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A CATEGORICAL GENERALIZATION OF A THEOREM OF G.BIRKHOFF ON PRIMITIVE CLASSES OF UNIVERSAL ALGEBRAS

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A primitive class (cf [10]) of universal (or abstract) algebras of some finitary type τ is a class which consists exactly of all algebras of type τ in which certain equational relations hold true identically. More precisely, let F be any free algebra of type τ and let ρ be any binary relation on F. Let $\mathbb{C}(F, \rho)$ be the class of all algebras A of type τ such that for any homomorphism $\varphi : F \to A$ $x \rho y$ implies $x\varphi = y\varphi$ in A. Now, a primitive class P of algebras of type τ is simply a class for which there exist some F and ρ with $P = \mathbb{C}(F, \rho)$. A well-known theorem of G. Birkhoff (cf [1]) states that a class P of algebras of type τ is primitive if and only if it contains with every algebra A all its subalgebras and factor-algebras and if it is closed under formation of cartesian products.

Categorical methods seem to be especially convenient for investigating primitive classes of universal algebras and related questions (e.g., cf [7],[5],[11]). However, in the present paper we try to find a categorical generalization of the Birkhoff's theorem which would pass over the limits of categories of algebras. Really, there are categories without free joins which our theorem 1,15 does concern. We shall apply this theorem to a special class of models (relational systems), too. On the other side, some conditions have to be supposed to hold in categories for which our theorem will be proved, but most of them seem to be quite natural with respect to the aim we want to attain.

After having put down the following results the author got acquainted with the highly interesting paper [8] which seems to be closely related (especially its section 3) to the present work. However, the existence of zero morphisms supposed in [8] which seems to be quite essential for the whole paper is not supposed by us. Yet, we think that our system of conditions (β_i) $i = 1, 2, 3, 4$ (see 1,11) may prove useful even in some cases which cannot be treated by the use of kernel-techniques.

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1.1. Our notations do not differ essentially from those used in [6]. $\mathbb{C}$ being a category, the class of all its morphisms is denoted by the same $\mathbb{C}$. The symbols $obj\ \mathbb{C}$, $epi\ \mathbb{C}$, $mono\ \mathbb{C}$, $iso\ \mathbb{C}$ are used, respectively, for the class of all objects of $\mathbb{C}$, the subcategory of all epimorphisms of $\mathbb{C}$, the subcategory of all monomorphisms of $\mathbb{C}$, the subcategory of all isomorphisms (invertible morphisms) of $\mathbb{C}$. We point out that the composite of $\alpha : A \to B$ and $\beta : B \to C$ is written as $\alpha\beta$ (not $\beta\alpha$).

1.2. Let $\mathbb{C}$ be a category, $\mathbf{E} \subset epi\ \mathbb{C}$ and $\mathbf{M} \subset mono\ \mathbb{C}$ two subcategories. $[\mathbb{C}, \mathbf{E}, \mathbf{M}]$ is called a bicategory if and only if both conditions below are satisfied:

(i) $\mathbf{E} \cap \mathbf{M} = iso\ \mathbb{C}$.

(ii) Any $\alpha \in \mathbb{C}$ can be written in the form $\alpha = \nu\mu$ with $\nu \in \mathbf{E}$ and $\mu \in \mathbf{M}$; if $\nu\mu = \nu'\mu'$ with $\nu, \nu' \in \mathbf{E}$ and $\mu, \mu' \in \mathbf{M}$ then there exists some $\iota \in iso\ \mathbb{C}$ such that $\nu\iota = \nu'$ (hence $\mu = \iota\mu'$).

Bicategories were introduced by J.R. Isbell (cf [4]). The present definition accepts the modification due to Z. Semadeni (cf [13]).

1,3. The following assertions hold true in a bicategory $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ (cf [4],[13]).

Let $\alpha, \beta \in \mathbb{C}$. If $\alpha\beta \in \mathbb{E}$ then $\beta \in \mathbb{E}$. If $\alpha\beta \in \mathbb{M}$ then $\alpha \in \mathbb{M}$.

For let $\alpha\beta \in \mathbb{E}$. Following (ii) write $\alpha = \alpha'\alpha''$, $\beta = \beta'\beta''$ with $\alpha', \beta' \in \mathbb{E}$, $\alpha'', \beta'' \in \mathbb{M}$. Similarly, let $\alpha''\beta' = \gamma'\gamma''$ with $\gamma' \in \mathbb{E}$, $\gamma'' \in \mathbb{M}$. Now, $\alpha\beta = \alpha'\alpha''\beta'\beta'' = \alpha'\gamma'\gamma''\beta'' \in \mathbb{E}$ and hence, by (ii), $\gamma''\beta'' = \iota \in$ $\in \mathit{iso}\,\mathbb{C}$. We have $(\iota^{-1}\gamma'')\beta'' = 1$, $\beta''(\iota^{-1}\gamma'')\beta'' = \beta''$ and, as $\beta'' \in \mathbb{M}$, $\beta''(\iota^{-1}\gamma'') = 1$. Hence $\beta'' \in \mathit{iso}\,\mathbb{C}$ and $\beta = \beta'\beta'' \in \mathbb{E}$. The second part of 1,3 is proved dually.

1,4. Let $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ be a bicategory. Let us recall that any $P \in \mathit{obj}\,\mathbb{C}$ is called projective (in the sense of the bicategory - and this will be the only case considered in this paper) if and only if for any $\alpha : P \to A$ and any $\nu : B \to A$, $\nu \in \mathbb{E}$ there exists always some $\beta : P \to B$ with $\alpha = \beta\nu$.

1,5. Let $\mathbb{C}$ be a category. Any $S \in \mathit{obj}\,\mathbb{C}$ will be called semiinitial if and only if for any $A \in \mathit{obj}\,\mathbb{C}$ there exists at least one $\vartheta : S \to A$. (Initial and terminal objects are introduced by J.A. Zilber (cf [14]), see also S. MacLane [9]).

1,6. For any $\alpha, \beta \in \mathbb{C}$ we write $\alpha // \beta$ if and only if there exists some $\gamma \in \mathbb{C}$ with $\beta = \alpha\gamma$. Clearly $\alpha // \beta$ implies that α and β are coinitial (they have the same domain).

1,7. A star $\mathcal{S}$ in a category $\mathbb{C}$ is any non-void family

$\mathcal{S} = \{\alpha_\lambda ; \lambda \in \Lambda\}$ of coinitial morphisms $\alpha_\lambda : A \to$ $\to A_\lambda (\lambda \in \Lambda)$. We often write $\mathcal{S} : A \to \{A_\lambda ; \lambda \in \Lambda\}$. The non-void system Λ of indices may be a proper class in the sense of the Gödel-Bernays axiomatic set theory (cf [3]). If Λ is a set the star $\mathcal{S}$ and the family $\{A_\lambda ; \lambda \in \Lambda\}$ will be called <u>small</u>. If all morphisms α_λ of $\mathcal{S}$ belong to some class $\mathbb{L}$ we shall write $\mathcal{S} \subset \mathbb{L}$.

