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ASYMPTOTIC PROPERTIES OF DISPERSIONS OF THE DIFFERENTIAL EQUATIONS $y'' = q(t)y$

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§1. BASIC NOTIONS AND SOME PROPERTIES OF THE FIRST PHASE AND OF THE DISPERSION OF THE DIFFERENTIAL EQUATION $y'' = q(t)y$

In the following let us denote $R = (-\infty, \infty)$, $R^+ = (0, \infty)$, $a \in R$, $I = \langle a, \infty)$. For every positive integer n C_I^n denotes the set of all functions having continuous n-th derivative on I, C_I^0 denotes the set of all functions continuous on I. We assume the reader is familiar with the definition and properties of the first phase, of the fundamental dispersion of the first kind (in the following we write briefly dispersion) and of the central dispersion with the index i of the differential equation (q) (shortly equation (q))

$$y'' = q(t)\,y, \tag{q}$$

(see [1]). If α and φ are a first phase and the dispersion of the equation (q), $q \in C_I^0$ respectively, there hold the following equalities $\left(\{\alpha, t\} = \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} - \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2\right)$,

$$\alpha \circ \varphi(t) = \alpha(t) + \pi \operatorname{sgn} \alpha', \tag{1}$$

$$-\{\alpha, t\} - \alpha'^2(t) = q(t), \qquad t \in I. \tag{2}$$

Further if α is the first phase of two independent solutions u, v of the equation (q) and the Wronskian of u, v equals w, then the function $r\colon I \to R^+$, $r : t \to \sqrt{-\frac{w}{\alpha'(t)}}$ is a solution of the differential equation

$$r'' = qr + \frac{w^2}{r^3}, \tag{3}$$

on I.

In this paper we shall study the behaviour of the dispersion and of its derivatives of the disturbed equation

$$y'' = (q(t) - \Delta(t))\,y,$$

assuming that $\lim_{t\to\infty} \Delta(t) = 0$. In the papers [2], [3] analogous problems are studied and sufficient conditions for the validity of the relations $\lim_{t\to\infty} \varphi'(t) = 1$, $\lim_{t\to\infty} \varphi^{(j)}(t) = 0$ $(j = 2, 3, \dots, n + 3)$ are given.

In our considerations the following lemmas will be useful.

Lemma 1. *Let* $q \in C_I^0$, $\limsup_{t\to\infty} q(t) < 0$. *Let* φ *be the dispersion of the equation* (q). *Then there exists a number* K, $K \in R^+$ *such that there holds the inequality*

$$\varphi(t) - t \leqq K, \qquad t \in I.$$

Proof. Because of $\limsup_{t\to\infty} q(t) < 0$ then there exist numbers ε, b, $-\varepsilon \in R^+$, $b \geqq a$ such that we have $q(t) \leqq \varepsilon$ for $t \in \langle b, \infty)$. The function $x\colon I \to I$, $x\colon t \to t + \dfrac{\pi}{\sqrt{-\varepsilon}}$ is the dispersion of the equation $y'' = \varepsilon y$. Using Sturm's Comparison Theorem we get $\varphi(t) \leqq x(t)$ as $t \in \langle b, \infty)$. Thus, for $t \in \langle b, \infty)$ we have $\varphi(t) - t \leqq x(t) - t = \dfrac{\pi}{\sqrt{-\varepsilon}}$. Putting $k = \max_{\langle a, b\rangle} (\varphi(t) - t) > 0$, $K = \max\left(k, \dfrac{\pi}{\sqrt{-\varepsilon}}\right)$ we come to the inequality $\varphi(t) - t \leqq K$, $t \in I$ which was to be proved.

Lemma 2. *Let us assume that* $q \in C_I^n$, $\liminf_{t\to\infty} q(t) > -\infty$, $\limsup_{t\to\infty} q(t) < 0$. *For* $n > 0$, *let* $c \in R^+$ *be a number such that* $|q^{(k)}(t)| \leqq c$, $t \in I$, $k = 1, 2, \dots, n$. *Let* φ *be the dispersion of the equation* (q). *Then there exist numbers* k_1, k_2, $0 < k_1 \leqq k_2$ *such that the inequalities*

$$k_1 \leqq \varphi'(t) \leqq k_2, \qquad t \in I, \tag{4}$$

hold and the functions $\varphi^{(k)}(t)$, $k = 2, 3, \dots, n + 3$ *are bounded on* I.

Proof. From the assumptions $\liminf_{t\to\infty} q(t) > -\infty$, $\limsup_{t\to\infty} q(t) < 0$ it follows the existence of such numbers b, c_1, c_2, $b \geqq a$, $0 > c_1 \geqq c_2$ that for $t \in \langle b, \infty)$ it is $c_1 \geqq q(t) \geqq c_2$. For suitable numbers t_1, t_2, $t < t_1 < t_2 < \varphi(t)$ we have $\varphi'(t) = \dfrac{q(t_1)}{q(t_2)}$ (cf. [1], p. 115) and hence $\dfrac{c_1}{c_2} \leqq \varphi'(t) \leqq \dfrac{c_2}{c_1}$, $t \in \langle b, \infty)$. Let us write $d_1 = \max_{\langle a, b\rangle} \varphi'(t)$, $d_2 = \min_{\langle a, b\rangle} \varphi'(t)$. We see from the properties of φ that $d_2 \in R^+$. The inequaltities (4) are clearly valid if we pose $k_1 = \min\left(\dfrac{c_1}{c_2}, d_2\right)$, $k_2 = \max\left(\dfrac{c_2}{c_1}, d_1\right)$.

