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ON DESCRIPTIVE CLASSIFICATION OF SET-FUNCTORS I.

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The aim of the present paper is to study set-functors (functors from the category $\mathcal{S}$ of all sets into itself) in some detail, with respect to preserving of limits of several types of diagrams (equalizers, sets of fixed points, preimages, intersections, products and so on). Also, some notions and proof from [9], [10] are modified and generalized.

The paper has eight parts. In the first one the known definitions, facts and conventions are recalled. In the second one the distinguished pair of a functor is defined and some easy consequences are proved.
The categorial definitions of the preservation of preimages, finite intersections, sets of fixed points and their equivalent forms expressed by means of sets are given in the third part. The following two parts contain auxiliary propositions. In the fourth one, the functors without non-trivial separating subfunctors are considered, in the fifth one the heredity of the preserving of limits, and its "converse", is investigated. In the sixth part special functors are considered. The main results are proved in the last two

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Secondary -

parts, in the seventh and the eighth ones. Namely, we give a characterization of functors preserving preimages, equalizers, pull back diagrams, separating systems up to m (see III.9), products up to m, limits up to m, relations between these properties and many examples.

An investigation of preserving of coequalizers, push out diagrams, finite colimits etc. will follow soon in the forthcoming paper On descriptive classification of set-functors II.

I.

Conventions:

I.1. Set-theoretic conventions:

a) As usual, an ordinal number α is the set of all ordinal numbers $\beta < \alpha$; thus, $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{0,1\}$ etc. Cardinal numbers are the initial ordinal numbers.

b) If X is a set, the symbols ϑ_X, j_X designate the mappings $\vartheta_X : \emptyset \to X$, $j_X : X \to 1$; thus, $\vartheta_1 = j_0 \cdot v_X^0 : X \to 2$ or $v_X^1 : X \to 2$ are the constant mappings on 0 or 1, respectively. The identical mapping of X onto itself will be denoted by id_X.

c) As usual, a mapping $f : X \to Y$ is said to be an injection if $f(x) \neq f(y)$ whenever $x \neq y$, surjection if $f(X) = Y$, inclusion if $f(x) = x$ for all $x \in X$.

I.2. If K is a category, then K^{σ} denotes the class of its objects, K^{m} the class of its morphisms. If

$a, b \in K^{\sigma}$ then $K(a, b)$ denotes the set of all morphisms of K from a to b.

I.3. The category of sets (the empty set included) and all their mappings will be denoted by $\mathcal{S}$. $\mathcal{S}^*$ is the category of all non-empty sets and all their mappings.

I.4. Throughout this paper the word "functor" means always a covariant functor from $\mathcal{S}$ to $\mathcal{S}$.

I.5. Let P, M be sets, $\kappa : P \to M$ a mapping. Then $C_{P,\kappa,M}$ is the functor H given by formulas

$H(\emptyset) = P$ and if $X \neq \emptyset$, then $H(\vartheta_X) = \kappa$,

$H(X) = M$, $H(f) = id_M$ whenever $f : X \to Y$.

If $P \subset M$ and κ is the inclusion, we write simply $C_{P,M}$; if, moreover, $P = M$, we write C_M.

I.6. The identical functor of $\mathcal{S}$ onto itself will be denoted by I. If M is a set, we put $Q_M(-) = \mathcal{S}(M, -)$. Thus, Q_0 is naturally equivalent to C_1.

I.7. The functor C_0 is called <u>trivial</u>, the other functors are called <u>non-trivial</u>. If H is non-trivial then $H(X) \neq \emptyset$ whenever $X \neq \emptyset$. The domain-range-restriction of H to $\mathcal{S}^*$ will be denoted by H^*; thus, $H^* : \mathcal{S}^* \to \mathcal{S}^*$.

I.8. A functor G is called a subfunctor of a functor H if $G(X) \subset H(X)$ for every set X and the inclusions form a transformation of G in H. The expression in functors:

$$H = H_1 \cup H_2$$

means: H, H_1, H_2 are functors, H_1 and H_2 are subfunctors of H and $H(X) = H_1(X) \cup H_2(X)$ for every set X. The expression in functors $G = G_1 \cap G_2$ is obvious.