<u>1.8</u>. Let $\mathcal{S}$ be a star as in 1,7. Let $\gamma : C \to A$. Then $\gamma\mathcal{S}$ means the star $\gamma\mathcal{S} = \{\gamma\alpha_\lambda ; \lambda \in \Lambda\}$.

<u>1.9</u>. Let $\mathcal{T}$ be a star in a category $\mathbb{C}$ and let $\gamma \in \mathbb{C}$. We write $\gamma / \mathcal{T}$ if and only if there exists a star $\mathcal{S}$ in $\mathbb{C}$ with $\mathcal{T} = \gamma\mathcal{S}$.

<u>1.10</u>. Let us recall the definition of the <u>product</u> of some non-void and small family $\{A_\lambda ; \lambda \in \Lambda\}$ of objects of some category $\mathbb{C}$. By this product we mean any star $\mathcal{S} : A \to$ $\to \{A_\lambda ; \lambda \in \Lambda\}$ in $\mathbb{C}$ with the following property: for any star $\mathcal{T} : B \to \{A_\lambda ; \lambda \in \Lambda\}$ in $\mathbb{C}$ there exists exactly one $\vartheta : B \to A$ with $\mathcal{T} = \vartheta\mathcal{S}$. Any class $\mathbb{A} \subset obj\ \mathbb{C}$ is said to be closed under formation of products if and only if for any non-void and small family $\{A_\lambda ; \lambda \in \Lambda\}$ of objects in $\mathbb{A}$ there exists at least one product $\mathcal{S} : A \to$ $\to \{A_\lambda ; \lambda \in \Lambda\}$ with $A \in \mathbb{A}$.

<u>1.11</u>. Bicategories $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ which we shall mostly deal with will satisfy the following four conditions:

($\mathcal{B}_1$) $obj\ \mathbb{C}$ is closed under formation of products.

($\mathcal{B}_2$) For any star $\mathcal{S} \subset \mathbb{E}$ there exists always a small star $\mathcal{T} \subset \mathbb{E}$ satisfying both conditions below:

1) For any $\sigma \in \mathcal{S}$ there exist some $\tau \in \mathcal{T}$ and $\iota \in iso\ \mathbb{C}$ with $\sigma = \tau\iota$.

2) For any $\tau \in \mathcal{T}$ there exist some $\sigma \in \mathcal{S}$ and

$\iota \in iso\ \mathbb{C}$ with $\tau = \sigma\iota$.

($\mathcal{B}_3$) For any small star $\mathcal{T} \subset \mathbb{E}$ there exists some $\eta \in \mathbb{E}$ such that:

1) $\eta / \mathcal{T}$.

2) If $\nu / \mathcal{T}$ and $\nu \in \mathbb{E}$ then ν / η.

($\mathcal{B}_4$) There exists a projective and semiinitial object $P \in obj\ \mathbb{C}$ with the following property:

For any $A \in obj\ \mathbb{C}$ there exist a projective and semi initial object P_A and some $P'_A \in obj\ \mathbb{C}$ (which need not be projective or semiinitial) such that:

1) There exists some $\vartheta : P_A \to A$, $\vartheta \in \mathbb{E}$.

2) For any $\alpha : P'_A \to P_A$ there exist $\sigma : P_A \to P$ and $\tau : P \to P_A$ with $\alpha = \alpha\sigma\tau$.

3) If $\psi : P_A \to B$, $\nu : P_A \to C$, $\psi, \nu \in \mathbb{E}$, $\nu \not/ \psi$ then there exists some $\alpha : P'_A \to P_A$ with $\alpha\nu \not/ \alpha\psi$ ($\not/$ is the negation of $/$).

1,12. ($\mathcal{B}_4$) may be replaced by a stronger condition ($\mathcal{B}'_4$) obtained from ($\mathcal{B}_4$) by requiring, in addition, P'_A should be projective and semiinitial for all $A \in obj\ \mathbb{C}$.

1,13. Conditions ($\mathcal{B}_i$) $i = 1, 2, 3, 4$ may seem to be rather complicated. Yet, the following interpretation will give them a sufficiently clear sense.

Proposition. Let $\mathbb{C}$ be the category of all universal algebras of some fixed finitary type and of all homomorphisms from any such algebra into another. Then $[\mathbb{C}, epi\ \mathbb{C}, mono\ \mathbb{C}]$ is a bicategory satisfying the conditions ($\mathcal{B}_i$) $i = 1, 2, 3, 4$ of 1,11.

Proof: For $\mathbb{C}$ considered above $epi\ \mathbb{C}$, $mono\ \mathbb{C}$ respectively consists exactly of all homomorphisms onto (cf [2]), injective homomorphisms into. Thus (i) and (ii) from 1,2 are clearly satisfied. ($\mathcal{B}_1$) is obvious and ($\mathcal{B}_2$) follows by replacing every $\sigma \in \mathcal{S}$ by the corresponding natural homomorphism τ of the algebra onto its factor-algebra. ($\mathcal{B}_3$) may be proved by considering $\mathcal{T}$ as to consist of natural homomorphisms. The intersection of the corresponding congruence-relations is a congruence relation which gives a natural homomorphism η with both properties 1) and 2) required. The main difficulty is to prove ($\mathcal{B}_4$). It is easy to prove that projective and semiinitial objects in $[\,\mathbb{C}, \mathit{epi}\ \mathbb{C}, \mathit{mono}\ \mathbb{C}\,]$ are exactly all free algebras of the given type. For any non-void set X let $F(X)$ mean the free algebra in $\mathbb{C}$ having X for the set of free generators. Put $P = F(X)$ with $\mathit{card}\ X = \aleph_0$. Then, for any $A \in \mathit{obj}\ \mathbb{C}$, put $P_A = F(Y)$ with $\mathit{card}\ Y \geq \aleph_0$ so large that there exists some $\vartheta : P_A \to A$, $\vartheta \in \mathbb{E}$. For P'_A let us take $P'_A = F(Z)$ with two free generators $Z = \{x_1, x_2\}$. Since 1) in ($\mathcal{B}_4$) is already satisfied let us prove 2). Suppose $\alpha : F(Z) \to F(Y)$. Then there is obviously a finite subset $Y' \subset Y$ with $\mathfrak{X}^{\alpha} \subset F(Y')$ (as the type is finitary), hence $[F(Z)]^{\alpha} \subset F(Y')$. Taking some $X' \subset X$ with $\mathit{card}\ X' = \mathit{card}\ Y'$ and bijections $\sigma : Y' \to X'$, $\tau = \sigma^{-1} : X' \to Y'$ we extend them in any way to homomorphisms $\sigma : F(Y) \to F(X)$, $\tau : F(X) \to F(Y)$. Now, it is easy to see that $\alpha = \alpha\,\sigma\,\tau$. Finally, let us prove 3). Suppose $\psi : F(Y) \to B$, $\nu : F(Y) \to C$, ψ, $\nu \in \mathbb{E}$, with congruence-relations ρ_{ψ} and ρ_{ν} on $F(Y)$. Let $\nu \nmid \psi$. Then $\rho_{\nu} \not\subset \rho_{\psi}$ and hence we have two elements $f_1, f_2 \in F(Y)$ with $f_1\, \rho_{\nu}\, f_2$ and $f_1\ (\mathit{non}\ \rho_{\psi})\, f_2$. There is clearly a homomorphism $\alpha : F(Z) \to F(Y)$ with $x_1^{\alpha} = f_1$, $x_2^{\alpha} = f_2$. Now, $\alpha\,\nu\,\gamma = \alpha\,\psi$ for some $\gamma : C \to B$ would give $x_1^{\alpha\psi} = x_2^{\alpha\psi}$ because of $x_1^{\alpha\nu} = x_2^{\alpha\nu}$.

