v_{k+1}).$$

If u^* is a solution of $F(u) = 0$ from $[u_k, v_k]$, it yields

$$\begin{aligned} T_{1k}(v_k - u^*) &= v_k - u^* + B_{1k}[F(v_k) - S_{1k}(v_k - u^*)] \\ &\leqslant v_k - u^* - B_{1k}[F(u^*) - F(v_k) - S_{1k}(u^* - v_k)] \\ &\leqslant v_k - u^*, \end{aligned}$$

so that $v_k - u^* \geqslant z_{1k}$. But from this yields

$$u^* \leqslant v_{k+1}.$$

Analogously one shows

$$u_{k+1} \leqslant u^*,$$

and all assertions of the theorem are proved. □

Remark 1. The condition, that K_V has to be strongly minihedral, can be replaced by one of the following assumptions

V is a POB and W a real Banach space, K_V is regular. F, B_{ik} and S_{ik} are for $i = 1, 2$ and all $k \in \mathbb{N}_0$ continuous and the composed mappings $B_{ik}S_{ik}$ are inverse-isotone (comp. [8]).

V is a POB and W a real Banach space, K_V is normal. F and S_{ik} are for $i = 1, 2$ and all $k \in \mathbb{N}_0$ continuous and B_{ik} completely continuous and the composed mappings $B_{ik}S_{ik}$ are inverse-isotone (comp. [8]).

Remark 2. From [12] or [13] we have
If V is a POB with a normal cone, then we have the error estimate

$$\forall k \in \mathbb{N}_0 : \ \max\{\|u^* - u_k\|, \|u^* - v_k\|\} \leqslant C\|u_k - v_k\|.$$

If K_V is regular, then the sequences $\{u_k\}_{k\in\mathbb{N}_0}, \{v_k\}_{k\in\mathbb{N}_0}$ are convergent.

If F is continuous and the S_{ik} are uniformly bounded with respect to k, then the limits of the sequences are solutions of the equation $F(u) = 0$.

Remark 3. The order of convergence of our iteration methods is discussed in [8] and [12].

The results of Remark 1 were obtained by Potra in [7] for a certain class of iteration methods including the cases I. for (3), II. for (4) and IV. for (3) and (4). Potra assumes that $B_1 = B_2$ and $S_1 = S_2$, so that the same linear operator can be used to compute both iterated sequences. Then some of the conditions in I.–IV. can be simplified. This is also possible in our case III., which is not contained in [7] and which is treated in [8] under stronger conditions than in Theorem 2. By this way the cpu-time used to solve (3) and (4) is minimized in many cases. On a parallel computer this is of no importance, moreover the convergence can be improved for case I. if I. is replaced by

$$\text{I}^*.\ S_{2k} = S_2(u_k, v_{k+1})$$

and for II. by

$$\text{II}^*.\ S_{1k} = S_1(u_{k+1}, v_k).$$

In these cases one first computes v_1 or respectively, u_1, and afterwards simultaneously u_k and v_{k+1} or, respectively, v_k and u_{k+1}. The assumptions I. and II. have to be modified as follows, where the second condition in I. for u_0 becomes stronger and for v_0 is weakened as well as the fourth and the fifth condition (similar for II.).

Corollary 1. *Let F, D, V, W as well as S_i and B_i, $i = 1, 2$ be given as in Theorem 2. Instead of* I. *and* II. *we suppose that:*

