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WISHART DISTRIBUTIONS IN THE MULTIVARIATE GAUSS-MARKOFF MODEL WITH SINGULAR COVARIANCE MATRIX

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Summary. This paper concerns generalized quadratic forms for the multivariate case. These forms are used to test linear hypotheses of parameters for the multivariate Gauss-Markoff model with singular covariance matrix. Distributions and independence of these forms are proved.

Keywords: multivariate general linear Gauss-Markoff model, Wishart distribution, multinormal distribution, set of linear estimable parametric functions, test, quadratic form, singular covariance matrix

AMS classification: 62H10

1. INTRODUCTION

Let $(U, XB, \Sigma \otimes \sigma^2 V)$ denote multivariate Gauss-Markoff model, with known $V \geqslant 0$, where $\otimes$ is the symbol of Kronecker product of matrices, B is a matrix of unknown fixed parameters, X is a given known design matrix, U is a random matrix of observations with the expected value $\varepsilon(U) = XB$ and with a fixed non-singular matrix Σ, σ^2 is unknown positive scalar. Let us state that $T = V + XMX'$, where $M = M'$ is an arbitrary matrix such that $R(X) \subset R(T)$. The symbol $R(X)$ is used to denote the vector space spanned on the columns of the matrix X.

Problem of testing the hypothesis $LB = 0$ in the model $(U, XB, \Sigma \otimes V)$, $V \geqslant 0$ and in the model $(U, XBA, \Sigma \otimes I)$ is presented by Srivastava and Khatri ([7], pp. 170–193), G.A.F. Seber ([6], p. 423) considers testing hypothesis $LBA = 0$ in $(U, XB, \Sigma \otimes I)$ or equivalent by testing hypothesis $LB^* = 0$ in $(U^*, XB^*, \Sigma \otimes I)$.

K.V. Mardia, J.T. Kent and J.M. Bibby [1] transform testing hypothesis L$BA = D$ in $(U, XB, \Sigma \otimes I)$ into testing hypothesis L$B = D$ in $(UM, XBM, M'\Sigma M \otimes I)$.

The case when V is allowed to be singular is discussed in full by Rao and Mitra [4], but in the univariate case only. The case of a singular V has not received much explicit attention in literature, so three theorems concerning testing a set of hypotheses of the form LBA = 0 are presented and proved in this paper. Wilk's, Hotelling's, Pillai's or Roy's test can be used.

2. Distributions of quadratic forms

Let LBA be a set of linear estimable parametric functions, (Roy, [5]) where L and A are $a \times m$ and $p \times b$ matrices respectively.

The test of the estimable hypothesis

$$\mathrm{L}BA = 0 \tag{2.1}$$

can be deduced from the known Wilk's, Hotelling's or Roy's tests.

In constructing the corresponding test functions the following quadratic forms can be used:

$$S_H = (\mathrm{L}\hat{B}A)'L^-(\mathrm{L}\hat{B}A) = (\mathrm{L}\hat{B}A)'(\mathrm{L}C_4\mathrm{L}')^-(\mathrm{L}\hat{B}A), \tag{2.2}$$

$$S_E = A'U'C_1UA, \tag{2.3}$$

where

$$\hat{B} = (X'T^-X)^-X'T^-U = C_2'U = C_3U \tag{2.4}$$

$$\begin{cases} C_1 = T^- - T^-X(X'T^-X)^-X'T^- = T^-(I - XC_3) \\ C_2' = C_3 = (X'T^-X)^-X'T^- \end{cases} \tag{2.5}$$

$$L = (\mathrm{L}C_3)V(\mathrm{L}C_3)' = \mathrm{L}C_4\mathrm{L}' \tag{2.6}$$

(Oktaba, [2] (2.1), p. 179).

The symbol L^- is reserved for any choice of the g-inverse, i.e. the following relation holds:

$$LL^-L = L. \tag{2.7}$$

Let the symbol $W_b(\nu, \Sigma)$ denote the central Wishart distribution with ν degrees of freedom and with the dispersion matrix Σ. We recall that

$$\begin{pmatrix} V & X \\ X' & 0 \end{pmatrix}^{-} = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}.$$

Theorem 2.1. *If*

$$U \sim N_{np}(XB, \Sigma \otimes \sigma^2 V) \tag{2.8}$$

where $N_{np}(\ ,\ )$ denotes an np-variate normal distribution N_{np} with parameters defined in the "Introduction", LBA is a set of estimable linear combinations of parameters and the hypothesis (2.1) *is true, then*

$$S_H \sim W_b\left[r(L), \sigma^2 A'\Sigma A\right]$$

P r o o f. Subject to the assumptions that the hypothesis (2.1) is true and $\mathrm{L}BA$ is a set of estimable parametric functions, the quadratic form (2.2) can be presented in the form:

$$\begin{aligned} S_H &= (\mathrm{L}\hat{B}A)'L^-(\mathrm{L}\hat{B}A) = (\mathrm{L}\hat{B}A - \mathrm{L}BA)'L^-(\mathrm{L}\hat{B}A - \mathrm{L}BA) \\ &= (\mathrm{L}C_3UA - \mathrm{L}C_3XBA)'L^-(\mathrm{L}C_3UA - \mathrm{L}C_3XBA) \\ &= (UA - XBA)'(\mathrm{L}C_3)'L^-\mathrm{L}C_3(UA - XBA) \\ &= (UA - XBA)'D(UA - XBA) = \left[UA - \varepsilon(UA)\right]'D\left[UA - \varepsilon(UA)\right], \end{aligned} \tag{2.9}$$

where

$$D = \left[\mathrm{L}(X'T^-X)^-X'T^-\right]'L^-\left[\mathrm{L}(X'T^-X)^-XT^-\right] = (\mathrm{L}C_3)'L^-(\mathrm{L}C_3). \tag{2.10}$$