It would follow $f_1\Psi = f_2\Psi$ and $f_1 \rho_\Psi f_2$ in contradiction to our hypothesis. Hence $\alpha\nu \neq \alpha\psi$ and our proposition is proved. (Notice that we have actually proved $(\mathcal{B}'_4)$).

1,14. Let $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ be a bicategory satisfying $(\mathcal{B}_i)$ $i = 1, 2, 3, 4$. Let P be any of its projective and semiinitial objects. Let $\eta : P \to Q$, $\eta \in \mathbb{E}$. Then $\mathbb{C}(P, \eta)$ will mean the class of all $A \in obj\, \mathbb{C}$ such that for any $\vartheta : P \to A$ there is always η / ϑ. A class $\mathbf{P} \subset obj\, \mathbb{C}$ will be called primitive if and only if there exist some P and η with $\mathbf{P} = \mathbb{C}(P, \eta)$.

It is clear that in the case of the bicategory of all universal algebras of some fixed type (see 1,13) this categorical definition of a primitive class is equivalent to the usual one.

1,15. Theorem: Let the bicategory $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ satisfy $(\mathcal{B}_i)$ $i = 1, 2, 3, 4$ from 1,11 and let $\mathbf{P} \neq \emptyset$ be any class of its objects. Then $\mathbf{P}$ is primitive if and only if the following three conditions (P_i) $i = 1, 2, 3$ are satisfied:

(P_1) If $\mu : A \to B$, $\mu \in \mathbb{M}$, $B \in \mathbf{P}$ then $A \in \mathbf{P}$.

(P_2) If $\nu : A \to B$, $\nu \in \mathbb{E}$, $A \in \mathbf{P}$ then $B \in \mathbf{P}$.

(P_3) $\mathbf{P}$ is closed under formation of products.

Proof: Let $\mathbf{P}$ be primitive, $\mathbf{P} = \mathbb{C}(P, \eta)$, $\eta \in \mathbb{E}$, $\eta : P \to Q$. Let $\mu : A \to B$, $\mu \in \mathbb{M}$, $B \in \mathbf{P}$. We have to prove $A \in \mathbf{P}$. Let $\vartheta : P \to A$. Then $\vartheta\mu : P \to B$ and, as $B \in \mathbf{P}$, $\eta / \vartheta\mu$ and $\vartheta\mu = \eta\vartheta_1$ for some $\vartheta_1 : Q \to B$. Consider decompositions $\vartheta = \vartheta'\vartheta''$, $\vartheta_1 = \vartheta_1'\vartheta_1''$ with $\vartheta', \vartheta_1' \in \mathbb{E}$, $\vartheta'', \vartheta_1'' \in \mathbb{M}$. Then, by (ii), there exists some $\iota \in iso\, \mathbb{C}$ with $\vartheta' = \eta\vartheta_1'\iota$.

Hence η/ϑ', η/ϑ, $A \in \mathbb{P}$ and (P_1) is proved. Let $\nu : A \to B$, $\nu \in \mathbb{E}$, $A \in \mathbb{P}$. We have to prove $B \in \mathbb{P}$. Let $\vartheta : P \to B$. As P is projective (otherwise $\mathbb{C}(P,\eta)$ is not defined, see 1,14) we have $\vartheta = \vartheta_1 \nu$ for some $\vartheta_1 : P \to A$. Since $A \in \mathbb{P}$ we have η/ϑ_1 and η/ϑ. Hence (P_2) is proved. Let $\{A_\lambda ; \lambda \in \Lambda\}$ be a non-void and small family of objects in $\mathbb{P}$. By ($\mathcal{B}_1$) there exists its product $\mathcal{S} : S \to \{A_\lambda ; \lambda \in \Lambda\}$ in $\mathbb{C}$. In order to prove (P_3) it is sufficient to show that $S \in \mathbb{P}$. Let $\vartheta : P \to S$. Then $\eta / \vartheta \mathcal{S}$ and $\vartheta \mathcal{S} = \eta \mathcal{T}$ for some star $\mathcal{T}$. Since $\mathcal{S}$ is product we have $\mathcal{T} = \psi \mathcal{S}$ for some ψ. It follows $\vartheta \mathcal{S} = \eta \psi \mathcal{S}$ and, again by $\mathcal{S}$ being product, $\vartheta = \eta \psi$. Hence η/ϑ, $S \in \mathbb{P}$ and (P_3) is proved.

Suppose now that $\mathbb{P} \neq \emptyset$ is any class of objects satisfying (P_i) $i = 1, 2, 3$. We have to prove that $\mathbb{P}$ is primitive.

First take any semiinitial and projective object $\overline{P}$ in our bicategory and consider the star $\mathcal{S}$ of all $\vartheta \in \mathbb{E}$ with $\vartheta : \overline{P} \to A$, $A \in \mathbb{P}$. Because of $\mathbb{P} \neq \emptyset$, (ii) and (P_1) there exists really at least one ϑ with the above property. By ($\mathcal{B}_2$) we find a small star $\mathcal{T}$ with properties mentioned in ($\mathcal{B}_2$) and for this $\mathcal{T}$ we find $\overline{\eta} : \overline{P} \to \overline{Q}$ as required in ($\mathcal{B}_3$). From $\overline{\eta}/\mathcal{T}$, (ii) and (P_1) it follows easily that

(α) for any $\beta : \overline{P} \to B$ with $B \in \mathbb{P}$ it is always $\overline{\eta}/\beta$.

Hence $\mathbb{P} \subset \mathbb{C}(\overline{P}, \overline{\eta})$. We shall prove that

(β) $\overline{Q} \in \mathbb{P}$.