It is known that φ satisfies the equality (cf. [1], p. 123)

$$-\frac{1}{2}\frac{\varphi'''(t)}{\varphi'(t)}+\frac{3}{4}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2+\varphi'^2(t)\,q\circ\varphi(t)=q(t),\qquad t\in I,\tag{qq}$$

which may be written in the equivalent form

$$2\left(\frac{\varphi''(t)}{\varphi'(t)}\right)'-\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2=4(\varphi'^2(t)\,q\circ\varphi(t)-q(t)),\qquad t\in I.$$

Let us define $u\colon I\to R$, $u\colon t\to\dfrac{\varphi''(t)}{\varphi'(t)}$. Then u satisfies the equality

$$2u'(t)-u^2(t)=S(t),\ S(t)=4(\varphi'^2(t)\,q\circ\varphi(t)-q(t)),\ t\in I.\tag{5}$$

From the boundedness of q on I and from the inequalities (4) we deduce also the boundedness of S on I i.e. $|S(t)|\leqq K$, $t\in I$, $K\in R^+$. We are going now to show that even the boundedness of u follows from (5). It is easy to see that at the extremum-points of u there is $u^2=-S$ (u' equals zero at these points) and hence at these points $u^2\leqq K$. If ∞ is a accumulation point of the extremum-points of u, then u must be bounded on I. Let us now admit the function u is not bounded on I. Then u must be decreasing or increasing for t large enough and thus $\lim\limits_{t\to\infty}u(t)=-\infty$ or $\lim\limits_{t\to\infty}u(t)=\infty$. Hence $\lim\limits_{t\to\infty}\dfrac{\varphi''(t)}{\varphi'(t)}=-\infty$ or ∞. With respect to the inequalities (4) we have $\lim\limits_{t\to\infty}\varphi''(t)=-\infty$ or $\lim\limits_{t\to\infty}\varphi''(t)=\infty$. But $\varphi'>0$ on I and we obtain immediately $\lim\limits_{t\to\infty}\varphi''(t)=\infty$. This conclusion contradicts the boundedness of φ' on I. We have so the boundedness of u i.e. of $\dfrac{\varphi''}{\varphi'}$ and with respect to (4) also the boundedness of φ'' on I. Finally the boundedness of φ''' on I follows from (qq) and (4).

Let us now assume that $n>0$ and $\varphi^{(j)}$ are bounded on I for $j=1,2,\ldots,k+2$, $k\geqq 1$, $k\leqq n$. We shall prove that $\varphi^{(k+3)}$ is also bounded on I. The k-th derivative of (qq) gives

$$-\frac{1}{2}\left(\frac{\varphi'''(t)}{\varphi'(t)}\right)^{(k)}+\frac{3}{4}\left(\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2\right)^{(k)}=(q(t)-\varphi'^2(t)\,q\circ\varphi(t))^{(k)},\qquad t\in I$$

and this equality can be written in the form

$$-\frac{1}{2}\,\frac{\varphi^{(k+3)}(t)}{\varphi'(t)}=P(t),\qquad t\in I.$$

In this formula P is a fraction and its numerator consists of sums of products of functions $\varphi',\ldots,\varphi^{(k+2)}$, $q,\ldots,q^{(k)}$ multiplied by suitable numbers; in the denominator stands a power of φ'. Thus P is bounded on I and therefore also $\varphi^{(k+3)}$ is bounded on I. Lemma 2 is thus proved.

Lemma 3. *Let us assume that* $q \in C_I^0$, $\liminf_{t\to\infty} q(t) > -\infty$, $\limsup_{t\to\infty} q(t) < 0$. *Let us denote by* α *the first phase of the equation* (q). *Then there exists a constant* c, $c \in R^+$ *such that* $|\alpha'(t)| \leqq c$ *is on* I *if and only if a constant* b, $b \in R^+$ *exists such that* $b \leqq \leqq |\alpha'(t)|$ *is on* I.

Proof. With respect to the assumptions of this Lemma it is obviously possible to assume without loss of generality that two numbers $m \in R^+$, $M \in R^+$, $m \geqq M$ exist so that $-m \leqq q(t) \leqq -M$ on I.

Let α_1 be an arbitrary fixed first phase of the equation (q). Then there exists such a first phase ε_1 of the equation (-1) that the relation $\alpha_1(t) = \varepsilon_1 \circ \alpha(t)$ holds. Because of ε_1' being a periodic function with period π and $\varepsilon_1 \in C_I^3$, $\varepsilon_1' \neq 0$ on I, it follows from the last relation that $|\alpha_1'|$ is on I bounded from above (below) by a positive constant if $|\alpha'|$ is bounded from above (below) by a positive constant on I. Therefore, without loss of generality, it is possible to assume that $\operatorname{sgn} \alpha' = 1$ and that α is the first phase of independent solutions of (q) having the Wronskian $w = -1$. In this case the function $r: I \to R^+$, $r: t \to \sqrt{\dfrac{1}{\alpha'(t)}}$ satisfies the equality

$$r''(t) = q(t)\, r(t) + \frac{1}{r^3(t)}, \qquad t \in I. \tag{6}$$

Let us assume there exists $c \in R^+$ such that $0 < \alpha'(t) \leqq c$, $t \in I$ and let $\liminf_{t\to\infty} \alpha'(t) = = 0$. First we are going to show that no point t_0, $t_0 \in I$ for which $r(t_0) \neq \dfrac{1}{\sqrt[4]{-q(t_0)}}$ is a accumulation point of zero points of the function r'. In the opposite case we have $r' = r'' = 0$ at this point and from (6) there follows the equality $r(t_0) = \dfrac{1}{\sqrt[4]{-q(t_0)}}$.