$$
\begin{aligned}
\mathrm{I}^*.\ & B_1(v_0, v_0)F(u_0) \leqslant 0 \leqslant B_1(v_0, v_0)F(v_0) \\
& \forall z \in [u_0, v_0]:\ B_2(u_0, z)F(u_0) \leqslant 0 \\
& \forall u, v \in [u_0, v_0],\ u \leqslant v,\ z \in \{u, v\}: \\
& \quad 0 \leqslant B_1(z, z)[F(u) - F(v) - S_1(v, v)(u - v)] \\
& \forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v: \\
& \quad 0 \leqslant B_1(v, v)[F(u) - F(w) - S_2(u, v)(u - w)] \\
& \forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v: \\
& \quad 0 \leqslant B_2(u, v)[F(u) - F(w) - S_2(u, v)(u - w)] \\
& \forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v: \\
& \quad 0 \leqslant B_2(w, z)[F(u) - F(w) - S_2(u, v)(u - w)] \\
& \forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v: \\
& \quad 0 \leqslant B_2(u, z)[F(z) - F(v) - S_1(v, v)(z - v)] \\
\mathrm{II}^*.\ & \forall z \in [u_0, v_0]:\ 0 \leqslant B_1(z, v_0)F(v_0) \\
& B_2(u_0, u_0)F(u_0) \leqslant 0 \leqslant B_2(u_0, u_0)F(v_0) \\
& \forall u, v, w \in [u_0, v_0],\ u \leqslant w \leqslant v: \\
& \quad 0 \leqslant B_1(u, v)[F(u) - F(w) - S_1(u, v)(u - w)] \\
& \forall u, v, w, z \in [u_0, v_0],\ u \leqslant w \leqslant z \leqslant v: \\
& \quad 0 \leqslant B_1(w, z)[F(z) - F(v) - S_1(u, v)(z - v)] \\
& \forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v: \\
& \quad 0 \leqslant B_1(z, v)[F(u) - F(z) - S_2(u, u)(u - z)] \\
& \forall u, v \in [u_0, v_0],\ u \leqslant v,\ z \in \{u, v\}: \\
& \quad 0 \leqslant B_2(z, z)[F(u) - F(v) - S_2(u, u)(u - v)] \\
& \forall u, v, z \in [u_0, v_0],\ u \leqslant z \leqslant v: \\
& \quad 0 \leqslant B_2(u, u)[F(z) - F(v) - S_1(u, v)(z - v)].
\end{aligned}
$$

Then the assertions of Theorem 2 are valid.

Proof. The proof is done again for case I*. The existence of v_{k+1} follows as in the proof of Theorem 2. From the second condition in I*. it follows that $B_2(u_0, v_1)F(u_0) \leqslant 0$. For $k > 0$ this yields that $B_{2k}F(u_k) = -B_{2k}[F(u_{k-1}) - F(u_k) - S_{2k}(u_{k-1} - u_k)] \leqslant 0$.

Now it follows for $k \geqslant 0$ that $B_{2k}F(v_{k+1}) \geqslant 0$ because $0 \leqslant B_{2k}[F(v_{k+1}) - F(v_k) - S_{1k}(v_{k+1} - v_k)]$. By this way we obtain $T_{2k}(v_{k+1} - u_k) = v_{k+1} - u_k - B_{2k}[F(u_k) - F(v_{k+1}) - S_{2k}(u_k - v_{k+1})] - B_{2k}F(v_{k+1}) \leqslant v_{k+1} - u_k$, so that the existence of u_{k+1} with the property (5) can be shown as in the proof of Theorem 2. As in that proof it follows also that $B_{1k+1}F(u_{k+1}) \leqslant 0 \leqslant B_{1k+1}F(v_{k+1})$. The same inequalities for B_{2k+1}, namely $B_{2k}F(u_k) \leqslant 0 \leqslant B_{2k}F(v_{k+1})$, has been already shown.

Finally, the fact that every solution of $F(u) = 0$ from $[u_0, v_0]$ lies in $[u_k, v_k]$, $k \in \mathbb{N}$ follows as in the proof of Theorem 2 again. $\square$

If the mappings S_{ik} have additional properties in the sense of Def. 8 or Def. 9, we have one more corollary.

Corollary 2 ([12], [13]). *Let F, D, V, u_0 and v_0 be given as in Theorem 2, let W be a POS. Furthermore, let*

$$F(u_0) \leqslant 0 \leqslant F(v_0).$$

and let the mappings $S_i\colon [u_0, v_0] \times [u_0, v_0] \to L(V, W)$ have the following properties:
$\forall u, v \in [u_0, v_0]$ with $u \leqslant v$:

$$S_i(u, v)(u - v) \leqslant F(u) - F(v), \tag{6}$$

$S_i(u, v)$ is inverse-isotone and the image of V under $S_i(u, v)$ contains that of $[u_0, v_0]$ under F. Finally

$$\begin{array}{lllll} \text{I}'. & S_1(\cdot, v) & \textit{and} & S_2(u, \cdot) & \textit{are isotone.} \\ \text{II}'. & -S_1(\cdot, v) & \textit{and} & -S_2(u, \cdot) & \textit{are isotone.} \\ \text{III}'. & -S_1(\cdot, v) & \textit{and} & S_2(u, \cdot) & \textit{are isotone.} \\ \text{IV}'. & S_1(\cdot, v) & \textit{and} & -S_2(u, \cdot) & \textit{are isotone.} \end{array}$$

Then the assertions of Theorem 2 hold. Furthermore we have for all $k \in \mathbb{N}$: $F(u_k) \leqslant 0 \leqslant F(v_k)$.