Let the star $\mathcal{T}$ have the form $\mathcal{T} : \overline{P} \to \{B_\lambda ; \lambda \in \Lambda\}$ with $B_\lambda \in \mathbb{P}$ for all $\lambda \in \Lambda$. Let $\mathfrak{X} : X \to \{B_\lambda ; \lambda \in \Lambda\}$

be the product of the family $\{B_\lambda ; \lambda \in \Lambda\}$. By (P_3) we may suppose that $X \in \mathbb{P}$. As $\bar{\eta} / \mathcal{T}$ we have $\mathcal{T} = \bar{\eta}\,\mathcal{T}'$ for some star $\mathcal{T}'$ and, as $\mathcal{X}$ is the product, $\mathcal{T}' = \gamma \mathcal{X}$ for some γ. We consider some decomposition $\gamma = \gamma' \gamma''$ with $\gamma' \in \mathbb{E}$ and $\gamma'' \in \mathbb{M}$. Then we have $\mathcal{T} = \bar{\eta}\,\gamma' \gamma'' \mathcal{X}$. Since $\bar{\eta}\,\gamma' \in \mathbb{E}$ and $\bar{\eta}\,\gamma' / \mathcal{T}$ it follows by $(\mathcal{B}_3)$ that $\bar{\eta}\,\gamma' / \bar{\eta}$ and $\bar{\eta} = \bar{\eta}\,\gamma' \varkappa$ for some $\varkappa$. We have then $1 = \gamma' \varkappa$ and, using (i) and the second assertion of 1,3, we conclude $\gamma' \in \mathbb{M}$ and $\gamma = \gamma' \gamma'' \in \mathbb{M}$. But $\gamma : \bar{Q} \to X$ and $X \in \mathbb{P}$, hence by (P_1) $\bar{Q} \in \mathbb{P}$. Thus (β) is proved.

We have already proved $\mathbb{P} \subset \mathbb{C}(\bar{P}, \bar{\eta})$. Now, in place of an arbitrary $\bar{P}$ we take the semiinitial and projective object P from $(\mathcal{B}_4)$. Again, we find $\eta : P \to Q$ just in the same way as $\bar{\eta}$ was found for $\bar{P}$ and, again by (β), we have $Q \in \mathbb{P}$. There is $\mathbb{P} \subset \mathbb{C}(P, \eta)$, too. But for this special P quite $\mathbb{P} = \mathbb{C}(P, \eta)$ is true. To prove it suppose any $A \in \mathbb{C}(P, \eta)$. We want to show that $A \in \mathbb{P}$. Take P_A and P_A' by $(\mathcal{B}_4)$. Again, considering P_A instead of $\bar{P}$ we find $\eta_A : P_A \to Q_A$ in the same way as $\bar{\eta}$ was found for $\bar{P}$ and, again, $Q_A \in \mathbb{P}$. By $(\mathcal{B}_4)$ 1) there exists some $\vartheta \in \mathbb{E}$ with $\vartheta : P_A \to A$. We shall prove that η_A / ϑ.

Suppose that $\eta_A \not/ \vartheta$. Then, by $(\mathcal{B}_4)$ 3), we can find some $\alpha : P_A' \to P_A$ with $\alpha \eta_A \not/ \alpha \vartheta$. To this α we can find $\sigma : P_A \to P$, $\tau : P \to P_A$ such that $\alpha = \alpha \sigma \tau$ (see $(\mathcal{B}_4)$, 2)). As $Q \in \mathbb{P}$ we have, by (α), $\eta_A / \sigma \eta$ and $\sigma \eta = \eta_A \varkappa$ for some $\varkappa$. As $A \in \mathbb{C}(P, \eta)$ and $\tau \vartheta : P \to A$ we have $\eta / \tau \vartheta$

and $\tau\vartheta = \eta\vartheta_1$ for some ϑ_1 . Thus $\alpha\vartheta = \alpha\sigma\tau\vartheta =$ $= \alpha\sigma\eta\vartheta_1 = \alpha\eta_A \varkappa \vartheta_1$ and $\alpha\eta_A / \alpha\vartheta$ in contradiction to our hypothesis. Hence η_A / ϑ and $\vartheta = \eta_A \vartheta_2$ for some ϑ_2 .

Since $\vartheta \in \mathbb{E}$ it follows by the first assertion of 1,3 that $\vartheta_2 \in \mathbb{E}$. But $\vartheta_2 : Q_A \to A$ and $Q_A \in \mathbb{P}$. Hence by (P_2) we get $A \in \mathbb{P}$. Our theorem is proved.

1.16. **Remark.** Notice that if $\mathbb{P} \neq \emptyset$ is a primitive class of objects of a bicategory $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ satisfying $(\mathcal{B}_i)$ $i = 1, 2, 3, 4$ then it can be written in the form $\mathbb{P} = \mathbb{C}(P, \eta)$ where P is that projective and semi-initial object which is introduced in ($\mathcal{B}_4$).

2.

This section is intended to show one application more of the preceding investigations. It deals with a special class of models which we call $\mathcal{R}$ -systems. The main purpose of it is not to investigate models in general but to give some further illustration of the ideas of section 1.

2.1. Consider the covariant functor $Hom(I, X)$ of the category of all sets to itself (I being fixed). It assignes to every $\vartheta : X \to Y$ a unique $Hom(I, X) \to Hom(I, Y)$ denoted by $\overline{\vartheta}$. It is clear that for $\vartheta : X \to Y$ and $\psi : Y \to Z$ we have always $\overline{\vartheta\psi} = \overline{\vartheta}\,\overline{\psi}$.

2.2. An $\mathcal{R}$ -__system of type__ I is any system A of the form $A = \langle X, I, \mathcal{U} \rangle$ where $X, I, \mathcal{U}$ are non-void sets and $\mathcal{U} \subset Hom(I, X)$. $B = \langle Y, I, \mathcal{V} \rangle$ being some second $\mathcal{R}$ -system, the mapping $\vartheta : X \to Y$ will be called a __homomorphism__ of A into $\mathcal{B}$ (abbreviated by $\vartheta : A \to B$) if and only if $\mathcal{U}\,\overline{\vartheta} \subset \mathcal{V}$. If $\mathcal{U}\,\overline{\vartheta} = \mathcal{V}$, ϑ will be called __strong__.

2.3. It is clear that all $\mathcal{R}$-systems of some fixed type I and all homomorphisms of one $\mathcal{R}$-system into another form a category. We shall denote it by $\mathcal{C}_I$. Let $\mathbb{E}, \mathbb{M}$ mean, respectively, the subcategory of all strong homomorphisms onto, the subcategory of all injective homomorphisms. It is easy to prove that $[\mathbb{C}_I, \mathbb{E}, \mathbb{M}]$ is a bicategory.