Assuming r to be increasing on $I_1 = \langle t_1, \infty) \subset I$ we get with respect to the assumption $\liminf_{t\to\infty} \dfrac{1}{r^2(t)} = 0$ that $\lim_{t\to\infty} r(t) = \infty$. From (6) we have $2r'r'' = 2qrr' + \dfrac{2r'}{r^3}$ and integrating this identity between the limits t_1 and t $(t > t_1)$ we obtain

$$r'^2(t) - r'^2(t_1) = \int_{t_1}^{t} q(s)\,(r^2(s))'\,ds - \left(\frac{1}{r^2(t)} - \frac{1}{r^2(t_1)}\right)$$

By the Mean Value Theorem of the Integral Calculus ($r' \geqq 0$ on I_1) a number $\xi = \xi(t)$ exists such that

$$r'^2(t) - r'^2(t_1) = q(\xi)\,(r^2(t) - r^2(t_1)) - \left(\frac{1}{r^2(t)} - \frac{1}{r^2(t_1)}\right). \tag{7}$$

We have $-m \leqq q(\xi) \leqq -M$ and for this reason for $t \to \infty$ the expression on the right hand side of (7) tends to $-\infty$. But this conclusion contradicts the fact that the left hand side is bounded from below on I_1. If r is decreasing on I_1, then we have $\liminf_{t\to\infty} \frac{1}{r^2(t)} > 0$ and this contradicts our assumption.

For the validity of $\liminf_{t\to\infty} \alpha'(t) = 0$ it is thus necessary that the function r' changes its sign on I and even on every interval of the type $\langle b, \infty) \subset I$. There exist sequences $\{s_n\}$, $\{t_n\}$, $s_n < t_n$, $\lim_{t\to\infty} s_n = \infty$, $\lim_{t\to\infty} t_n = \infty$ of maxima and minima respectively of the function r and $r(s_n) \neq \frac{1}{\sqrt[4]{-q(s_n)}}$, $\lim_{n\to\infty} \frac{1}{r^2(s_n)} = 0$. Let us assume t_n are the least numbers having this properties.

Using a procedure analogous to that of the first part of the proof of this Lemma it is possible to prove – integrating (6) – the existence of a sequence $\{\eta_n\}$, $s_n < \eta_n < t_n$ such that for every n, $n = 1, 2, 3, \ldots$ the equality

$$r'^2(t_n) - r'^2(s_n) = q(\eta_n)(r^2(t_n) - r^2(s_n)) - \left(\frac{1}{r^2(t_n)} - \frac{1}{r^2(s_n)}\right)$$

holds. Because of $r'(s_n) = r'(t_n) = 0$ for every positive integer n, we obtain from the last equality

$$\frac{1}{r^2(s_n)\, r^2(t_n)} = -q(\eta_n).$$

It follows immediately from $-m \leqq q(\eta_n) \leqq -M$, $\lim_{n\to\infty} \frac{1}{r^2(s_n)} = 0$ that $\lim_{n\to\infty} \frac{1}{r^2(t_n)} = \lim_{n\to\infty} \alpha'(t_n) = \infty$ holds but this contradicts the inequality $\alpha' \leqq c$. Thus, if $|\alpha'(t)| \leqq c$, we must have $\liminf_{t\to\infty} |\alpha'(t)| > 0$. Let us suppose a constant $b \in R^+$ exists so that there is $\alpha'(t) \geqq b$ on I. Assuming moreover the relation $\limsup_{t\to\infty} \alpha'(t) = \infty$ we see immediately that r cannot be increasing on $I_1 = \langle t_1, \infty) \subset I$ because this yields a contradiction by $\limsup_{t\to\infty} \alpha'(t) = \limsup_{t\to\infty} \frac{1}{r^2(t)} < \infty$. If r decreases on I we must necessarily have with respect to $\limsup_{t\to\infty} \alpha'(t) = \infty : \lim_{t\to\infty} r(t) = 0$. Analogously to the first part of the proof it is now possible to deduce (7) with a suitably chosen number $\xi \in (t_1, t)$. From this it follows directly that the right hand side tends to $-\infty$ if $t \to \infty$ whenever the left hand side remains to be bounded from below. In case r is neither increasing nor decreasing on any interval of the type $I_1 = \langle t_1, \infty) \subset I$ we obtain a contradiction to our assumption $\limsup_{t\to\infty} \alpha'(t) = \infty$ by using a procedure analogous to the foregoing part of the proof.

Remarks.

1. The condition $b \in R^+$, $|\alpha'(t)| \geqq b$ on I is equivalent to the condition that each solution of (q) is bounded on I. This follows directly from the fact that $\dfrac{\cos\alpha(t)}{\sqrt{|\alpha'(t)|}}$, $\dfrac{\sin\alpha(t)}{\sqrt{|\alpha'(t)|}}$ are independent solutions of (q) (cf. [5], Lemma 1).

2. From [4], p. 63 we see that the assumption $\limsup_{t\to\infty} q(t) < 0$, $\liminf_{t\to\infty} q(t) > -\infty$ cannot quarantee the boundedness of each solution of (q) on I.

§2. ASYMPTOTICAL PROPERTIES OF DISPERSIONS AND THEIR DERIVATIVES

Let us first prove the following

Lemma 4. *Assume that $q \in C_I^0$ and the inequality $-m \leqq q(t) \leqq -M$ are fulfilled on I with $m \in R^+$, $M \in R^+$. Let us denote by α the first phase of* (q) *and assume that $|\alpha'(t)| \geqq b$, $t \in I$, $b \in R^+$. For every λ, $\lambda \in R^+$ let us $\lambda\alpha$ be the first phase of the equation* (q_λ). *If φ denotes the dispersion of the equation (q) and φ_λ the dispersion of the equation* (q_λ), *then the following inequality*

$$|\varphi(t) - \varphi_\lambda(t)| \leqq \left|\frac{1-\lambda}{\lambda}\right| \frac{\pi}{b}, \qquad t \in I,$$

is satisfied.

