Archivum Mathematicum

Irena Rachůnková

On a coincidence of central dispersions of the first and second kind in connection with periodic solutions of the differential equation $y' = q(t)y$

Archivum Mathematicum, Vol. 11 (1975), No. 4, 205--216

Persistent URL: http://dml.cz/dmlcz/104859

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XI: 205—216, 1975

ON A COINCIDENCE OF CENTRAL DISPERSIONS OF THE FIRST AND SECOND KIND IN CONNECTION WITH PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION $y'' = q(t)y$

IRENA RACHŮNKOVÁ, Olomouc
(Received August 14, 1974)

This paper will be devoted to the study of the properties of phases and dispersions of the 2nd order differential equation $y'' = q(t)y$. In the first part we shall describe the set of all increasing phases of all the differential equations whose every solution is half-periodic with exactly one zero on the interval of the periodlength. There is found a connection between these differential equations and those having the basic central dispersions of the 1st and 2nd kind coinciding on the interval $(-\infty, \infty)$.

In the second part there is derived a necessary and sufficient condition for a coincidence of the n-th central dispersions of the 1st and 2nd kind on the interval $(-\infty, \infty)$. Moreover, there is described the set of all increasing phases of all the differential equations whose every solution is periodic (n even) or half-periodic (n odd) and has exactly n zeros on the interval of the periodlength. Further, properties of this set and its subsets are investigated.

The paper is closed with establishing a connection between the foregoing differential equations and such equations having the n-th central dispersions of the 1st and 2nd kind coinciding on the interval $(-\infty, \infty)$.

1. Basic concepts and relations used in this paper are taken from [1], where they are defined and proved. For completeness, we give below a brief summary of them.

We shall consider a both-side oscillatory differential equation

$$(q) \qquad y'' = q(t)y,$$

where the carrier $q(t)$ is a continuous function on the interval $(-\infty, \infty)$, that is, $q(t) \in C^0$. Let $u(t)$, $v(t)$ be a base of the differential equation (q), that is, a pair of linearly independent solutions of (q). A function α, continuous on $(-\infty, \infty)$ and satisfying the relation

$$\tan \alpha(t) = u(t)/v(t)$$

everywhere where $v(t) \neq 0$, is called the first phase of (q) corresponding to the base $u(t)$, $v(t)$ (henceforth a phase of (q)). For every phase α of the differential equation

(q) there holds $\alpha \in C^3$, $\alpha'(t) \neq 0$ for $t \in (-\infty, \infty)$. The converse is valid, too. Namely, the function α satisfying the property

$$\alpha \in C^3, \qquad \alpha'(t) \neq 0 \qquad \text{for } t \in (-\infty, \infty)$$

is a phase of the differential equation (q) where q is determined by the relation

$$q(t) = -\{\tan \alpha, t\} = -\{\alpha, t\} - (\alpha'(t))^2 = -(1/2)\,\alpha'''/\alpha' + (3/4)\,(\alpha''/\alpha')^2 - (\alpha')^2.$$

Let $t_0 \in (-\infty, \infty)$, and y be a nontrivial solution of (q), whereby $y(t_0) = 0$. Let $\varphi(t_0) \in (-\infty, \infty)$ be the first zero of the solution y lying on the right of t_0. Then φ is called the basic central dispersion of the 1st kind of the differential equation (q) (henceforth the basic central dispersion). Similarly, if $\varphi_n(t_0)$ $[\varphi_{-n}(t_0)]$ is the n-th zero of the solution y lying on the right [on the left] of t_0, the function $\varphi_n[\varphi_{-n}]$ is called the n-th [-n-th] central dispersion of the 1st kind of (q) (henceforth n-th [-n-th] central dispersion).

If α is a phase of the differential equation (q) and φ is the 1st kind basic central dispersion of the differential equation (q), then Abel's equation

$$\alpha(\varphi(t)) = \alpha(t) + \pi \, . \, \operatorname{sgn} \alpha'$$

is satisfied on the whole interval $(-\infty, \infty)$. Similarly the n-th dispersion φ_n, $n = 0, \pm 1, \pm 2, \ldots$, satisfies

$$\alpha(\varphi_n(t)) = \alpha(t) + n\pi \operatorname{sgn} \alpha'.$$

The following theorems are valid in the sequel.