2.4. The theory of $\mathcal{R}$-systems is much similar to the theory of universal algebras. We mention some most important facts needed below.

2.5. Let $\mathcal{A} = \langle X, I, \mathcal{U} \rangle$ be an $\mathcal{R}$-system. We define, for any equivalence-relation Σ on X, a new $\mathcal{R}$-system $A_\Sigma = \langle X_\Sigma, I, \mathcal{U}_\Sigma \rangle$ by the following: X_Σ is the factor-set of X by Σ; the natural mapping $\tau : X \to X_\Sigma$ (with $x \in x\tau$ for all $x \in X$) induces $\overline{\tau} : Hom(I, X) \to Hom(I, X_\Sigma)$; we put $\mathcal{U}_\Sigma = \mathcal{U}\overline{\tau}$. Hence $\tau : A \to A_\Sigma$ is a homomorphism (called natural) and $\tau \in \mathbb{E}$. A_Σ is called the factor- $\mathcal{R}$-system of A with respect to Σ.

2.6. Let $\sigma : A \to B$, $\sigma \in \mathbb{E}$. Let Σ, τ be, respectively, the equivalence-relation on X, the natural homomorphism $\tau : A \to A_\Sigma$ corresponding to σ (hence $\tau = \sigma . \sigma^{-1}$). Then, there is obviously an isomorphism $\iota : A_\Sigma \to B$ with $\sigma = \tau\iota$. It is clear that in the bicategory $[\mathbb{C}_I, \mathbb{E}, \mathbb{M}]$ the condition ($\mathcal{B}_2$) from 1,11 holds true. Really, we need only to replace every $\sigma \in \mathcal{S}$ of the given star $\mathcal{S} \subset \mathbb{E}$ by the corresponding natural homomorphism $\tau \in \mathbb{E}$.

2.7. The condition ($\mathcal{B}_3$) for $[\mathbb{C}_I, \mathbb{E}, \mathbb{M}]$ is also easy to be proved. First of all, we may suppose that the small star $\mathcal{T} = \{\tau_\lambda ; \lambda \in \Lambda\} \subset \mathbb{E}$ consists of natural homomorphisms τ_λ , each corresponding to some equivalence-relation Σ_λ . Then $\Sigma = \bigcap_{\lambda\in\Lambda} \Sigma_\lambda$ is an equivalence-relation and the natural homomorphism η corresponding to Σ has both properties required in ($\mathcal{B}_3$).

2.8. Let $\{A_\lambda ; \lambda \in \Lambda\}$ be a small and non-void family of objects in $\mathbb{C}_I$, $A_\lambda = \langle X_\lambda , I , \mathcal{U}_\lambda \rangle$ for all $\lambda \in \Lambda$. Define X and $\mathcal{U}$ by cartesian products $X = \prod_{\lambda\in\Lambda} X_\lambda$, $\mathcal{U} = \prod_{\lambda\in\Lambda} \mathcal{U}_\lambda$. Observing that each $u \in \mathcal{U}$ may be considered as a mapping of I into X we obtain an $\mathcal{R}$-system $A = \langle X, I, \mathcal{U}\rangle$. It is easy to prove that the star of projections $\pi_\lambda : A \to A_\lambda$ is the categorical product of the family $\{A_\lambda ; \lambda \in \Lambda\}$. Hence the condition ($\mathcal{B}_1$) is proved for $[\, \mathbb{C}_I , \mathbb{E} , \mathbb{M} \,]$.

2.9. Consider a non-void set S . For any $s \in S$ let $\underline{s}$ be the mapping of I into $I \times S$ defined by the formula $i\underline{s} = \langle i , s \rangle$ for all $i \in I$. Let $\underline{S}$ mean the set of all $\underline{s}$ with $s \in S$. Then the system $F_S = \langle I \times S, I, \underline{S} \rangle$ is an $\mathcal{R}$-system. More general, let D be any set with $D \cap$ $\cap (I \times S) = \emptyset$. We denote by $F_{S,D}$ the $\mathcal{R}$-system $F_{S,D} =$ $= \langle (I \times S) \cup D , I, \underline{S} \rangle$.

2.10. The system $F_{S,D}$ from 2,9 has the following property: $A = \langle X, I, \mathcal{U}\rangle$ being any $\mathcal{R}$-system and $\psi : \underline{S} \to \mathcal{U}$ any mapping, there exists always a homomorphism $\alpha : F_{S,D} \to A$ with the restriction $(\bar{\alpha} / \underline{S}) = \psi$. If $\alpha_1 : F_{S,D} \to A$ is another such homomorphism then $(\alpha / I \times S) = (\alpha_1 / I \times S)$. Really, define α by $\langle i, s\rangle\alpha = i(\underline{s}\ \psi)$ for all $i \in I$ and all $s \in S$ and $(\alpha / D) : D \to X$ in any way. Then, as $i(\underline{s}\,\bar{\alpha}) = (i\underline{s})\alpha = \langle i, s\rangle\alpha = i(\underline{s}\ \psi)$ for all $i \in I$, we have $\underline{s}\,\bar{\alpha} = \underline{s}\,\psi$ and $(\bar{\alpha} / \underline{S}) = \psi$. Hence $\alpha : F_{S,D} \to A$ is a homomorphism. For any $\alpha_1 : F_{S,D} \to A$ with $(\bar{\alpha}_1 / \underline{S}) = \psi = (\bar{\alpha} / \underline{S})$ we have $i(\underline{s}\,\bar{\alpha}) = i(\underline{s}\,\bar{\alpha}_1)$

and, using the above equations, $\langle i, s\rangle \alpha = \langle i, s\rangle \alpha_1$ for all $i \in I$ and all $s \in S$. Notice that if, in addition, some $\chi : D \to X$ is given, then there exists a unique $\alpha : F_{S,D} \to A$ with $(\bar{\alpha} / \underline{S}) = \psi$ and $(\alpha / D) = \chi$.

2,11. From 2,10 it follows easily that any $F_{S,D}$ is semi-initial in $[\mathbb{C}_I, \mathbb{E}, \mathbb{M}]$. Moreover, any $F_{S,D}$ is projective in $[\mathbb{C}_I, \mathbb{E}, \mathbb{M}]$. Really, let $\beta : F_{S,D} \to B = \langle Y, I, \mathcal{V}\rangle$ and $\nu : A = \langle X, I, \mathcal{U}\rangle \to B$, $\nu \in \mathbb{E}$. As $\mathcal{U}\bar{\nu} = \mathcal{V}$ there exists always some $\psi : \underline{S} \to \mathcal{U}$ with $(\bar{\beta}/\underline{S}) = \psi\bar{\nu}$. Again, as $X\nu = Y$, there exists some $\chi : D \to X$ with $(\beta / D) = \chi\nu$. Now, following 2,10, there exists some $\alpha : F_{S,D} \to A$ with $(\bar{\alpha} / \underline{S}) = \psi$ and $(\alpha / D) = \chi$. To prove $\beta = \alpha\nu$, it is sufficient to show that $(\bar{\beta}/\underline{S}) = (\bar{\alpha}\bar{\nu}/\underline{S})$ and $(\beta / D) = (\alpha\nu / D)$. But these assertions are both clearly satisfied by the above construction.