Proof. From $q_\lambda(t) = -\{\lambda\alpha, t\} - \lambda^2\alpha'^2(t)$ it follows that $q_\lambda(t) = q(t) + (1-\lambda^2)\,\alpha'^2(t)$. With respect to the fact that the functions α, φ, φ_λ fulfil the following identities

$$\alpha \circ \varphi(t) = \alpha(t) + \pi \operatorname{sgn} \alpha',$$
$$\lambda\alpha \circ \varphi_\lambda(t) = \lambda\alpha(t) + \pi \operatorname{sgn} \alpha', \ t \in I,$$

we obtain (α^{-1} means the inverse function to α) $\varphi(t) = \alpha^{-1} \circ (\alpha(t) + \pi \operatorname{sgn} \alpha')$, $\varphi_\lambda(t) = \alpha^{-1} \circ \left(\alpha(t) + \frac{\pi}{\lambda} \operatorname{sgn} \alpha'\right)$. Let us define: $\psi_\lambda : I \to R$, $\psi_\lambda : t \to \varphi(t) - \varphi_\lambda(t)$. Then $\psi_\lambda(t) = \alpha^{-1} \circ (\alpha(t) + \pi \operatorname{sgn} \alpha') - \alpha^{-1} \circ \left(\alpha(t) + \frac{\pi}{\lambda} \operatorname{sgn} \alpha'\right)$. By the Mean Value Theorem there exists a number, ξ, $\xi = \xi(t)$ such that

$$\psi_\lambda(t) = \alpha^{-1\prime}(\xi)\left(\pi \operatorname{sgn} \alpha' - \frac{\pi}{\lambda} \operatorname{sgn} \alpha'\right).$$

From the relation $\alpha^{-1\prime}(t) = \dfrac{1}{\alpha' \circ \alpha^{-1}(t)}$ we obtain the inequality $|\psi_\lambda(t)| \leqq \leqq \left|\dfrac{1-\lambda}{\lambda}\right| \dfrac{\pi}{b}$, $t \in I$ which was to be proved.

Theorem. *Assume that* $p \in C_I^0$, $q \in C_I^0$, $\liminf_{t\to\infty} q(t) > -\infty$, $\limsup_{t\to\infty} q(t) < 0$, $\lim_{t\to\infty} (q(t) - p(t)) = 0$. *Let us denote by* ψ, φ *the dispersions of the equations* (p), (q) *respectively. If every solution of* (q) *is bounded on I, then*

$$\lim_{t\to\infty} (\varphi(t) - \psi(t)) = 0.$$

If moreover $n = 0$ *or* n *is a positive integer,* $p \in C_I^n$, $q \in C_I^n$, $\lim_{t\to\infty} (q(t) - p(t))^{(j)} = 0$, $j = 0, 1, \ldots, n$, q *has in I the* $(n+1)$ *st derivative with bounded* $q^{(j)}$, $j = 1, 2, \ldots, n+1$, $t \in I$, *then from the validity of the relation*

$$\lim_{t\to\infty} (\varphi(t) - \psi(t))' = 0$$

it follows also the validity of

$$\lim_{t\to\infty} (\varphi(t) - \psi(t))^{(j)} = 0, j = 0, 1, \ldots, n+3.$$

Proof. Without loss of generality it is possible to assume q satisfying the inequality $-m \leqq q(t) \leqq -M$ on I with suitably chosen numbers $m \in R^+$, $M \in R^+$, $m \geqq M$. Let us choose $\varepsilon \neq 0$, $\varepsilon \in R$, $\varepsilon < M$ and denote by ω_ε the dispersion of the equation $(q + \varepsilon)$. By Sturm's Comparison Theorem we obtain for $\varepsilon \in R^+ (-\varepsilon \in R^+)$ the inequality $\varphi(t) < \omega_\varepsilon(t)$ $(\varphi(t) > \omega_\varepsilon(t))$ on I. We are going to prove that if ε tends to 0 then the system of function $\omega_\varepsilon(t)$ uniformly tends to the function $\varphi(t)$ on I. With α denoting the first phase of (q) we obtain from Lemma 3 and from Remark 1 the existence of constants b, c, $b \in R^+$, $c \in R^+$, $b < c$ such that $b \leqq |\alpha'(t)| \leqq c$, $t \in I$. For every λ, $\lambda \in R^+$ we denote (q_λ) the equation having the first phase $\lambda\alpha$. φ_λ denotes then the dispersion of this equation. From the proof of Lemma 4 it follows that $q_\lambda(t) = q(t) + (1 - \lambda^2)\,\alpha'^2(t)$. Thus, to every number $\varepsilon \neq 0$, $\varepsilon < b^2$ there exists such a number λ, $\lambda = \lambda(\varepsilon)$ that on I there hold the relations

$$q_\lambda(t) - q(t) = (1 - \lambda^2)\,\alpha'^2(t) \leqq \varepsilon \qquad \text{for} \qquad -\varepsilon \in R^+$$
$$q_\lambda(t) - q(t) = (1 - \lambda^2)\,\alpha'^2(t) \geqq \varepsilon \qquad \text{for} \qquad \varepsilon \in R^+$$

Clearly it is $\lambda > 1$ for $-\varepsilon \in R^+$ and $\lambda < 1$ for $\varepsilon \in R^+$. Moreover the numbers $\lambda = \lambda(\varepsilon)$ can be chosen in such a manner that $\lim_{\varepsilon\to 0} \lambda(\varepsilon) = 1$ which will be assumed in the following. From Lemma 4 we obtain the following inequality

$$|\varphi_{\lambda(-\varepsilon)}(t) - \varphi_{\lambda(\varepsilon)}(t)| \leqq |\varphi_{\lambda(-\varepsilon)}(t) - \varphi(t)| + |\varphi_{\lambda(\varepsilon)}(t) - \varphi(t)| \leqq$$
$$\leqq \left(\left|\frac{1 - \lambda(-\varepsilon)}{\lambda(-\varepsilon)}\right| + \left|\frac{1 - \lambda(\varepsilon)}{\lambda(\varepsilon)}\right|\right)\frac{\pi}{b}, \qquad t \in I, \qquad \varepsilon \in R^+.$$