Theorem 1.1. *The set $\bar{G}$ of all phases of all oscillatory differential equations (q) with an operation of composition of functions forms a group.*

Theorem 1.2. *The set $\bar{E}$ of all phases corresponding to the equation (-1) is a subgroup of the group $\bar{G}$. It is called a basic subgroup.*

Theorem 1.3. *Let $\bar{G}/_r\bar{E}$ be a righthanded decomposition of the group $\bar{G}$. Then any class of this decomposition is formed by exactly all the phases belonging to an appropriate equation (q).*

Every equation (q) has an infinite number (continuum) of countable phase systems $\ldots < \alpha_{-2} < \alpha_{-1} < \alpha_0 < \alpha_1 < \alpha_2 \ldots$, every system belonging to exactly one base of the equation (q). Hence the set of all bases of the differential equation (q) is equivalent to the set of all countable phase systems of this differential equation.

Theorem 1.4. *If it holds $w < 0[w > 0]$ on $(-\infty, \infty)$ for the Wronskian w of the base u, v of the differential equation (q) then all the phases of the corresponding phase system are simultaneously increasing [decreasing].*

Thus, if we choose a base u, v of the equation (q) such that the corresponding Wronskian $w < 0$, and then perform all the transformations of this base the determinant of which is greater than zero, we obtain exactly all the bases to which exactly all the systems of the increasing phases correspond. Every class of the decomposition $\bar{G}/_r\bar{E}$ can be therefore decomposed into two equivalent subsets: the set of all increasing phases and the set of all decreasing phases of the differential equation (q). Consequently the basic subgroup $\bar{E}$, too, can be decomposed into the (normal) subgroup E of all increasing phases of the differential equation (-1) and the coset of all decreasing phases of that equation.

Theorem 1.5. *The subset G of the group $\bar{G}$ consisting of exactly all increasing phases of all oscillatory equations (q) is a normal subgroup of the group $\bar{G}$.*

This evidently implies that the following theorems hold.

Theorem 1.6. *Every class of the (righthanded) decomposition $G/_rE$ of the group G is formed by exactly all increasing phases belonging to the appropriate equation (q).*

Let us define the $1-1$ mapping $\Phi : G/_rE \to \bar{G}/_r\bar{E}$ by $\Phi(E) = \bar{E}$, $\Phi(\alpha E) = \alpha\bar{E}$ for each $\alpha \in G$. Corresponding classes belong to the same differential equation (q).

Theorem 1.7. *In the group $\bar{G}$ the subset $\bar{H}$ of all elementary phases, that is, the subset of all phases satisfying the condition*

$$\alpha c = c_{\operatorname{sgn}\alpha'}\alpha, \qquad \text{where } c(t) = t + \pi, \qquad c(t)_{\operatorname{sgn}\alpha'} = t + \operatorname{sgn}\alpha' \,.\, \pi$$

forms a subgroup. It holds $\bar{G} \supset \bar{H} \supset \bar{E}$.

The group $\bar{H}$ can again be decomposed into the subgroup H of all increasing elementary phases and the coset of all decreasing elementary phases.

It is evident that for any phase $\alpha \in H$ there holds

$$\alpha c = c\alpha. \tag{c, c}$$

The cyclic group C of the phases $c_n(t) = t + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ is a subgroup of the group E and it holds $G \supset H \supset E \supset C$, where C is the centre of H.

2. In this section we shall be concerned exclusively with increasing phases, that is, with the groups G, H, E, C; we shall therefore drop the word "increasing" in the writing and shall simply say "phases".

Theorem 2.1. *H is the group of phases of exactly all the equations whose basic central dispersion is c, that is $\varphi(t) = t + \pi$.*

Proof: Let $\alpha \in H$. Then $\alpha c = c\alpha$ and from Abel's equation $\alpha\varphi = c\alpha$ we obtain $\varphi = \alpha^{-1}c\alpha = \alpha^{-1}\alpha c = c$. Let $\varphi = c$. This gives us $\alpha\varphi = c\alpha$ leading to $\alpha c = c\alpha$ and consequently $\alpha \in H$. It holds (see [1]) that if an equation has one elementary phase, then all its phases are elementary ones, too.

Let us consider the group G with the subgroup H and let us form the decomposition $G/_rH$. It holds $G/_rH > G/_rE$, that is, $G/_rH$ is a superposition of the decomposition $G/_rE$.

Theorem 2.2. *Let $\mathscr{P}$ be the set of all phases from G belonging to those equations* *having the same basic central dispersion $\varphi = t + k$; $k > 0$, const. Then $\mathscr{P}$ forms exactly one class in $G/_r H$.*

Proof. Let us write $\boldsymbol{k}(t) = t + k$. Then it holds for any phase $f \in \mathscr{P}$ (based on Abel's equation)

$$(k, c) \qquad f\boldsymbol{k} = cf,$$

and conversely, any phase f with the property (k, c) belongs to $\mathscr{P}$, because it is a phase of a differential equation with the basic central dispersion $\varphi = \boldsymbol{k}(t) = t + k$. Namely, $\varphi = f^{-1}cf = f^{-1}f\boldsymbol{k} = \boldsymbol{k}$.