2,12. Let $A = \langle X, I, \mathcal{U}\rangle$ be given. Then there exist always some $F_{S,D}$ and some $\vartheta : F_{S,D} \to A$ with $\vartheta \in \mathbb{E}$. Really, there are clearly sets S, D with some surjections (mappings onto) $\psi : \underline{S} \to \mathcal{U}$, $\chi : D \to X$ and with $D \cap (I \times S) = \emptyset$. Finding to ψ and χ an $\alpha : F_{S,D} \to A$ by 2,10 we see immediately that $\alpha \in \mathbb{E}$.

2,13. Proposition. The bicategory $[\mathbb{C}_I, \mathbb{E}, \mathbb{M}]$ (for definition see 2,3) satisfies the conditions $(\mathcal{B}_i)$ $i = 1, 2, 3, 4$ from 1,11.

Proof: $(\mathcal{B}_i)$ $i = 1, 2, 3$ were proved in 2,8, 2,6 and 2,7. For to prove $(\mathcal{B}_4)$ put $P = F_{S,D}$ with $S = \{s_1, s_2, s_3, s_4\}$ and $D = \{d_1, d_2\}$ and $P'_A = F_{T,E}$ with $T = \{t_1, t_2\}$ and $E = \{e_1, e_2\}$ for each $A \in obj\,\mathbb{C}$. By 2,12, find to any $A \in obj\,\mathbb{C}_I$ some

$\vartheta : F_{S_A, D_A} \to A$, $\vartheta \in E$ and put $P_A = F_{S_A, D_A}$. Hence, by 2,11, P and P_A are semiinitial and projective. As 1) in $(\mathcal{B}_4)$ is already satisfied, let us prove 2). Let $\alpha : P_A' \to$ $\to P_A$. Let L be the set consisting of $\underline{t_1}\,\overline{\alpha}$, $\underline{t_2}\,\overline{\alpha}$ and of all $\underline{s_A} \in \underline{S_A}$ such that $e_1 \alpha = \langle i, s_A \rangle$ or $e_2 \alpha = \langle i, s_A \rangle$ for some $i \in I$. Let K be the set of all $k \in D_A$ with $e_1 \alpha = k$ or $e_2 \alpha = k$. Hence $L \subset \underline{S_A}$, $K \subset D_A$ and, $\operatorname{card} L \leq 4$, $\operatorname{card} K \leq 2$.

Now, we can clearly find mappings $\psi_1 : \underline{S_A} \to \underline{S}$, $\psi_2 : \underline{S} \to \underline{S_A}$, $\chi_1 : D_A \to D$, $\chi_2 : D \to D_A$ with $(\psi_1 \psi_2 / L) = 1$ and $(\chi_1 \chi_2 / K) = 1$ (1 is the identity mapping). Following 2,10, we can find homomorphisms $\sigma : P_A \to P$ and $\tau : P \to P_A$ with $(\overline{\sigma}/\underline{S_A}) = \psi_1$, $(\sigma / D_A) = \chi_1$, $(\overline{\tau}/\underline{S}) = \psi_2$, $(\tau / D) = \chi_2$. For to prove that $\alpha = \alpha \sigma \tau$ it is sufficient to prove $(\overline{\alpha}/\underline{T}) = (\overline{\alpha \sigma \tau}/\underline{T})$ and $(\alpha / E) = (\alpha \sigma \tau / E)$. But for any $\underline{t} \in \underline{T}$ we have $\underline{t}\,\overline{\alpha \sigma \tau} = (\underline{t}\,\overline{\alpha})\overline{\sigma \tau} = (\underline{t}\,\overline{\alpha})\psi_1 \psi_2 = \underline{t}\,\overline{\alpha}$ because of $\underline{t}\,\overline{\alpha} \in L$. If $e \in E$ and $e\alpha \in D_A$ then $e\alpha \in K$ and $e\alpha\sigma\tau = e\alpha\chi_1\chi_2 = e\alpha$. If $e \in E$ and $e\alpha = \langle i, s_A \rangle$ for some $i \in I$ and some $s_A \in S_A$ then $\underline{s_A} \in L$, $e\alpha\sigma\tau = (i\,\underline{s_A})\sigma\tau = i[\underline{s_A}\,\overline{\sigma\tau}] = i[\underline{s_A}\psi_1\psi_2] = i\,\underline{s_A} = e\alpha$. Hence $(\alpha / E) = (\alpha\sigma\tau / E)$ and 2) is proved.

Finally, to prove 3) in $(\mathcal{B}_4)$ consider $\psi : P_A \to B$, $\nu : P_A \to C$, $\psi, \nu \in E$ and $\nu \neq \psi$. Following 2,6 we may obviously suppose that ψ and ν are natural homomorphisms, $B = (P_A)_{\underline{\psi}}$, $C = (P_A)_H$ for some equivalence-relations $\underline{\psi}$ and H on the set $M = (I \times S_A) \cup D_A$. $\nu \neq \psi$ implies $H \not\subset \underline{\psi}$ and hence there exist some $m_1, m_2 \in M$ with $m_1 H m_2$ and $m_1 (\mathrm{non}\ \underline{\psi}) m_2$. Now, it is sufficient to show that there exist some $n_1, n_2 \in N = (I \times T) \cup E$ and some homomorphism $\alpha : P'_A \to P_A$ with $n_1\alpha = m_1$ and $n_2\alpha = m_2$. Really, it follows then that $n_1\alpha\nu = n_2\alpha\nu$ and $n_1\alpha\psi \neq n_2\alpha\psi$ so that $\alpha\nu \neq \alpha\psi$. Essentially, there are three cases to be considered. First, when $m_1 = \langle i', s'_A \rangle$, $m_2 = \langle i'', s''_A \rangle$ for some $i', i'' \in I$, $s'_A, s''_A \in S_A$. Then put $n_1 = \langle i', t_1 \rangle$, $n_2 = \langle i'', t_2 \rangle$ and $\alpha : P'_A \to P_A$ choose so that $\underline{t_1}\bar{\alpha} = \underline{s'_A}$, $\underline{t_2}\bar{\alpha} = \underline{s''_A}$. Second, when $m_1 = \langle i', s'_A \rangle$, $m_2 \in D_A$. Then put $n_1 = \langle i', t_1 \rangle$, $n_2 = e_1$ and α choose so that $\underline{t_1}\bar{\alpha} = \underline{s'_A}$, $e_1\alpha = m_2$. Third, when $m_1, m_2 \in D_A$. Then clearly $n_1 = e_1$, $n_2 = e_2$ and $e_1\alpha = m_1$, $e_2\alpha = m_2$. Herewith our proposition is proved. (Notice that we have actually proved $(\mathcal{B}'_4)$ (see 1,12).)