From the relation $\lim_{\varepsilon\to 0} \lambda(\varepsilon) = 1$, the last inequality and the straightforward inequalities

$$\varphi_{\lambda(-\varepsilon)}(t) \leqq \omega_{-\varepsilon}(t) < \varphi(t) < \omega_\varepsilon(t) \leqq \varphi_{\lambda(\varepsilon)}(t), \quad t \in I, \quad \varepsilon \in R^+,$$

we get immediately for ε tending to zero the uniform convergence on I of the system of functions $\varphi_{\lambda(-\varepsilon)}(t) - \varphi_{\lambda(\varepsilon)}(t)$ to the function which equals identically zero on I. The system $\omega_\varepsilon(t)$ converges hence uniformly on I to the function $\varphi(t)$ for ε tending to zero. It follows further from $\lim_{t\to\infty} (q(t) - p(t)) = 0$ that for $\varepsilon \in R^+$, $\varepsilon < M$ there exists $t_\varepsilon \in I$ such that $q(t) - \varepsilon \leqq p(t) \leqq q(t) + \varepsilon$ is valid on $I_\varepsilon = \langle t_\varepsilon, \infty)$. Thus on every interval I_ε the inequalities $\omega_{-\varepsilon}(t) \leqq \psi(t) \leqq \omega_\varepsilon(t)$ are satisfied. With respect to the fact that the systems of functions $\omega_\varepsilon(t) - \omega_{-\varepsilon}(t)$ and $\omega_\varepsilon(t)$ converge uniformly on I to the function which is zero identically and to $\varphi(t)$ respectively as ε tends to 0 we obtain directly $\lim_{t\to\infty} (\varphi(t) - \psi(t)) = 0$ and the first part of the Theorem is proved.

The second part of the Theorem is proved by the mathematical induction. By the first part of the Theorem we have $\lim_{t\to\infty} (\varphi(t) - \psi(t)) = 0$ and by our assumption also $\lim_{t\to\infty} (\varphi(t) - \psi(t))' = 0$. For φ, ψ the following identities hold (cf. [1], p. 123)

$$-\frac{1}{2}\frac{\varphi'''(t)}{\varphi'(t)} + \frac{3}{4}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2 + \varphi'^2(t)\, q \circ \varphi(t) = q(t)$$

$$-\frac{1}{2}\frac{\psi'''(t)}{\psi'(t)} + \frac{3}{4}\left(\frac{\psi''(t)}{\psi'(t)}\right)^2 + \psi'^2(t)\, p \circ \psi(t) = p(t), \qquad t \in I.$$

From this identities it follows immediately

$$-\frac{1}{2}\left(\frac{\varphi'''(t)}{\varphi'(t)} - \frac{\psi'''(t)}{\psi'(t)}\right) + \frac{3}{4}\left(\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2 - \left(\frac{\psi''(t)}{\psi'(t)}\right)^2\right) + \varphi'^2(t)\, q \circ \varphi(t) -$$
$$- \psi'^2(t)\, p \circ \psi(t) = q(t) - p(t), \qquad t \in I. \tag{8}$$

By the Mean Value Theorem there exists $\xi = \xi(t)$ situated between $\varphi(t)$ and $\psi(t)$ such that there hold the equalities

$$\varphi'^2(t)\, q \circ \varphi(t) - \psi'^2(t)\, p \circ \psi(t) = (q \circ \psi(t) + q'(\xi)\,(\varphi(t) - \psi(t))\, \varphi'^2(t) -$$
$$- \psi'^2(t)\, p \circ \psi(t) = (q \circ \psi(t) - p \circ \psi(t))\, \varphi'^2(t) + (\varphi'^2(t) - \psi'^2(t))\, p \circ \psi(t) +$$
$$+ (\varphi(t) - \psi(t))\, \varphi'^2(t)\, q'(\xi). \tag{9}$$

We know from Lemma 2 that the functions φ', ψ' are bounded on I and because of our assumptions p, q, q' are also bounded on I we obtain from (9) respecting the relations $\lim_{t\to\infty} (q(t) - p(t)) = 0$, $\lim_{t\to\infty} (\varphi(t) - \psi(t))^{(j)} = 0$ $(j = 0, 1)$

$$\lim_{t\to\infty} (\varphi'^2(t)\, q \circ \varphi(t) - \psi'^2(t)\, p \circ \psi(t)) = 0 \tag{10}$$

Defining $f_1 : I \to R$, $f_1 : t \to 4\varphi'^2(t)\,\psi'^2(t)\,(q(t) - p(t) + \psi'^2(t)\,p \circ \psi(t) - \varphi'^2(t)\,q \circ \circ\,\varphi(t))$, we have $\lim_{t\to\infty} f_1(t) = 0$ and it is possible to write the identity (8) in the form

$$-2\varphi'(t)\,\psi'(t)\,(\varphi'''(t)\,\psi'(t) - \psi'''(t)\,\varphi'(t)) + 3(\varphi''(t)\,\psi'(t) - \varphi'(t)\,\psi''(t))\,(\varphi'(t)\psi'(t))' = = f_1(t),\ t \in I. \tag{11}$$

Putting now $f_2 : I \to R$, $f_2 : t \to -\dfrac{1}{2\varphi'^2(t)\,\psi'(t)}\,(2\varphi'(t)\,\psi'(t)\,\varphi'''(t)\,(\psi(t) - \varphi(t))' - 3\varphi''(t)\,(\psi(t) - \varphi(t))'\,(\varphi'(t)\,\psi'(t))' + f_1(t))$ we have $\lim_{t\to\infty} f_2(t) = 0$ and using following identities

$$\varphi'''\psi' - \psi'''\varphi' = \varphi'(\varphi - \psi)''' + \varphi'''(\psi - \varphi)',$$
$$\varphi''\psi' - \varphi'\psi'' = \varphi''(\psi - \varphi)' + \varphi'(\varphi - \psi)'',$$

we obtain finally

$$(\varphi(t) - \psi(t))''' - \frac{3}{2}(\varphi(t) - \psi(t))''\,\frac{(\varphi'(t)\,\psi'(t))'}{\varphi'(t)\,\psi'(t)} = f_2(t), \qquad t \in I. \tag{12}$$