Next for an arbitrary $f \in \mathscr{P}$ and $h \in H$ it holds $hf\boldsymbol{k} = hcf = chf$ and therefore $hf \in \mathscr{P}$ which results in $Hf \subset \mathscr{P}$. Conversely, if there is an arbitrary phase $g \in \mathscr{P}$, $f \in \mathscr{P}$, then $\boldsymbol{k}^{-1}f^{-1} = f^{-1}c^{-1}$ and consequently $gf^{-1} = g\boldsymbol{k}\boldsymbol{k}^{-1}f^{-1} = cgf^{-1}c^{-1}$ which means $gf^{-1}c = cgf^{-1}$ and finally $gf^{-1} \in H$. Therefore $g \in Hf$ and so $\mathscr{P} \subset Hf$.

Theorem 2.3. *To any function $\boldsymbol{k}(t) = t + k$, $k > 0$, const., there exists exactly one differential equation (q) with the constant carrier $q = -(\pi/k)^2$ whose basic central dispersion $\varphi = \boldsymbol{k}$.*

Proof. We show first that the differential equation $y'' = -(\pi/k)^2\, y$ has $\varphi = t + k$. For this it suffices to find one phase of this differential equation satisfying the condition (k, c). The considered equation $(-(\pi/k)^2)$ has, for instance, the base $u = \sin(\pi/k)\, t$, $v = \cos(\pi/k)\, t$. The corresponding system of phases α_n has a form $\alpha_n = (\pi/k)\, t + n\pi$, $n = 0, \pm 1, \ldots$ An arbitrary phase α_n of this system satisfies the condition (k, c) and following this we can write $\varphi = \alpha_n^{-1} c \alpha_n = \alpha_n^{-1}\alpha_n \boldsymbol{k} = \boldsymbol{k}(t) = t + k$.

Next we see that the mapping $k \to -(\pi/k)^2$ is a $1-1$ mapping of the set of all positive numbers k onto the set of all negative numbers $-(\pi/k)^2$. This, of course, implies that there exists exactly one equation with a constant carrier for every function $\boldsymbol{k}(t) = t + k$.

With the foregoing theorems we can now state a theorem as follows:

Theorem 2.4. *Let $\mathscr{P} \in G/_r H$ be the class of all the phases satisfying the condition (k, c). This yields*

$$\mathscr{P} = H\,.\,f,$$

where f is an arbitrary (increasing) phase of the differential equation $y'' = -(\pi/k)^2\, y$.

In [2] there is derived the following

Theorem 2.5. *The differential equation (q) has the dispersion φ satisfying the equation*

$$\varphi_n(t) = t + k$$

for a positive integer n and for all $t \in (-\infty, \infty)$ if and only if every solution of (q) is *periodic (n even) of half-periodic (n odd) with period k and has exactly n zeros on the interval* $[0, k)$.

In the special case $n = 1$ we have (see also [2]):

Theorem 2.6. *The differential equation (q) has the dispersion* φ *satisfying*

$$\varphi(t) = t + k$$

for every $t \in (-\infty, \infty)$ *if and only if every solution of the differential equation (q) is half-periodic with period k and has exactly one zero on the interval* $[0, k)$.

From Theorems 2.2, 2.4 and 2.6 it follows

Theorem 2.7. *Let* $\mathcal{P}$ *be the set of all phases of all the equations whose every solution is half-periodic with period k and has exactly one zero on the interval* $[0, k)$. *Then*

$$\mathcal{P} = H \,.\, f,$$

where f is an arbitrary phase of the differential equation $y'' = -(\pi/k)^2\, y$.

Next it holds

Theorem 2.8. *Let* $\mathcal{R}$ *be the set of all phases of all the equations whose every solution is half-periodic with exactly one zero on the halfclosed interval of the appropriate periodlength. Thus we arrive at*

$$\mathcal{R} = \bigcup_{k \in R^+} H \,.\, f_k,$$

where R^+ *is the set of all positive real numbers and* f_k *is an arbitrary phase of the differential equation* $y'' = -(\pi/k)^2\, y$.

In addition to all this, let us now suppose at the differential equation (q) that $q(t) \in C^2$, $q(t) < 0$ for each $t \in (-\infty, \infty)$.

Let $t_0 \in (-\infty, \infty)$, y be a nontrivial solution of (q) wherein $y'(t_0) = 0$. Let $\psi(t_0) \in \in (-\infty, \infty)$ be the first zero of the function y' lying on the right of t_0. Then ψ is called the basic central dispersion of the 2nd kind of the differential equation (q).

Similarly, if $\psi_n(t_0)$ $[\psi_{-n}(t_0)]$ is the n-th zero of the function y' lying on the right [left] of t_0, then the function $\psi_n[\psi_{-n}]$ is called the n-th [-n-th] central dispersion of the 2nd kind of the differential equation (q).