Using (2.8) we obtain that

$$UA \sim N_{nb}(XBA, A'\Sigma A \otimes \sigma^2 V). \tag{2.11}$$

By the definition of Wishart distribution (cf. Rao, [3], p. 534) we have $S_H \sim W_b\left[r(L), \sigma^2 A'\Sigma A\right]$ if and only if

$$VDVDV = VDV \quad \text{and} \quad r(L) = \mathrm{tr}(VD), \tag{2.12}$$

where $tr(A)$ denotes the trace of the matrix A.

We will prove now that the relation (2.12) holds. In fact,

$$VDVDV = V(LC_3)'L^-LC_3V(LC_3)'L^-LC_3V$$
$$= V(LC_3)'L^-LL^-LC_3V = V(LC_3)'L^-LC_3V = VDV, \tag{2.13}$$

and moreover,

$$\operatorname{tr}(VD) = \operatorname{tr}\left[V(LC_3)'L^-LC_3\right] = \operatorname{tr}\left[L^-LC_3V(LC_3)'\right]$$
$$= \operatorname{tr}(L^-L) = r(L).$$

Hence the result (2.12) follows. □

Let $(V\vdots X)$ denote a partitioned matrix.

Theorem 2.2. *Subject to the assumption* (2.8) *we have*

$$S_E \sim W_b\left[r(V\vdots X) - r(X), \sigma^2 A'\Sigma A\right]$$

where S_E *is as defined in* (2.3).

Proof. By the relation

$$X'C_1 = X'\left[T^- - T^-X(X'T^-X)^-X'T^-\right] = 0 \tag{2.14}$$

the matrix (2.3) can be presented in the form

$$S_E = (UA)'C_1(UA) = (UA)'(T^- - T^-X(X'T^-X)^-X'T^-)UA$$
$$= (UA - XBA)'\left[T^- - T^-X(X'T^-X)^-X'T^-\right](UA - XBA)$$
$$= \left[UA - \varepsilon(UA)\right]'C_1\left[UA - \varepsilon(UA)\right]. \tag{2.15}$$

In virtue of (2.11) the matrix (2.15) has the Wishart distribution with the parameters as in Theorem 2.2 if and only if (cf. proof of Theorem 2.1)

$$VC_1VC_1V = VC_1V \tag{2.16}$$

and

$$\operatorname{tr} VC_1 = r(V\vdots X) - r(X). \tag{2.17}$$

Let us see that conditions (2.16) and (2.17) are fulfilled. In fact, by (2.14) and the definition of the matrix T we have $VC_1 = TC_1$ and the right hand side of (2.16) can be written as

(2.18)
$$\begin{aligned} VC_1VC_1V &= TC_1TC_1V \\ &= T\big[T^- - T^-X(X'T^-X)^-X'T^-\big]T\big[T^- - T^-X(X'T^-X)^-X'T^-\big]V \\ &= T\big[T^- - T^-X(X'T^-X)^-X'T^-\big]V = TC_1V = VC_1V. \end{aligned}$$

The relations (2.18) show that S_E has the central Wishart distribution.

Now we prove (2.17). By the definition of T and (2.14) we obtain

$$\begin{aligned} \operatorname{tr} VC_1 &= \operatorname{tr} TC_1 = \operatorname{tr} \big[TT^- - TT^-X(X'T^-X)^-X'T^-\big] \\ &= \operatorname{tr} TT^- - \operatorname{tr} X(X'T^-X)^-X'T^- = r(T) - \operatorname{tr}(X'T^-X)^-X'T^-X \\ &= r(V\vdots X) - r(X'T^-X) = r(V\vdots X) - r(X). \end{aligned} \tag{2.19}$$

□

Theorem 2.3. *If the assumptions of Theorems 2.1 and 2.2 concerning matrices S_H and S_E are fulfilled, then S_H and S_E are stochastically independent.*

Proof: A necessary and sufficient condition for S_H and S_E to be independently distributed is

$$VDVC_1V = 0, \tag{2.20}$$

where C_1 and D are defined as in (2.5) and (2.10) respectively. In virtue of (2.14), $X'T^-T = X'$, we have

$$\begin{aligned} VDVC_1V = VDTC_1V &= V\big[\mathrm{L}(X'T^-X)^-X'T^-\big]'L^-\big[\mathrm{L}(X'T^-X)^-X'T^-\big]TC_1V \\ &= V\big[\mathrm{L}(X'T^-X)^-XT^-\big]'L^-\mathrm{L}(X'T^-X)^-X'C_1V = 0. \end{aligned}$$

It means that the condition (2.20) holds for S_H and S_E, so the forms considered are independent. □

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