Let us introduce the function $Y : I \to R$, $Y : t \to (\varphi(t) - \psi(t))'$. It is possible to show $\lim_{t\to\infty} Y'(t) = 0$ as follows. There is $\lim_{t\to\infty} Y(t) = 0$ (from assumptions) and from (12) we see that Y is a solution of the equation

$$z'' - \frac{3}{2}(\ln \varphi'\psi')'\,z' = f_2. \tag{13}$$

By means of substitution $y : t \to (\varphi'(t)\,\psi'(t))^{-3/4}\,z(t)$ the equation (13) becomes

$$y'' + \left(-\frac{21}{16}(\varphi'\psi')^{-2}(\varphi'\psi')'^2 + \frac{3}{4}(\varphi'\psi')^{-1}(\varphi'\psi')''\right)y = F_1. \tag{14}$$

Here $F_1 : t \to (\varphi'(t)\,\psi'(t)^{-3/4}f_2(t)$ and $\lim_{t\to\infty} F_1(t) = 0$. Having performed the transformations just described it is clear that there exists such a solution y_1 of the equation (14) that $Y(t) = (\varphi'(t)\,\psi'(t))^{3/4}y_1(t)$, $t \in I$ holds and $\lim_{t\to\infty} y_1(t) = 0$. Let us pose finally $F : t \to F_1(t) - \left(-\dfrac{21}{16}(\varphi'(t)\,\psi'(t))^{-2}\,(\varphi'(t)\,\psi'(t))'^2 + \dfrac{3}{4}(\varphi'(t)\,\psi'(t))^{-1}\,.\,(\varphi'(t)\,\psi'(t))''\right)y_1(t) - y_1(t)$. We have $\lim_{t\to\infty} F(t) = 0$ and

$$y_1''(t) - y_1(t) = F(t), \qquad t \in I.$$

Thus y_1 is a solution of the equation

$$y'' - y = F. \tag{15}$$

and it is possible to write (here b_1, b_2 denote suitable constants)

$$y_1(t) = b_1e^t + b_2e^{-t} + \frac{1}{2}\int_a^t (e^{t-s} - e^{s-t})F(s)\,ds =$$

$$= e^t\left(b_1 + \frac{1}{2}\int_a^t e^{-s}F(s)\,ds\right) + e^{-t}\left(b_2 - \frac{1}{2}\int_a^t e^sF(s)\,ds\right).$$

From $\lim\limits_{t\to\infty} y_1(t) = 0$ and $\lim\limits_{t\to\infty} \dfrac{b_2 - \frac{1}{2}\int_a^t e^sF(s)\,ds}{e^t} = 0$ we obtain $b_1 = -\frac{1}{2}\int_a^\infty e^{-s}F(s)\,ds$.
Hence

$$y_1(t) = -\frac{e^t}{2}\int_t^\infty e^{-s}F(s)\,ds + b_2e^{-t} - \frac{e^{-t}}{2}\int_a^t e^sF(s)\,ds$$

and further immediately

$$y_1'(t) = -\frac{e^t}{2}\int_t^\infty e^{-s}F(s)\,ds + \frac{F(t)}{2} - b_2e^{-t} + \frac{e^{-t}}{2}\int_a^t e^sF(s)\,ds - \frac{F(t)}{2} =$$

$$= -\frac{e^t}{2}\int_t^\infty e^{-s}F(s)\,ds - b_2e^{-t} + \frac{e^{-t}}{2}\int_a^t e^sF(s)\,ds.$$

The last identity and the relations $\lim\limits_{t\to\infty} \dfrac{\int_t^\infty e^{-s}F(s)\,ds}{e^{-t}} = \lim\limits_{t\to\infty} F(t) = 0$, $\lim\limits_{t\to\infty} \dfrac{\int_a^t e^sF(s)\,ds}{e^t} =$ $= \lim\limits_{t\to\infty} F(t) = 0$ imply $\lim\limits_{t\to\infty} y_1'(t) = 0$. From the identity $Y'(t) = \frac{3}{4}(\varphi'(t)\,\psi'(t))^{-1/4}\,.$ $.(\varphi'(t)\,\psi'(t))'\,y_1(t) + (\varphi'(t)\psi'(t))^{3/4}y_1'(t)$ it follows $\lim\limits_{t\to\infty}(\varphi(t) - \psi(t))'' = \lim\limits_{t\to\infty} Y'(t) = 0$. The latter and (12) finally give $\lim\limits_{t\to\infty}(\varphi(t) - \psi(t))''' = 0$.

Let us assume that the equatilies $\lim\limits_{t\to\infty}(\varphi(t) - \psi(t))^{(j)} = 0$ are valid for $j = 0, 1, \ldots, k, k < n + 3$, $k \geqq 3$. In the proof of thisTheorem we have deduced the formula

$$(\varphi(t) - \psi(t)''' - \frac{3}{2}(\varphi(t) - \psi(t))''\,\frac{(\varphi'(t)\,\psi'(t))'}{\varphi'(t)\,\psi'(t)} =$$

$$= -\frac{1}{2\varphi'^2(t)\,\psi'(t)}(2\varphi'(t)\,\psi'(t)\,\varphi'''(t))(\psi(t) - \varphi(t))' - 3\varphi''(t)(\psi(t) - \varphi(t))'(\varphi'(t)\psi'(t))' +$$
$$+ 4\varphi'^2(t)\,\psi'^2(t)\,(q(t) - p(t) + \psi'^2(t)\,p \circ \psi(t) - \varphi'^2(t)\,q \circ \varphi(t)), \quad t \in I.$$

Calculating the $(k-2)$nd derivative of the last equality and recpecting Lemma 2 and assumptions of our Theorem we see that for proving the equality $\lim_{t\to\infty}(\varphi(t) - \psi(t))^{(k+1)} = 0$ it is enough to prove the validity of the relations

$$\lim_{t\to\infty}(\psi'^2(t)\, p\circ\psi(t) - \varphi'^2(t)\, q\circ\varphi(t))^{(j)} = 0, \qquad j = 0, 1, \dots, k-2.$$