It will be always pointed out when central dispersions of the 2nd kind are being discussed. The simple notion of dispersion will mean a dispersion of the 1st kind all the time.

A carrier $q(t)$ is called an F-carrier if for the basic central dispersion φ of the 1st kind and for the basic central dispersion ψ of the 2nd kind there holds $\varphi = \psi$ for each $t \in (-\infty, \infty)$. (See [1].)

Theorem 2.9. *q is an F-carrier if and only if the dispersion* φ *satisfies the equation* $\varphi(t) = t + k$, $k > 0$, *const.*

This theorem is derived in [3]. From last two theorems it follows

Theorem 2.10. *Let $\mathscr{F}$ be the set of all phases of all the equations with F-carriers (i.e. with coinciding basic central dispersions of the 1st and 2nd kinds). Then*

$$\mathscr{F} = \{\alpha \in \mathscr{R}: \{\alpha, t\} + (\alpha')^2 > 0, \alpha \in C^5 \text{ for each } t \in (-\infty, \infty)\}.$$

This means that all F-carriers can be characterized by the phases of all negative elementary carriers from C^2 and by the phases of all negative constant carriers.

3. Again suppose that in addition there holds $q(t) \in C^2$, $q(t) < 0$ for each $t \in \in (-\infty, \infty)$.

Let $u(t)$, $v(t)$ be a base of the differential equation (q). A function β continuous on $(-\infty, \infty)$ and satisfying the relation

$$\tan \beta(t) = u'(t)/v'(t)$$

for $v'(t) \neq 0$ is called the 2nd phase of the differential equation (q). For an arbitrary second phase β of the differential equation (q) there holds: $\beta \in C^1$, $\beta'(t) \neq 0$ for $t \in \in (-\infty, \infty)$.

It will be always pointed out when the 2nd phases are being discussed. The simple notion of phase will mean the 1st phase all the time.

If β is the second phase of (q) and ψ the basic central dispersion of the 2nd kind of (q) then there holds Abel's equation

$$\beta(\psi(t)) = \beta(t) + \pi \operatorname{sgn} \beta'$$

for each $t \in (-\infty, \infty)$. Similarly for the n-th dispersion of the 2nd kind ψ_n, $n = 0, \pm 1, \pm 2, \dots$, there holds

$$\beta(\psi_n(t)) = \beta(t) + n\pi \operatorname{sgn} \beta'.$$

By a polar function of a base u, v of (q) we mean the function $\vartheta = \beta - \alpha$, $t \in \in (-\infty, \infty)$, where α and β are the first and the second phases of the base u, v, respectively. The phases α and β are either both increasing or both decreasing (see [1]).

We now define a function $h(\alpha)$ on $(-\infty, \infty)$ by $h(\alpha) = \vartheta\alpha^{-1}(\alpha) = \vartheta(t)$. The function h is called a normed polar function of the 1st kind (see [1, § 6]).

Let φ_n and ψ_n be the n-th central dispersions of the 1st and 2nd kinds of (q), respectively. If it holds $\varphi_n = \psi_n$ for $t \in (-\infty, \infty)$ then the carrier q will be called an F_n-carrier.

Theorem 3.1. *A carrier q is an F_n-carrier if and only if a normed polar function of the 1st kind h is periodic with period $n\pi$.*

Proof. a) Let q be an F_n-carrier. Then $\varphi_n = \psi_n$ and we can write $h[\alpha + \varepsilon n\pi] = h\alpha(\varphi_n) = \beta(\varphi_n) - \alpha(\varphi_n) = \beta(\psi_n) - \alpha(\varphi_n) = [\beta(t) + \varepsilon n\pi] - [\alpha(t) + \varepsilon n\pi] = h\alpha(t)$. ($\varepsilon = \operatorname{sgn} \alpha' = \operatorname{sgn} \beta'$.) b) Let $h[\alpha + n\pi] = h(\alpha)$ for $\alpha \in (-\infty, \infty)$. Then for each $t \in (-\infty, \infty)$ there holds $\beta(\varphi_n(t)) = \alpha\varphi_n(t) + h\alpha\varphi_n(t) = \alpha(t) + \varepsilon n\pi + h[\alpha(t) + \varepsilon n\pi] = \alpha(t) + \varepsilon n\pi + h\alpha(t) = \beta(t) + \varepsilon n\pi$, which leads to $\varphi_n(t) = \psi_n(t)$.