Proof. Since the S_{ik} are inverse-isotone, they are also injective and because $F([u_0, v_0]) \subset S_{ik}V$, for each $k \in \mathbb{N}$ there exist positive inverses $S_{ik}^{-1}\colon S_{1k}V \cap S_{2k}V \to V$. Setting $B_{ik} := S_{ik}^{-1}$ it remains to show that $F(x) - F(y) - S(v, w)(u - z) \leqslant 0$ with x, y, v, w, u, z being chosen corresponding to the inequalities in I.–IV. This can easily be done in case I. except for the fifth condition for $z = v$. But only a simple modification of the proof of Theorem 2 using the inverse-isotony of S_{1k} is necessary to show the assertion (comp. Satz 2 in Section 2.1 in [12]). $\square$

Remark 1. The conditions I′. and II′. are fulfilled if F is *ddo-convex* with $S_1 = S_2 = S$ or $S_1 = S_2 = -S$. (6) is fulfilled by a *ddo*, a *gddo* as well as by a *qddo*.

Remark 2. Rem. 1 and 2 after Theorem 2 remain valid for Corollary 2.

Most of the theorems on monotone iteration methods are special cases of Theorem 2 and its corollaries (comp. [8], [12]) except some methods in [7], [8], [10], [11], [14], Brown's method [2] and the interval-iteration methods.

Many examples of nonlinear operator equations which can be treated via Theorem 2 and its corollaries can be found in the cited references on monotone including iteration methods. Especially in [9] a simple example is given, where Theorem 2 can be applied, but the S_{ik} are not inverse-isotone. In [1] and [12] examples are given where $S_1 \neq S_2$ and moreover F is not o-convex. In the next chapter we apply our theorem to coupled systems of nonlinear two-point boundary value problems.

3. Application to nonlinear systems of two-point boundary value problems

In this section problems of the following kind are solved

$$
\begin{aligned}
y'' &= f(\cdot, y, z)\\
z'' &= g(\cdot, y, z)\\
y(0) = y(b) &= z(0) = z(b) = 0,
\end{aligned}
$$

where $f, g\colon [0,b] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $b \in \mathbf{R}$, $i = 1, 2$ are continuous functions.

First we consider the following class of nonlinear systems:

$$y'' = y^2 + y + cz^2 - f \tag{7}$$

$$z'' = dy^2 + z^2 + 2z - g \tag{8}$$

with $b = 1$, positive real constants c, d and nonnegative functions f, g. It is easy to see that $\underline{y} \equiv \underline{z} \equiv 0$ is a subsolution for (7), (8) and $\overline{y} = c_1 t(1-t)$, $\overline{z} = c_2 t(1-t)$, $t \in [0,1]$ are supersolutions, if $2c_1 \geqslant \max\limits_{t\in[0,1]} f(t)$ and $2c_2 \geqslant \max\limits_{t\in[0,1]} g(t)$.

It will be shown, that Theorem 2 is applicable to the discrete problem corresponding to (7), (8), while monotonicity results like Corollary 2 are not, if we use the "natural" *ddo*

$$
S_1[(u,v);(\tilde u,\tilde v)] := S_2[(u,v);(\tilde u,\tilde v)] := S[(u,v);(\tilde u,\tilde v)] :=
$$

$$
\left(\begin{array}{cccccccc}
\frac{2}{h^2}+1+u_1+\tilde u_1 & -\frac{1}{h^2} & 0 & \cdots & c(v_1+\tilde v_1) & 0 & \cdots & 0\\
-\frac{1}{h^2} & \frac{2}{h^2}+1+u_2+\tilde u_2 & -\frac{1}{h^2} & 0 & \cdots & c(v_2+\tilde v_2) & 0 & \cdots\\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+1+u_n+\tilde u_n & 0 & \cdots & c(v_n+\tilde v_n)\\
d(u_1+\tilde u_1) & 0 & \cdots & 0 & \frac{2}{h^2}+2+v_1+\tilde v_1 & -\frac{1}{h^2} & 0 & \cdots\\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
0 & \cdots & 0 & d(u_n+\tilde u_n) & \cdots & 0 & -\frac{1}{h^2} & \frac{2}{h^2}+2+v_n+\tilde v_n
\end{array}\right).
$$