I.9. Natural equivalence of functors will be denoted by $\simeq$. G is said to be a factor functor of H if there is an epitransformation $\nu : H \to G$.

I.10. Disjoint union of functors: let $\mathcal{J}$ be a set, H_ι be functors; we shall write $H = \bigvee_{\iota \in \mathcal{J}} H_\iota$ iff $H = \bigcup_{\iota \in \mathcal{J}} G_\iota$, $G_\iota \simeq H_\iota$ for every $\iota \in \mathcal{J}$ and if $\iota, \iota' \in \mathcal{J}$, $\iota \neq \iota'$, then $G_\iota \cap G_{\iota'} = C_0$.

I.11. A functor H is called <u>connected</u> if $card\, H(1) = 1$. Maximal connected subfunctors of a functor are called its <u>components</u>. If H is a non-trivial functor, put $H_a(X) = [H(j_X)]^{-1}(a)$ for every $a \in H(1)$; then H_a is a component of H and $H = \bigcup_{a \in H(1)} H_a = \bigvee_{a \in H(1)} H_a$.

I.12. If H is a functor and $f \neq \vartheta_X$ is an injection (or a surjection), then $H(f)$ is also an injection (or a surjection, respectively) (see [8]). $H(\vartheta_X)$ need not be injections, of course.

I.13. If H is a functor and $i : A \to X$ is an inclusion, we shall write $H(A)_X$ instead of $[H(i)](H(A))$. Thus, $H(A)_X \subset H(X)$.

I.14. For every functor it holds:

if $A, B \subset X$, $A \cap B \neq \emptyset$, then $H(A \cap B)_X = H(A)_X \cap$ $\cap H(B)_X$ (see [10], Proposition 2.1).

I.15. A functor H is said to be <u>separating</u> (see [9]) if $A, B \subset X$, $A \cap B = \emptyset$ implies $H(A)_X \cap H(B)_X = \emptyset$. Every functor H may be expressed as $H = H_s \vee H_d$ where H_s is separating and H_d has no non-trivial separating subfunctor (see [9], Statement 4.3).

I.16. Let H be a functor, $x \in H(X)$. Then $H_{\langle x, X\rangle}$ is the subfunctor G of H defined by $G(Y) = \{[H(f)](x) ; f : X \to Y\}$ for $Y \neq \emptyset$, $G(\emptyset) = \{a \in H(\emptyset) ; [H(\vartheta_X)](a) = x\}$ x).

II.

II.1. <u>Definition</u>: Let H be a functor, $x \in H(X)$. A pair $\langle x, X\rangle$ will be called <u>distinguished</u> iff $(H_{\langle x, X\rangle})^* \simeq C_1^*$.

II.2. <u>Proposition</u>: For every $x \in H(\emptyset)$, $\langle x, \emptyset\rangle$ is distinguished.

<u>Proof</u>: It is evident.

<u>Note:</u> Thus, if H is separating, then $H(\emptyset) = \emptyset$.

II.3. <u>Lemma</u>: Let $X \neq \emptyset$, $x \in H(X)$. $\langle x, X\rangle$ is distinguished iff the following conditions are satisfied:

a) $[H(f)](x) = x$ for all $f : X \to X$;

b) $[H(v_X^0)](x) = [H(v_X^1)](x)$.

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x) The definition of $H_{\langle x, X\rangle}$ differs from that given in [10] in the value $G(\emptyset)$

Note: If $\mathrm{card}\ X = 1$ then a) holds trivially. If $\mathrm{card}\ X > 1$, a) implies b).

Proof: If $X \neq \emptyset$ and $\langle x, X \rangle$ is distinguished, then a) b) hold trivially. Conversely, let a),b) hold. We have to prove that $[H(g)](x) = [H(g')](x)$ for every $g, g' : X \to Y$. Put $a = [H(j_X)](x)$, $b = [H(g)](x)$, $b' = [H(g')](x)$. Choose an $h : 1 \to X$. Evidently, $x = [H(h)](a)$. If $g \circ h = g' \circ h$, then $b = b'$. If $g \circ h \neq g' \circ h$, there exists an $l : 2 \to Y$ with $g \circ h = l \circ v_1^0$, $g' \circ h = l \circ v_1^1$. Consequently, $b = [H(g \circ h)](a) = [H(l \circ v_1^0)] = [H(l \circ v_1^1)](a) = [H(g' \circ h)](a) = b'$.