3.

3,1. Consider a bicategory $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ and let $\mathbb{P} \neq \emptyset$ be any class of objects in $\mathbb{C}$ satisfying conditions $(\mathcal{P}_i)$, $i = 1, 2$ from 1,15. Let $\mathbb{C}'$ be the full subcategory of $\mathbb{C}$ with $obj\ \mathbb{C}' = \mathbb{P}$. Put $\mathbb{E}' = \mathbb{E} \cap \mathbb{C}'$, $\mathbb{M}' = \mathbb{M} \cap \mathbb{C}'$. Then $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$ is a bicategory.

3,2. Theorem: Let $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ be a bicategory satisfying the conditions $(\mathcal{B}_1)$ $(\mathcal{B}_2)$ $(\mathcal{B}_3)$ $(\mathcal{B}'_4)$. Let $\mathbb{P} \neq \emptyset$ be any primitive class of objects of $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$. Then the bicategory $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$, corresponding to $\mathbb{P}$ in the sense of 3,1, satisfies the same conditions $(\mathcal{B}_1)$ $(\mathcal{B}_2)$ $(\mathcal{B}_3)$ $(\mathcal{B}'_4)$.

Proof: $(\mathcal{B}_1)$ is satisfied for $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$ by theorem 1,15. $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$ hold true in $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$ as they hold true in $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ and, again, because of theorem 1,15. Thus we have only to prove $(\mathcal{B}'_4)$.

In the proof of theorem 1,15 a method has been described of how to get to any projective and semiinitial object $\overline{P}$

some $\overline{\eta} : \overline{P} \to \overline{Q}$, $\overline{\eta} \in \mathbb{E}$ satisfying (α) and (β). By ($\mathcal{B}_4'$) which is supposed to hold in $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ we have some fixed P and to any $A \in obj\ \mathbb{C}$ some P_A and P_A'. By the above method find $\eta : P \to Q$, $\eta_A : P_A \to Q_A$, $\eta_A' : P_A' \to Q_A'$ so that by (β) $Q, Q_A, Q_A' \in \mathcal{P} = obj\ \mathbb{C}'$. We claim that ($\mathcal{B}_4'$) is satisfied for $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$ with these Q, Q_A, Q_A' (in place of P, P_A, P_A' in the wording of ($\mathcal{B}_4'$)), for all $A \in obj\ \mathbb{C}'$.

Really, Q is semiinitial in $\mathbb{C}'$ by (α). Q is projective in $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$ because P is projective in $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ and $\eta \in \mathbb{E}$. Similarly, Q_A and Q_A' are projective and semiinitial in $[\mathbb{C}', \mathbb{E}', \mathbb{M}']$. For any $A \in obj\ \mathbb{C}'$ there exists always some $\vartheta : Q_A \to A$, $\vartheta \in \mathbb{E}$. Really, we have $\vartheta' : P_A \to A$ for some $\vartheta' \in \mathbb{E}$ and, by (α), $\vartheta' = \eta_A \vartheta$, $\vartheta : Q_A \to A$. Now, using the first assertion of 1,3, we get $\vartheta \in \mathbb{E}$ and, of course, $\vartheta \in \mathbb{E}'$. Hence 1) in ($\mathcal{B}_4'$) is proved. Let $\alpha : Q_A' \to Q_A$. As P_A' is projective, there exists some $\alpha' : P_A' \to P_A$ with $\alpha' \eta_A = \eta_A' \alpha$. Now, by ($\mathcal{B}_4'$) for $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ we have $\alpha' = \alpha' \sigma' \tau'$ for some $\sigma' : P_A \to P$, $\tau' : P \to P_A$. By (α), $\eta_A / \sigma' \eta$ and thus $\sigma' \eta = \eta_A \sigma$ for some $\sigma : Q_A \to Q$. Similarly, $\eta / \tau' \eta_A$ and $\tau' \eta_A = \eta \tau$ for some $\tau : Q \to Q_A$. Now, $\eta_A' \alpha = \alpha' \eta_A = \alpha' \sigma' \tau' \eta_A = \alpha' \sigma' \eta \tau = = \alpha' \eta_A \sigma \tau = \eta_A' \alpha \sigma \tau$, hence $\alpha = \alpha \sigma \tau$. Thus, 2) in ($\mathcal{B}_4'$) is proved. Finally, let us have $\nu, \psi \in \mathbb{E}'$, $\psi : Q_A \to B$, $\nu : Q_A \to C$, $\nu \nmid \psi$. Then clearly $\eta_A \nu \nmid \eta_A \psi$. By ($\mathcal{B}_4'$) for $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ there exists some $\alpha' : P_A' \to P_A$ with $\alpha' \eta_A \nu \nmid \alpha' \eta_A \psi$. By ($\alpha$) we have $\eta_A' / \alpha' \eta_A$ and $\alpha' \eta_A = \eta_A' \alpha$ for some $\alpha : Q_A' \to Q_A$. Now, $\alpha \nu / \alpha \psi$.

would imply $\eta'_A \alpha \nu \,/\, \eta'_A \alpha \psi$ and $\alpha' \eta_A \nu \,/\, \alpha' \eta_A \psi$ in contradiction to the above result. Hence $\alpha \nu \not/ \alpha \psi$ and 3) in $(\mathcal{B}'_4)$ is proved.

3,3. As the bicategories of propositions 1,13 and 2,13 satisfy conditions $(\mathcal{B}_1)(\mathcal{B}_2)(\mathcal{B}_3)(\mathcal{B}'_4)$ these conditions are satisfied by bicategories corresponding to primitive classes of universal algebras or $\mathcal{R}$-systems.

4.

In this section we want to indicate some relations between the present investigations and some of those ideas which concern the concept of independent sets in the sense of J. Schmidt (cf [12]). Let X be any subset of an algebra A. B being any algebra isotypic to A, X is called B-independent if and only if any mapping $\varphi : X \to B$ can be extended to a homomorphism $\bar{\varphi} : \mathcal{Cl}(X) \to B$ where $\mathcal{Cl}(X)$ means the closure of X in A. We want to find a categorical equivalent to this concept (and to some others) and we guess the present way may turn out to be an appropriate one.

4,1. Let $[\,\mathbb{C}, \mathbb{E}, \mathbb{M}\,]$ and $[\,\mathbb{C}', \mathbb{E}', \mathbb{M}'\,]$ be two bicategories and let the first one satisfy the conditions $(\mathcal{B}_i)$ $i = 1, 2, 3, 4$ from 1,11. Consider a covariant functor $F : \mathbb{C} \to \mathbb{C}'$ and suppose that

(F_1) If $\alpha \in \mathbb{E}$ then $F(\alpha) \in \mathbb{E}'$.