It is easy to see that $S[(u,v),(\tilde{u},\tilde{v})]$ is not an M-matrix, since the $n+1$-st diagonal consists of positive elements. Moreover, its inverse therefore contains negative elements and thus it cannot be inverse-isotone. Now we verify the assumptions of Theorem 2. We do this for the sake of brevity for the simplest case of discretization by finite differences, i.e. $h=\frac{1}{2}$, but it is true even for each $h=\frac{1}{n}$, $1<n\in\mathbb{N}$. The system to solve is

$$x_1^2+9x_1+cx_2^2=f(0.5)$$
$$dx_1^2+10x_2+x_2^2=g(0.5)$$

with

$$S(x,\tilde{x})=\begin{pmatrix} x_1+\tilde{x}_1+9 & c(x_2+\tilde{x}_2) \\ d(x_1+\tilde{x}_1) & x_2+\tilde{x}_2+10 \end{pmatrix}$$

Now a simple calculation shows that if $f,g\in[0,2]$, then I. of Theorem 2 is fulfilled for $c\in(0,3.125)$ and $d\in(0,3.2)$. Similar conditions for c,d ensure the applicability of our theorem if $f,g>2$ (see the condition for c_1 and c_2 above) or for smaller h. Since the cone obtained by the "natural" componentwise partial order of $\mathbb{R}^n$ is strongly minihedral, the assumptions of Theorem 2 are fulfilled for the discrete problem.

As a numerical example we solve (7), (8) with finite differences, $h=\frac{1}{100}$ for $c=d=3$ and $f(t)=2*\sin\pi t$, $g(t)=1+t$.

Tab. 1 Inclusions for Example (7), (8)

t	$\underline{x}_1$	$\underline{x}_2$	$y(t)$	$\overline{x}_2$	$\overline{x}_1$
0.1	0.053324	0.054648	0.054649	0.054650	0.055503
0.3	0.139083	0.142710	0.142712	0.142714	0.145015
0.5	0.171529	0.176084	0.176087	0.176088	0.178895
0.7	0.138893	0.142484	0.142486	0.142488	0.144648
0.9	0.053217	0.054520	0.054522	0.054522	0.055292

t	$\underline{x}_1$	$\underline{x}_2$	$z(t)$	$\overline{x}_2$	$\overline{x}_1$
0.1	0.048027	0.049154	0.049156	0.049157	0.049710
0.3	0.114723	0.117812	0.117816	0.117818	0.119294
0.5	0.142660	0.146546	0.146549	0.146551	0.148360
0.7	0.127939	0.131020	0.131022	0.131024	0.132456
0.9	0.059509	0.060630	0.060632	0.060632	0.061161

In [3] the following predator-prey problem arising from the Volterra-Lotka model is presented

$$u''+u(2-u-v)=0 \tag{9}$$

$$v''+v(3+u-6v)=0 \tag{10}$$

where $b := \pi$. The existence of a positive solution is proved by Theorem 4 in [3] by constructing a supersolution and a subsolution, but the particular super- and subsolution given there are wrong. One obtains correctly

$$\underline{u}_0(x) = \frac{4}{7}\sin x, \quad \overline{u}_0 \equiv 2,$$
$$\underline{v}_0(x) = \frac{3}{7}\sin x, \quad \overline{v}_0 \equiv \frac{5}{6}.$$

The supersolution does not satisfy the boundary condition, but nevertheless our method can be applied and using the *qddo* S_i

$$S_i[(u,v);(\tilde{u},\tilde{v})] := \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2 + (u+\tilde{u}) - v & -\tilde{u} \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 3 + 6(v+\tilde{v}) \end{pmatrix}$$

condition I$'$. from Corollary 2 is fulfilled. Moreover, by [5] the *qddo*

$$\tilde{S}_i[(u,v);(\tilde{u},\tilde{v})] := \begin{pmatrix} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (u+\tilde{u}) - v & -\tilde{u} \\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 6(v+\tilde{v}) \end{pmatrix}$$

is a-priori inverse-isotone and fulfils I$'$. Here $V = C^2[0,\pi]\times C^2[0,\pi]$ with the pointwise order in each component and $D = \{(u,v) \in V \mid u(0) = v(0) = u(\pi) = v(\pi) = 0\}$. The fixed-point result of Theorem 1 applied to T_{ik} in the proof of Theorem 2 is obtained by using the equivalent system of integral equations (comp. [4], Section 7.4). We obtain the improved supersolution

$$\overline{u}_1(x) = -\frac{154}{85}\cosh\sqrt{\frac{17}{6}}x + \beta\sinh\sqrt{\frac{17}{6}}x + \frac{2}{5}\cosh\sqrt{7}x - \frac{2}{5}\alpha\sinh\sqrt{7}x + \frac{24}{17}$$
$$\overline{v}_1(x) = \frac{5}{6}\left[1 - \cosh\sqrt{7}x + \alpha\sinh\sqrt{7}x\right],$$