Moreover we have

$$\psi'^2(t)\, p\circ\psi(t) - \varphi'^2(t)\, q\circ\varphi(t) = \psi'^2(t)\,(p\circ\psi(t) - q\circ\psi(t)) + \\ + (\psi(t) + \varphi(t))'\,(\psi(t) - \varphi(t))'\, q\circ\psi(t) + \varphi'^2(t)\,(q\circ\psi(t) - q\circ\varphi(t)),\ t\in I$$

and hence, with respect to Lemma 2, to the assumptions of our Theorem and to the assumptions $\lim_{t\to\infty}(\varphi(t) - \psi(t))^{(j)} = 0$, $j = 0, 1, \dots, k$, it is sufficient to prove the validity ot the equalities

$$\lim_{t\to\infty}(q\circ\psi(t) - q\circ\varphi(t))^{(j)} = 0, \qquad j = 0, 1, \dots, k-2. \tag{16}$$

By assumption the function q' is bounded on I and $\lim_{t\to\infty}(\varphi(t) - \psi(t)) = 0$. Thus we have $\lim_{t\to\infty}(q\circ\psi(t) - q\circ\varphi(t)) = 0$. Further, the following identities are valid:

$$(q\circ\varphi(t) - q\circ\psi(t))' = (q'\circ\varphi(t) - q'\circ\psi(t))\,\varphi'(t) + \\ + (\varphi(t) - \psi(t))'\, q'\circ\psi(t), \qquad t\in I. \tag{17}$$

$$(q\circ\varphi(t) - q\circ\psi(t))^{(j)} = \sum_{i=0}^{j-1}\binom{j-1}{i}(q'\circ\varphi(t) - q'\circ\psi(t))^{(i)}\,\varphi^{(j-i)}(t) + \\ + ((\varphi(t) - \psi(t))'\, q'\circ\psi(t))^{(j-1)}, \qquad t\in I, \qquad j = 2, \dots, k-2. \tag{18}$$

By the assumptions $\lim_{t\to\infty}(\varphi(t) - \psi(t))^{(j)} = 0, j = 0, 1, \dots, k$ and the functions $q^{(j)}$, $j = 0, 1, \dots, n+1$ are bounded on I. Thus from (17) and (18) we have $\left(\binom{0}{0} := 1\right)$

$$\lim_{t\to\infty}(q\circ\varphi(t) - q\circ\psi(t))^{(j)} = \lim_{t\to\infty}\sum_{i=0}^{j-1}\binom{j-1}{i}(q'\circ\varphi(t) - q'\circ\psi(t))^{(i)}\,\varphi^{(j-i)}(t), \\ (j = 1, 2, \dots, k-2).$$

Since the functions $\varphi^{(j)}$, $j = 1, 2, \dots, n+3$ are bounded on I (by Lemma 2 we see from the last equalities that for proving (16) it is sufficient to prove the validit of the identities

$$\lim_{t\to\infty}(q'\circ\varphi(t) - q'\circ\psi(t))^{(j)} = 0, \qquad j = 0, 1, \dots, k-3. \tag{19}$$

Let us now observe that (19) differs from (16) in the manner that q' stands instead of q and subscript j changes from 0 to $k-3$. Similarly to the first part of the proof the identities (17) and (18) hold, where instead of q, q' we now write q', q'' and j in (18) changes between the limits 2 and $k - 3$. Then it is easy to prove that $\lim_{t\to\infty} (q' \circ \varphi(t) - q' \circ \psi(t)) = 0$. Proceeding in this manner we come, having performed a finite number of steps, to the remaining equality $\lim_{t\to\infty} (q^{(k-2)} \circ \varphi(t) - q^{(k-2)} \circ \psi(t)) = 0$. This last equality can be deduced immediately from the Mean Value Theorem, from the boundedness of $q^{(k-1)}$ and from the validity of $\lim_{t\to\infty} (\varphi(t) - \psi(t)) = 0$.

Remark. Some conditions sufficient for $\lim_{t\to\infty} (\varphi(t) - \psi(t))' = 0$ are derived in the papers [2], [3].

Corollary. *Let i be a positive integer and ψ_i, φ_i the central dispersions of the first kind with the index i of the equations* (p), (q) *respectively.*

If the assumptions of the first part of our Theorem are satisfied, then

$$\lim_{t\to\infty} (\varphi_i(t) - \psi_i(t)) = 0.$$

If the assumptions of the second part of our Theorem are satisfied, then from the validity of the relation

$$\lim_{t\to\infty} (\varphi_i(t) - \psi_i(t))' = 0$$

follows also the validity of

$$\lim_{t\to\infty} (\varphi_i(t) - \psi_i(t))^{(j)} = 0, \qquad j = 0, 1, \ldots, n + 3. \tag{20}$$

Proof. By the first part of the Theorem $(\varphi_1 = \varphi, \psi_1 = \psi) \lim_{t\to\infty} (\varphi_1(t) - \psi_1(t)) = 0$. If $\lim_{t\to\infty} (\varphi_i(t) - \psi_i(t)) = 0$ for $i = k$, $k \geqq 1$, then by the known formulas (cf. [1], p. 105) $\varphi_i(t) = \underbrace{\varphi \circ \ldots \circ}_{i} \varphi(t)$, $\psi_i(t) = \underbrace{\psi \circ \ldots \circ}_{i} \psi(t)$ we get immediately $\varphi_{k+1}(t) - \psi_{k+1}(t) = \varphi \circ \varphi_k(t) - \psi \circ \psi_k(t) = (\varphi \circ \varphi_k(t) - \varphi \circ \psi_k(t)) + (\varphi \circ \psi_k(t) - \psi \circ \psi_k(t)) = \varphi'(\eta) (\varphi_k(t) - \psi_k(t)) + (\varphi \circ \psi_k(t) - \psi \circ \psi_k(t))$ where η lies between $\varphi_k(t)$ and $\psi_k(t)$. From Lemma 2 and from $\lim_{t\to\infty} (\varphi_k(t) - \psi_k(t)) = 0$ we obtain then $\lim_{t\to\infty} (\varphi_{k+1}(t) - \psi_{k+1}(t)) = 0$. The first part of the Corollary is proved.