II.4. Proposition: Let H be a functor, $A, B \subset X$, $A \cap B = \emptyset$. Then for every $x \in H(A)_X \cap H(B)_X$ the pair $\langle x, X \rangle$ is distinguished.

Proof: The proposition holds trivially for $A = \emptyset$ or $B = \emptyset$. Let A, B be non empty, let $x \in H(A)_X \cap H(B)_X$. Consequently $x = [H(i_A)](\bar{a}) = [H(i_B)](\bar{b})$ for some $\bar{a} \in H(A)$, $\bar{b} \in H(B)$, where $i_A : A \to X$, $i_B : B \to X$ are the inclusions. Choose $a \in A$, $b \in B$ and denote by $c_a : 1 \to A$ or $c_b : 1 \to B$ the constant mappings onto a or b, respectively. Let $r_A : X \to A$ and $r_B : X \to B$ be mappings with $r_A \circ i_A = id_A$, $r_A \circ i_B = c_a \circ j_B$, $r_B \circ i_A = c_b \circ j_A$, $r_B \circ i_B = id_B$. Since $j_X \circ i_A = j_A$, $j_X \circ i_B = j_B$, we have $y = [H(j_X)](x) = [H(j_A)](\bar{a}) = [H(j_B)](\bar{b})$. Then $[H(c_a)](y) = [H(c_a \circ j_B)](\bar{b}) = [H(r_A \circ i_B)](\bar{b}) = [H(r_A)](x) = [H(r_A \circ i_A)](\bar{a}) = \bar{a}$ and analogously $[H(c_b)](y) = \bar{b}$. Let $h : 2 \to X$ be the mapping with $i_A \circ c_a = h \circ v_1^0$, $i_B \circ c_b = h \circ v_1^1$. We have $x = [H(h \circ v_1^0)](y) = [H(h \circ v_1^1)](y)$. $H(h)$ is an injection and hence $[H(v_1^0)](y) = [H(v_1^1)](y)$. Thus, $\langle y, 1 \rangle$ is distinguished. Since $x = [H(h \circ v_1^0)](y)$, $\langle x, X \rangle$ is also distinguished.

II.5. <u>Definition</u>. A distinguished pair $\langle x, X \rangle$ of a functor H will be called <u>regular</u> if there is an $a \in H(\emptyset)$ with $[H(\vartheta_X)](a) = x$. A functor H will be called <u>regular</u> if every its distinguished pair is regular.

II.6. <u>Propotision</u>: A functor H is regular iff $H(A)_X \cap \cap H(B)_X = H(A \cap B)_X$ for all X, $A \subset X$, $B \subset X$.

<u>Proof</u>: If $A \cap B \neq \emptyset$, then every functor satisfies the equality. If $A \cap B = \emptyset$, use the previous proposition.

III.

III.1. An equalizer of morphisms f, g will be denoted by $m = eq(f, g)$.

<u>Definition</u>: A functor H is said <u>to preserve sets of fixed points</u> if $H(m) = eq(H(f), H(g))$ whenever $m = eq(f, g)$ and f is a monomorphism.

III.2. <u>Proposition</u>: A functor H preserves equalizers iff x)

a) all $H(\vartheta_X)$ are injections;

b) $H(A)_X = \{x \in H(X); [H(f)](x) = [H(g)](x)\}$ for every $f, g : X \to Y$, where $A = \{x \in X; f(x) = g(x)\}$.

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x) The functors preserving difference kernels are defined in [10] as those that satisfy b). Thus, this notion differs from preserving equalizers defined purely categorially.

Proof is evident.

III.3. Proposition: A functor H preserves sets of fixed points iff

a) all $H(\vartheta_X)$ are injections;

b) if $f: X \to X$, $A = \{x \in X ; f(x) = x\}$, then $H(A)_X = \{z \in H(X) ; [H(f)](z) = z\}$.