where

$$\alpha = \frac{\cosh\sqrt{7}\pi - 1}{\sinh\sqrt{7}\pi}$$
$$\beta = \frac{1}{\sinh\sqrt{\frac{17}{6}}\pi}\left(\frac{154}{85}\cosh\sqrt{\frac{17}{6}}\pi - \frac{2}{5}\cosh\sqrt{7}\pi + \frac{2}{5}\alpha\sinh\sqrt{7}\pi - \frac{24}{17}\right),$$

which fulfils the boundary conditions and reduces the error in the $||\cdot||_\infty$-norm (comp. Rem. 2 after Theorem 2 with $C = 1$ for $C[0,\pi]$) by more than a half. Further approximations cannot be computed analytically. For $k > 1$ numerical methods must be used.

Since the computation of the supersolutions is independent of the subsolutions, we first determine the supersolutions up to five correct digits and use the best computed supersolution instead of $\overline{u}_k$ and $\overline{v}_k$ to determine the subsolutions, so we need 24

Tab. 2 Supersolutions for Example (9), (10)

k	$\overline{u}_k(0.25\pi)$	$\overline{u}_k(0.5\pi)$	$\overline{v}_k(0.25\pi)$	$\overline{v}_k(0.5\pi)$
0	2.00000	2.00000	0.83334	0.83334
1	0.94787	1.16806	0.72740	0.80722
2	0.75789	0.98744	0.60115	0.71954
3	0.65241	0.86873	0.51853	0.64966
4	0.59106	0.79349	0.46744	0.59827
5	0.55375	0.74616	0.43547	0.56258
6	0.53044	0.71626	0.41505	0.53854
7	0.51567	0.69723	0.40181	0.52259
8	0.50623	0.68506	0.39317	0.51206
9	0.50019	0.67726	0.38750	0.50513
10	0.49631	0.67226	0.38378	0.50048
11	0.49383	0.66905	0.38134	0.49759
12	0.49224	0.66691	0.37974	0.49563
13	0.49122	0.66569	0.37869	0.49435
...	...	...	...	...
20	0.48957	0.66356	0.37681	0.49209
...	...	...	...	...
27	0.48952	0.66350	0.37676	0.49199

iteration steps, which is less than with the original method. The mesh size is $h = \frac{\pi}{256}$, so that the discretization error is smaller than 10^{-5}. Again the iteration stops if five digits are stable (after rounding) and the inclusion error becomes

$$||\overline{u}_{27} - \underline{u}_{24}||_\infty \leqslant 10^{-5}, ||\overline{v}_{27} - \underline{v}_{24}||_\infty \leqslant 10^{-5}.$$

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Tab. 3 Subsolutions for Example (9), (10)

k	$\underline{u}_k(0.25\pi)$	$\underline{u}_k(0.5\pi)$	$\underline{v}_k(0.25\pi)$	$\underline{v}_k(0.5\pi)$
0	0.40406	0.57142	0.30304	0.42857
1	0.43301	0.59356	0.32863	0.44101
2	0.45125	0.61432	0.34454	0.45463
3	0.46359	0.62987	0.35488	0.46566
4	0.47210	0.64087	0.36195	0.47390
5	0.47790	0.64841	0.36676	0.47971
6	0.48181	0.65351	0.37003	0.48372
7	0.48444	0.65691	0.37225	0.48664
8	0.48618	0.65197	0.37374	0.48828
9	0.48734	0.66067	0.37475	0.48952
10	0.48810	0.66166	0.37542	0.49035
...	...	...	...	...
18	0.48947	0.66344	0.37670	0.49192
...	...	...	...	...
24	0.48951	0.66349	0.37674	0.49197

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Zusammenfassung

AFFIN-INVARIANTE MONOTONE ITERATIONSVERFAHREN MIT ANWENDUNG AUF SYSTEME NICHTLINEARER ZWEIPUNKTRANDWERTPROBLEME

RUDOLF L. VOLLER

In der vorliegenden Arbeit wird ein neuer Satz über monoton einschließende Iterationsverfahren bewiesen. Die Voraussetzungen, die an die zu behandelnden Operatoren gestellt werden, sind affin-invariant, und topologieabhängige Eigenschaften werden weder von den zugrundegelegten Räumen noch von den behandelten Operatoren verlangt. Ferner kommen wir ohne Invers-isotonie aus. Als Beispiele werden Systeme schwach nichtlinearer gewöhnlicher Differentialgleichungen mit Zwei-Punkt-Randbedingungen behandelt.

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