Theorem 3.2. *A carrier q is an F_n-carrier if and only if the n-th central dispersion φ_n has the form*

$$\varphi_n = t + k, k \text{ const.}$$

Proof. Let us choose a number $t_0 \in (-\infty, \infty)$ and let us put $\alpha_0 = \alpha(t_0)$, $\alpha'_0 = \alpha'(t_0)$. Then (see [1, § 6]) $\alpha'(t) = \alpha'_0 \exp\left(-2\int_{\alpha_0}^{\alpha} \cot h(\varrho)\, d\varrho\right)$ and in the points $\alpha(t) = \alpha$, $\alpha^{-1}(\alpha) = t \in (-\infty, \infty)$ it holds

$$t = t_0 + \frac{1}{\alpha'_0}\int_{\alpha_0}^{\alpha}\left(\exp 2\int_{\alpha_0}^{\sigma} \cot h(\varrho)\, \mathrm{d}\varrho\right)\mathrm{d}\sigma.$$

Substituting $\varphi_n(t)$ for t into the last equation and using Abel's equation $\alpha(\varphi_n(t)) = \alpha(t) + \varepsilon n\pi$. ($\alpha$ may be either increasing or decreasing; $\varepsilon = \operatorname{sgn} \alpha'$.) We arrive at

$$\varphi_n(t) = t_0 + \frac{1}{\alpha'_0}\int_{\alpha_0}^{\alpha+\varepsilon n\pi}\left(\exp 2\int_{\alpha_0}^{\sigma} \cot h(\varrho)\, \mathrm{d}\varrho\right)\mathrm{d}\sigma.$$

t_0 is an arbitrary number from $(-\infty, \infty)$ so that we can write

$$\varphi_n(t) = t + \frac{1}{\alpha'_0}\int_{\alpha}^{\alpha+\varepsilon n\pi}\left(\exp 2\int_{\alpha_0}^{\sigma} \cot h(\varrho)\, \mathrm{d}\varrho\right)\mathrm{d}\sigma.$$

After differentiation and with some modification we get

$$\varphi'_n(t) = \exp 2\int_{\alpha}^{\alpha+\varepsilon n\pi} \cot h(\varrho)\, \mathrm{d}\varrho$$

and further

$$\frac{\varphi''_m(t)}{\varphi'_n(t)} = 2\alpha'_0[\cot h(\alpha + \varepsilon n\pi) - \cot h(\alpha)] \exp\left(-2\int_{\alpha_0}^{\alpha} \cot h(\varrho)\, \mathrm{d}\varrho\right).$$

By Theorem 3.1 it holds $[\cot h(\alpha + \varepsilon n\pi) - \cot h(\alpha)] = 0$; herefrom $\varphi''_n(t) = 0$, hence $\varphi'_n(t) = c$ and $\varphi_n(t) = ct + k$.

It remains to prove $c = 1$. Let us consider the sequence $\{\varphi_n(t)\}$ of n-th dispersions. It holds $\varphi_n(t) \to +\infty$ for $n \to +\infty$ and $\varphi_n(t) \to -\infty$ for $n \to -\infty$.

Thus also the selected sequence $\varphi_{n.m} \to +\infty$ for $m \to +\infty$ and $\varphi_{n.m} \to -\infty$ for $m \to -\infty$. Evidently, $\varphi_{n.m} = c^m t + k(c^m - 1)/(c - 1)$. Let $c > 1$. Then for $m \to +\infty$, $\varphi_{n.m} \to +\infty$, but for $m \to -\infty$, $\varphi_{n.m} = c^m t + k(c^m - 1)/(c - 1) \to (-k)/(c - 1) > -\infty$, a contradiction.

Suppose that $c < 1$. Then for $m \to -\infty$, $\varphi_{n.m} \to -\infty$, but for $m \to +\infty$ $\varphi_{n.m} \to k/(1 - c) < +\infty$, a contradiction.

For the purpose of fulfilling the conditions of the convergence it is necessary that $c = 1$ and consequently $\varphi_n = t + k$.

If $\varphi_n(t) = t + k$, then $\varphi_n''(t) = 0$ and thus $\cot[h(\alpha + \varepsilon n\pi)] = \cot(h(\alpha))$. By Theorem 3.1 $q(t)$ is an F_n-carrier.

Remark. Let us look now at a case of an oscillatory differential equation (q) with an interval of definition (a, b) where a resp. b is a finite number. We shall now show that if it holds $\varphi_n = \psi_n$ on (a, b) then $\varphi_n(t) = ct + k$, where $c > 1$ resp. $c < 1$ c, k const.

In fact let for instance $\varphi_n = \psi_n$ on (a, ∞). This gives us $\varphi_n = ct + k$, c, k const. (The proof is analogous to that of Theorem 3.2.) Consequently $\varphi_{nm} = c^m t + k(c^m - 1)/(c - 1)$.