(F_2) If $\alpha \in \mathbb{M}$ then $F(\alpha) \in \mathbb{M}'$.

(F_3) If the star $\mathcal{S} = \{\pi_\lambda ; \lambda \in \Lambda\} : S \to \{A_\lambda ; \lambda \in \Lambda\}$ in $\mathbb{C}$ is the product of $\{A_\lambda ; \lambda \in \Lambda\}$ then the star $F(\mathcal{S}) = \{F(\pi_\lambda) ; \lambda \in \Lambda\} : F(S) \to \{F(A_\lambda) ; \lambda \in \Lambda\}$ in $\mathbb{C}'$ is the product of $\{F(A_\lambda) ; \lambda \in \Lambda\}$.

4,2. Assume 4,1 and $A \in obj\ \mathbb{C}$. Let X be a projective object in $[\mathbb{C}', E', \mathsf{M}']$ and let $\mu : X \to F(A)$. Then we shall say that A is <u>generated</u> by μ if and only if the following two conditions hold:

(G_1) If $B \in obj\ \mathbb{C}$, $\alpha, \beta : A \to B$, $\mu F(\alpha) = \mu F(\beta)$ then $\alpha = \beta$.

(G_2) If $B, C \in obj\ \mathbb{C}$, $\alpha : A \to B$, $\varkappa : C \to B$, $\varkappa \in \mathsf{M}$, $\mu F(\alpha) = \lambda F(\varkappa)$ for some $\lambda : X \to F(C)$ then $\alpha = \alpha_1 \varkappa$ for some $\alpha_1 : A \to C$.

<u>4,3</u>. Assume 4,1 and 4,2. Especially, let A be generated by $\mu : X \to F(A)$. Let $B \in obj\ \mathbb{C}$. Then μ will be called B -<u>independent</u> if and only if for any $\vartheta : X \to F(B)$ there exists always some $\bar{\vartheta} : A \to B$ such that $\mu F(\bar{\vartheta}) = \vartheta$. It is clear that $\bar{\vartheta}$ is then uniquely determined by ϑ as one can see from (G_1). Denote by $\underline{ind\ \mu}$ the class of all B such that μ is B -independent.

<u>4,4</u>. <u>Theorem</u>: Assume 4,1 and 4,2. Especially, let A be generated by $\mu : X \to F(A)$. Then the class $ind\ \mu$ is primitive provided that it is non-void.

<u>Proof</u>: Theorem 1,15 shows that assuming $ind\ \mu \neq \emptyset$ we need only to prove that $(\mathcal{P}_i)$ $i = 1, 2, 3$ are satisfied for $ind\ \mu$.

Suppose $B \in ind\ \mu$, $\varkappa : C \to B$, $\varkappa \in \mathsf{M}$ and let us show that $C \in ind\ \mu$. Suppose $\vartheta : X \to F(C)$. Then $\vartheta F(\varkappa) : X \to F(B)$ and, as $B \in ind\ \mu$, we have $\vartheta F(\varkappa) = \mu F(\alpha)$ for some $\alpha : A \to B$. Now, by (G_2), we have $\alpha = \alpha_1 \varkappa$ for some $\alpha_1 : A \to C$. Thus $\vartheta F(\varkappa) = \mu F(\alpha_1) F(\varkappa)$ and, as $F(\varkappa) \in \mathsf{M}'$, it follows $\vartheta = \mu F(\alpha_1)$. Hence μ is C -independent and $C \in ind\ \mu$. $(\mathcal{P}_1)$ is proved.

Suppose again $B \in ind\, \mu$, $\nu : B \to C$, $\nu \in \mathbf{E}$ and let us show that $C \in ind\, \mu$. Suppose $\vartheta : X \to F(C)$. As X is projective and $F(\nu) \in \mathbf{E}'$ we have $\vartheta = \vartheta_1 F(\nu)$ for some $\vartheta_1 : X \to F(B)$. As $B \in ind\, \mu$ there must exist some $\overline{\vartheta_1} : A \to B$ with $\vartheta_1 = \mu F(\overline{\vartheta_1})$. Now, $\overline{\vartheta_1}\nu : A \to C$ and $\mu F(\overline{\vartheta_1}\nu) =$ $= \mu F(\overline{\vartheta_1}) F(\nu) = \vartheta_1 F(\nu) = \vartheta$. Hence μ is C-independent and $C \in ind\, \mu$. ($\mathcal{P}_2$) is proved.

Finally, let $B_\lambda \in ind\, \mu$ for all $\lambda \in \Lambda$ and let the star $\mathcal{S} = \{\pi_\lambda ; \lambda \in \Lambda\} : S \to \{B_\lambda ; \lambda \in \Lambda\}$ be the product of this system. We shall show that $S \in ind\, \mu$. Suppose $\vartheta : X \to F(S)$. Then $\vartheta F(\pi_\lambda) : X \to F(B_\lambda)$ and as $B_\lambda \in ind\, \mu$ we have $\vartheta F(\pi_\lambda) = \mu F(\rho_\lambda)$ for some $\rho_\lambda : A \to B_\lambda$. As $\mathcal{S}$ is the product we have $\rho_\lambda =$ $= \psi\pi_\lambda$ $(\lambda \in \Lambda)$ for some $\psi : A \to S$. Thus $\mu F(\psi) F(\mathcal{S}) =$ $= \vartheta F(\mathcal{S})$ and, as $F(\mathcal{S})$ is product by (F_3), it turns out that $\mu F(\psi) = \vartheta$. Hence, μ is S-independent and $S \in ind\, \mu$. ($\mathcal{P}_3$) is proved.

4,5. Let $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ be any bicategory satisfying ($\mathcal{B}_i$) $i = 1, 2, 3, 4$, and let $\mathbb{P} \neq \emptyset$ be any primitive class of its objects. Hence we may write $\mathbb{P} = \mathbb{C}(P, \eta)$ for some projective object P. Taking for F the identity functor $F : [\mathbb{C}, \mathbb{E}, \mathbb{M}] \to [\mathbb{C}, \mathbb{E}, \mathbb{M}]$, then, with respect to F, Q is generated by $\eta : P \to Q$. For (G_1) is clear and (G_2) follows easily from (ii) in 1,2. Now, one can easily see that $ind\, \eta = \mathbb{C}(P, \eta) = \mathbb{P}$. Hence every primitive class $\mathbb{P} \neq \emptyset$ of objects in $[\mathbb{C}, \mathbb{E}, \mathbb{M}]$ can be obtained in the way of theorem 4,4 when choosing a suitable functor F.

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