Let the assumptions of the second part of the Theorem be satisfied. By the first part of the Corollary the equalities (20) are valid for $j = 0$, $i = 1, 2, 3, \ldots$. Let us suppose these equalities are true also for $j = 0, 1, \ldots, k$, $0 \leqq k < n + 3$, $i = 1, 2, 3, \ldots$. We are going to prove that under these assumptions the equalities (20) are valid also for $j = k + 1$, $i = 1, 2, 3, \ldots$. From the Theorem there follows the relation

$$\lim_{t\to\infty} (\varphi(t) - \psi(t))^{(k+1)} = 0.$$

Let us assume that $\lim_{t\to\infty} (\varphi_i(t) - \psi_i(t))^{(k+1)} = 0$ holds for $i = 1, 2, \ldots, s$, $s \geqq 1$. From the following chain of identities

$$\begin{aligned}(\varphi_{s+1}(t) - \psi_{s+1}(t))^{(k+1)} &= (\varphi \circ \varphi_s(t) - \psi \circ \psi_s(t))^{(k+1)} = \\ &= (\varphi_s'(t)\, \varphi' \circ \varphi_s(t) - \psi_s'(t)\, \psi' \circ \psi_s(t))^{(k)} = \\ = \sum_{j=0}^{k} \binom{k}{j} \big(\varphi_s^{(k+1-j)}(t)\, &(\varphi' \circ \varphi_s(t))^{(j)} - \psi_s^{(k+1-j)}(t)\, (\psi' \circ \psi_s(t))^{(j)}\big) = \\ &= \sum_{j=0}^{k} \binom{k}{j} (\varphi' \circ \varphi_s(t))^{(j)} (\varphi_s(t) - \psi_s(t))^{(k+1-j)} + \\ &+ \sum_{j=0}^{k} \binom{k}{j} \psi_s^{(k+1-j)}(t) (\varphi' \circ \varphi_s(t) - \psi' \circ \psi_s(t))^{(j)}\end{aligned}$$

it can be deduced that for proving $\lim_{t\to\infty} (\varphi_{s+1}(t) - \psi_{s+1}(t)(({}^{k+1}) = 0$ it is sufficient to prove

$$\lim_{t\to\infty} (\varphi' \circ \varphi_s(t) - \psi' \circ \psi_s(t))^{(j)} = 0, \qquad j = 0, 1, \ldots, k. \tag{21}$$

From the relations

$$\begin{aligned}\varphi' \circ \varphi_s(t) - \psi' \circ \psi_s(t) &= (\varphi' \circ \varphi_s(t) - \psi' \circ \varphi_s(t)) + (\psi' \circ \varphi_s(t) - \psi' \circ \psi_s(t)) = \\ &= (\varphi' \circ \varphi_s(t) - \psi' \circ \varphi_s(t)) + \psi''(\xi)\, (\varphi_s(t) - \psi_s(t)),\end{aligned}$$

(with ξ between φ_s and ψ_s) we see at once that (21) holds for $j = 0$. We have further $(j > 0)$

$$\begin{aligned}(\varphi' \circ \varphi_s(t) - \psi' \circ \psi_s(t))^{(j)} &= (\varphi_s'(t)\, \varphi'' \circ \varphi_s(t) - \psi_s'(t)\, \psi'' \circ \psi_s(t))^{(j-1)} = \\ &= \sum_{l=0}^{j-1} \binom{j-1}{l} (\varphi'' \circ \varphi_s(t))^{(l)} (\varphi_s(t) - \psi_s(t))^{(j-l)} + \\ &+ \sum_{l=0}^{jj1} \binom{j-1}{l} \psi_s^{(j-l)}(t)\, (\varphi'' \circ \varphi_s(t) - \psi'' \circ \psi_s(t))^{(l)}.\end{aligned}$$

With respect to the assumptions of the induction and to Lemma 2 it is sufficient for proving (21) to verify the equalities

$$\lim_{\to\infty} (\varphi'' \circ \varphi_s(t) - \psi'' \circ \psi_s(t))^{(j)} = 0, \qquad j = 0, 1, \ldots, k - 1. \tag{22}$$

We see that the form of the equalities (22) can be obtained formally from the equalities (21) where instead of $\varphi'(\psi')$ there is $\varphi''(\psi'')$ and in (22) $j = 0, 1, \ldots, k - 1$.

Thus we see that performing a procedure analogous to that before we reduce our problem (analogous to the proof of our Theorem) to the proof of the validity of

$$\lim_{t \to \infty} (\varphi^{(k+1)} \circ \varphi_s(t) - \psi^{(k+1)} \circ \psi_s(t)) = 0.$$

But this proof is straightforward by using the assumptions of induction, Lemma 2, the boundedness of the function $q^{(n+1)}$ on I and the equality

$$\begin{aligned} &\varphi^{(k+1)} \circ \varphi_s(t) - \psi^{(k+1)} \circ \psi_s(t) = \\ &= (\varphi^{(k+1)} \circ \varphi_s(t) - \varphi^{(k+1)} \circ \psi_s(t)) + (\varphi^{(k+1)} \circ \psi_s(t) - \psi^{(k+1)} \circ \psi_s(t)) = \\ &= (\varphi^{(k+1)} \circ \psi_s(t) - \psi^{(k+1)} \circ \psi_s(t)) + \varphi^{(k+2)}(\eta)(\varphi_s(t) - \psi_s(t)) \end{aligned}$$

where η is situated between $\varphi_s(t)$ and $\psi_s(t)$.

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