In this case the points a, ∞ are the accumulation points of the set of all zeros of an appropriate differential equation and thus for the sequence $\{\varphi_n\}$ of dispersions it holds $\varphi_n(t) \to \infty$ for $n \to \infty$, $\varphi_n(t) \to a$ for $n \to -\infty$ and for each t.

And for the selected sequence $\{\varphi_{n.m}\}$ it holds, too, that $\varphi_{n.m} \to +\infty$ for $m \to +\infty$, $\varphi_{n.m} \to a$ for $m \to -\infty$. From the relation $c^m t + k(c^m - 1)/(c - 1) \to \infty$ for $m \to \infty$ follows the inequality $c \geqq 1$. From the relation $c^m t + k(c^m - 1)/(c - 1) \to a$ for $m \to -\infty$ we get $c \neq 1$ and therefore $c > 1$ must hold. Likewise for $\varphi_n = \psi_n$ on $(-\infty, b)$. Here the equation (q) under consideration has no F_n-carrier.

For the sake of simplicity, let us now consider the groups of increasing phases only, i.e. the groups G, H, E, C.

Let $H_n \subset G$ be the set of all phases α satisfying the condition

$$(c_n, c_n) \qquad \alpha c_n = c_n \alpha, \qquad \text{where} \qquad c_n = t + n\pi, n > 0.$$

Theorem 3.3. *H_n with the composition of functions is a group.*

Proof. Let $\alpha_1, \alpha_2, \alpha \in H_n$. This leads to $\alpha_1\alpha_2 c_n = \alpha_1 c_n \alpha_2 = c_n \alpha_1 \alpha_2$, $c_n^{-1}\alpha^{-1} = \alpha^{-1}c_n^{-1} \Rightarrow \alpha^{-1}c_n = c_n\alpha^{-1}$, and we find that $\alpha_1\alpha_2 \in H_n$, $\alpha^{-1} \in H_n$.

Let $\mathscr{L}_n \subset G$ be the set of all phases α satisfying the condition

$$(c, c_n) \qquad \alpha c = c_n \alpha.$$

Lemma. *Let α_1, α_2 be arbitrary phases in $\mathscr{L}_n$. Then $\alpha_1\alpha_2^{-1} \in H_n$, $\alpha_2^{-1}\alpha_2 \in H$.*

Proof. $\alpha_1 c = c_n\alpha_1 \Rightarrow c^{-1}\alpha_1^{-1} = \alpha_1^{-1}c_n^{-1}$; $\alpha_2 c = c_n\alpha_2 \Rightarrow c^{-1}\alpha_2^{-1} = \alpha_2^{-1}c_n^{-1}$. $\alpha_1\alpha_2^{-1} = \alpha_1 c c^{-1}\alpha_2^{-1} = c_n\alpha_1\alpha_2^{-1}c_n^{-1}$, thus $\alpha_1\alpha_2^{-1}c_n = c_n\alpha_1\alpha_2^{-1}$, i.e. $\alpha_1\alpha_2^{-1} \in H_n$. $\alpha_1^{-1}\alpha_2 = \alpha_1^{-1}c_n^{-1}c_n\alpha_2 = c^{-1}\alpha_1^{-1}\alpha_2 c$, thus $\alpha_1^{-1}\alpha_2 c = c\alpha_1^{-1}\alpha_2$, i.e. $\alpha_1^{-1}\alpha_2 \in H$.

Theorem 3.4. *$\mathscr{L}_n = H_n \,.\, \alpha$, where α is an arbitrary phase satisfying the condition (c, c_n).*

Proof. a) Let $f \in \mathscr{L}_n$, i.e. $fc = c_n f$. Then by the foregoing lemma $f\alpha^{-1} \in H_n$ from which we arrive at $f \in H_n\alpha$.

b) Let $f = h\alpha$, where $h \in H_n$. Then $fc = h\alpha c = hc_n\alpha = c_n h\alpha = c_n f$.

Theorem 3.5. $\mathscr{L}_n = \alpha H$, *where α is an arbitrary phase satisfying the condition* (c, c_n).

Proof. a) Let $f \in \mathscr{L}_n$. Then again by our lemma we can write $\alpha^{-1} f \in H$ and therefore $f \in \alpha H$.

b) Let $f = \alpha h$, where $h \in H$. Then $fc = \alpha hc = \alpha ch = c_n \alpha h = c_n f$.

The following theorem is a consequence of Theorems 3.4, 3.5 and 2.5.

Theorem 3.6. *The set of all phases of all the equations whose solution is periodic* (n *even*) *or half-periodic* (n *odd*) *with period* π *and has exactly* n *zeros on the interval* $[0, \pi)$ *is* $\mathscr{L}_n$.

