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A THEOREM ON MAPPINGS

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The theorem in question is of purely combinatorial character and a quite easy one. Probably, it has already appeared in the literature, at least implicitly. However, not having found an explicit reference, the present author preferred publishing a possibly well-known result to undertaking a long search; the more so, as the proof is short and there are applications to topology.

Theorem. Let X be a class and let f be a mapping of X (into some class) such that $fx = x$ for no $x \in X$.

Then there exist disjoint classes X_0, X_1, X_2 such that $X_0 \cup X_1 \cup X_2 = X$ and $f[X_i] \cap X_i = \emptyset$, $i = 0,1,2$.

Proof. We may suppose that $X \neq \emptyset$. For any $x \in X$ let $A(x)$ denote the class of all $y \in X$ such that, for some $m \in N$, $n \in N$ we have $f^m x = f^n y$ (we put, of course, $f^0 x = x$). Clearly, (i) any two classes $A(x_1)$, $A(x_2)$ either coincide or are disjoint, (ii) for any $x \in X$, $f[A(x)] \subset \subset A(x)$. Therefore, it is sufficient to prove the theorem for each of the classes $A(x)$.

Thus we may suppose that, for any $x \in X$, $y \in X$, there are m, n such that $f^m x = f^n y$. Choose an element $a \in X$. For any $x \in X$ denote by $m(x)$ the least $m \in N$ such that $f^m a = f^n x$ for some $n \in N$; denote by $n(x)$ the least $n \in N$ such that $f^n x = f^{m(x)} a$.

Clearly, for any $x \in X$ with $n(x) > 0$, we have $m(fx) = mx$, $n(fx) = n(x) + 1$. It is easy to see that there exists at most one $b \in X$ such that $n(b) = 0$ (i.e. $b = f^k a$ for some $k \in N$) and $m(fb) \neq m(b) + 1$. Put $X_0 = (b)$ if such an element b exists, $X_0 = \emptyset$ if this is not the case. Then

$$m(fx) + n(fx) = m(x) + n(x) + 1$$

whenever $x \in X - X_0$.

Now let X_1, respectively X_2 consist of all $x \in X - X_0$ such that $m(x) + n(x)$ is odd, respectively even. It is clear that

$$X_0 \cup X_1 \cup X_2 = X, \quad f[X_i] \cap X_i = \emptyset \text{ for } i = 0,1,2.$$

Remarks. 1) Clearly, we have used a strong form of the axiom of choice. If X is supposed to be a set, a current weak form is sufficient. 2) In certain cases two sets may do (i.e., we may put $X_0 = \emptyset$). A necessary and sufficient condition for this is the following: there are no distinct $x_1, x_2, \ldots, x_n \in X$, n odd, with $fx_i = x_{i+1}$ for $i = 1, \ldots, n-1$, $fx_n = x_1$.

The following assertion is a simple example of a topological proposition obtained immediately from the above theorem. Observe that if X, Y are completely regular spaces and $f : X \to Y$ is a continuous mapping, we shall denote by $\overline{f}$ the extension of f to a mapping of βX into βY.

<u>Proposition 1</u>. If D is a discrete space, and $f : D \to D$, then the set of fixed points of $\overline{f}$ coincides with the closure of the set of fixed points of f.

Using Proposition 1, a short proof can be given of the following result (see Z. Frolík, Fixed points of maps of βN, to appear in Bull.Acad.Polon.Sci.).

Proposition 2. Let f be a homeomorphism of βN into $\beta N - N$. Then f has no fixed point.

Proof. It is easy to see that there exist G_k, $k \in N$, such that (i) $\{G_k\}$ is disjoint, $\bigcup \{G_k\} = N$, (ii) $k \in G_k$ for no k, (iii) $(fk) \cup G_k$ is a neighborhood of fk in $N \cup f[N]$. For every $n \in N$ put $hn = fk$ where $n \in G_k$; thus h is a mapping of N onto $f[N]$. Put $g = f^{-1} \circ h$; then, by (ii), $gn \neq n$ for $n \in N$ and therefore, by Proposition 1, $\bar{g}$ has no fixed point.

Since $fk \in G_k$ and $hn = fk$ for $n \in G_k$, we have $\bar{h}(fk) = fk$; hence $\bar{h}\xi = \xi$ whenever $\xi \in \overline{f[N]}$.

Now suppose there is a $\xi \in \beta N$ with $f\xi = \xi$. Then $\bar{h}(f\xi) = f\xi = \xi$; hence ξ is a fixed point of $\bar{g}$, which is a contradiction.

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