$\mathscr{L}_n = H_n\alpha = \alpha H$, *where α is an arbitrary phase satisfying the condition* (c, c_n).

Remark. The differential equation (q) has the n-th central dispersion $\varphi_n = t + k$ if and only if $\alpha(t + k) = \alpha(t) + n\pi$, i.e. $\alpha \boldsymbol{k} = c_n \alpha$, where α is an arbitrary increasing phase of (q). This follows directly from Abel's equations.

Let us consider the classes $\mathscr{P}^{(n)} \in G/_r H$ of the phases satisfying the conditions (c_n, c), i.e. $f \in \mathscr{P}^{(n)} : fc_n = cf$. Between the system of the classes $\mathscr{P}^{(n)} \in G/_r H$ and that of the classes $\mathscr{L}_n \in G/_l H$ there exists a $1-1$ correspondence $\mathscr{P}^{(n)} = H\alpha \leftrightarrow \alpha^{-1} H = \mathscr{L}_n$, with $\alpha c_n = c\alpha$. (Evidently $\mathscr{P}^{(1)} = \mathscr{L}_1 = H$.)

Theorem 3.7. *The set of all phases of all the equations whose every solution is half-periodic with period* $n\pi$ *and has exactly one zero on* $[0, n\pi)$ *is the right coset* $H\alpha$ *in the decomposition* $G/_r H$ *and the set of all phases of all the equations whose every solution is periodic* (n *even*) *or half-periodic* (n *odd*) *with period* π *having exactly* n *zeros on the interval* $[0, \pi)$ *is the corresponding left coset* $\alpha^{-1}H$ *in the decomposition* $G/_l H$.

In other words, all the phases of the equations with periodic (n even) or half-periodic (n odd) solutions with period π and exactly n zeros on $[0, \pi)$ can be determined by means of the elementary phases and of the phases of equations with the constant carriers $q = -(1/n^2)$.

Theorem 3.8. *Let* $\mathscr{Q}_n$ *be the set of all phases of all the equations whose every solution is periodic* (n *even*) *or half-periodic* (n *odd*) *with period* k *and exactly* n *zeros on the interval* $[0, k)$. *Then it holds*

$$\mathscr{Q}_n = \mathscr{L}_n f = H_n h_n f = H_n f_n = h_n H f,$$

where the phase f *and* h_n *and* f_n *satisfy the conditions* (k, c) *and* (c, c_n) *and* (k, c_n), *respectively.*

Proof. a) Let $g \in \mathscr{Q}_n$; then by Theorem 2.5 $g\boldsymbol{k} = c_n g$, where $\boldsymbol{k}(t) = t + k$. Under the assumption $f\boldsymbol{k} = cf$, thus $\boldsymbol{k}^{-1} f^{-1} = f^{-1} c^{-1}$; continuing we obtain $g \,. f^{-1} = g\boldsymbol{k}\boldsymbol{k}^{-1} f^{-1} = c_n \,. g \,. f^{-1} c^{-1}$, therefore $gf^{-1} \in \mathscr{L}_n$ and consequently $g \in \mathscr{L}_n f$.

b) Let $g = \alpha f$, where $\alpha \in \mathscr{L}_n$, $f\boldsymbol{k} = cf$. Then $g\boldsymbol{k} = \alpha f\boldsymbol{k} = \alpha cf = c_n \alpha f = c_n g$, hence $g \in \mathscr{Q}_n$. This proves that $\mathscr{Q}_n = \mathscr{L}_n f$.

The remaining equalities follow from the foregoing theorems.

The following theorem was proved in [2]:

Theorem 3.9. *A differential equation* (q), $q \in C^0$, *has only periodic or half-periodic solutions with period* π, *with exactly* n *zeros on* $[0, \pi)$ *and, moreover, there exists a non-trivial solution* y *of* (q) *such that* $a + k\pi/n$, $k = 0, \pm 1, \pm 2, \dots$ *are all zeros of* y, *and* $|y'(a + k\pi/n)| = 1/A = \text{const.}$ *for every integer* k, *if and only if*

$$q(t) = f''(t) + f'^2(t) + 2nf'(t) \,.\, \cot[n(t - a)] - n^2,$$

where $f \in C^2$, $f(t + \pi) = f(t)$, $f(a + k\pi/n) = f'(a + k\pi/n) = 0$ *for all integers* k, *and*

$$\int_0^\pi (e^{-2f(t)} - 1)/\sin^2[n(t - a)]\,\mathrm{d}t = 0.$$

Then the solution y *can be written as*

$$y : t \to \frac{e^{f(t)}}{nA}(-1)^{n-1} \sin n(t - a).$$

Let us now consider the function

$$f(t) = -(1/2)\ln[1 - (1/2)\sin 2(t - a)\sin^2 n(t - a)].$$

This function has properties as follows:

a) $f(t)$ has continuous derivatives of an arbitrary order; thus $f(t) \in C^\infty$;

b) $f(t + \pi) = -(1/2)\ln[1 - (1/2)\sin 2(t + \pi - a)\sin^2 n(t + \pi - a)] = -(1/2)\ln[1 - (1/2)\sin 2(t - a)\sin^2 n(t - a)] = f(t)$;

c) $$f'(t) = (1/2)\frac{\cos 2(t - a)\sin^2 n(t - a) + (n/2)\sin 2(t - a)\sin 2n(t - a)}{1 - (1/2)\sin 2(t - a)\sin^2 n(t - a)};$$

$$f(a + k\pi/n) = -(1/2)\ln[1 - (1/2)\sin(2k\pi/n)\sin^2 k\pi] = 0,$$

$$f'(a + k\pi/n) = (1/2)\frac{\cos(2k\pi/n)\sin^2 k\pi + (n/2)\sin(2k\pi/n)\sin 2k\pi}{1 - (1/2)\sin(2k\pi/n)\sin^2 k\pi} = 0;$$

d) $$\int_0^\pi (e^{-2f(t)} - 1)/\sin^2 n(t - a)\,\mathrm{d}t =$$

$$= \int_0^\pi -(1/2)\sin 2(t - a)\sin^2 n(t - a)/\sin^2 n(t - a)\,\mathrm{d}t =$$

$$= (1/2)\int_0^\pi -\sin 2(t - a)\,\mathrm{d}t = (1/2)[\cos 2(t - a)]_0^\pi = 0.$$

From the above we can see that the function $f(t) = -(1/2) \ln [1 - (1/2) \sin 2(t - a) \sin^2 n(t - a)]$ satisfies the conditions of Theorem 3.9 and consequently the equation with the carrier defined with the aid of this function

$$q(t) = f''(t) + f'^2(t) + 2nf'(t) \cot [n(t - a)] - n^2 \ (\overset{\text{def.}}{=} q_n(t, a))$$

has only half-periodic or periodic solutions, with period π and exactly n zeros on $[0, \pi)$.

Thus we can state the following theorem.

Theorem 3.10. *Let $\mathcal{Q}_n$ be the set of all phases of all the differential equations whose every solution is periodic (n even) or half-periodic (n odd) with period k and exactly n zeros on $[0, k)$. Then*

$$\mathcal{Q}_n = \alpha_n Hf,$$

where H is the elementary phases group, α_n an arbitrary phase of the equation with a carrier $q_n(t, a)$ and f an arbitrary phase of the equation with the carrier $-(\pi/k)^2$.

Next it holds

Theorem 3.11. *Let $\mathcal{R}_n$ be the set of all phases of all the equations whose every solution is periodic (n even) or half-periodic (n odd) having exactly n zeros on the halfclosed interval of the appropriate periodlength. Then*

$$\mathcal{R}_n = \bigcup_{k \in R^+} \alpha_n H f_k,$$

where R^+ is the set all positive real numbers, α_n and f_k arbitrary phases of the differential equation with a carrier $q_n(t, a)$ and $-(\pi/k)^2$, respectively.

In Theorems 3.10 and 3.11 we can also use as the phase α_n any phase of the equation $y'' = -n^2 y$.

From Theorems 3.2, 3.8 and 3.11 we arrive at

Theorem 3.12. *Let $\mathcal{F}_n$ be the set of all phases of all the equations with F_n-carriers (i.e. of the equations with the coinciding n-th central dispersions φ_n, ψ_n of the 1st and 2nd kind). Then we can write*

$$\mathcal{F}_n = \{\alpha \in \mathcal{R}_n : \{\alpha, t\} + (\alpha')^2 > 0, \qquad \alpha \in C^5 \qquad \text{for each } t \in (-\infty, \infty)\}.$$

Thus we see that all F_n-carriers can be characterized with the aid of phases of all the negative elementary carriers from C^2, next with the aid of phases of the negative carriers $q_n(t, a)$ and finally with the aid of phases of the negative constant carriers.

REFERENCES

[1] Borůvka O., Lineare Differentialtransformationen 2. Ordnung, DVW, Berlin 1967.

[2] Neuman F., Linear differential equations of the second order and their applications, Rend. di Mat., vol. 4, ser. VI (1971), 559—617.

[3] Laitoch M., Sovpadenije central'nych dispersij 1-go i 2-go roda, sootvetstvujuščich differencial'nomu uravněniju vtorogo porjadka $y'' = Q(x)\,y$, Czech. Math. J. 6(81) (1956), 365—380.

I. Rachůnková
771 46 Olomouc, Palacký University
Czechoslovakia