Proof: I. Let H preserve sets of fixed points. Then a) evidently holds because it is easy to find mappings f, g, f monomorphism, with $\vartheta_X = eq(f, g)$. If $f: X \to X$ is a mapping, $A = \{x \in X ; f(x) = x\}$, $i: A \to X$ is the inclusion, then $i = eq(f, id_X)$, consequently $H(i) = eq(H(f), id_{H(X)})$. This implies $H(A)_X = \{z \in H(X) ; [H(f)](z) = z\}$.

II. Let $H \neq C_0$ satisfy a),b). Let $f, g: X \to Y$ be mappings, f a monomorphism. If $X = \emptyset$ then $f = g$, consequently $H(eq(f,g)) = eq(H(f), H(g))$. Let X be non-empty.

1) If either $g(X) \subset f(X)$ or g is non-constant, we can choose a mapping $h: Y \to X$ such that $h \circ f = id_X$ and $h \circ g(x) = x$ iff $f(x) = g(x)$. Put $A = \{x \in X; f(x) = g(x)\}$, $B = \{z \in H(X); [H(f)](z) = [H(g)](z)\}$. One can prove that $H(A)_X = B$.

2) Let g be a constant mapping on $y_0 \in Y - f(X)$. Then $\vartheta_X = eq(f, g)$. We may suppose H connected. It is sufficient to consider the following cases:

a) H is separating. Then necessarily $[H(f)](H(X)) \cap [H(g)](H(X)) = \emptyset$. Consequently, $H(\vartheta_X) = \vartheta_X = eq(H(f), H(g))$.

b) H has no nontrivial separating subfunctor. The equality $H(\vartheta_X) = eq_X(H(f), H(g))$ will be proved if we prove

α) $[H(f)](x) = [H(g)](x)$ for at most one point $x \in H(X)$;

β) $H(\emptyset) \neq \emptyset$.

α) follows easily from the fact that g factors through $j_X : X \longrightarrow 1$ and $H(f)$ is a monomorphism. To prove β), use the fact that there is a monotransformation $\mu : C_1^* \longrightarrow H^*$. Consequently, the mapping $H(v)$ has a fixed point, where $v: 2 \longrightarrow 2$, $v(0) = 1$, $v(1) = 0$. Thus, $\emptyset = \{x \in 2 ;\ v(x) = x\}$, $H(\emptyset)_2 = \{x \in H(2) ;$ $[H(v)](x) = x\} \neq \emptyset$, consequently $H(\emptyset) \neq \emptyset$.

III.4. <u>Convention</u>: The diagram

$$\begin{tikzcd}[column sep=small] & \cdot \arrow[dr, "g_1"] & \\ \cdot \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & \cdot \\ & \cdot \arrow[ur, "g_2"'] & \end{tikzcd}$$

will be designated by $\begin{pmatrix} f_1 , g_1 \\ f_2 , g_2 \end{pmatrix}$.

<u>Definition</u>: A functor H is said <u>to preserve preimages</u> (or <u>to preserve finite intersections</u>) if

$\begin{pmatrix} H(f_1) , H(g_1) \\ H(f_2) , H(g_2) \end{pmatrix}$ is a pullback diagram whenever $\begin{pmatrix} f_1 , g_1 \\ f_2 , g_2 \end{pmatrix}$ is a pullback and g_1 is a monomorphism (or g_1 and g_2 are monomorphisms, respectively).

III.5. <u>Proposition</u>: A functor H preserves finite intersecsections iff

a) all $H(\vartheta_X)$ are monomorphisms;

b) H is regular.

Proof: Let H preserve finite intersections. If X is a set, choose a set $Y \neq \emptyset$ with $X \cap Y = \emptyset$ and denote by $i_X : X \to X \cup Y$, $i_Y : Y \to X \cup Y$ the inclusions. Since $\begin{pmatrix} \vartheta_X , & i_X \\ \vartheta_Y , & i_Y \end{pmatrix}$ is a pull-back diagram,

$\begin{pmatrix} H(\vartheta_X) , & H(i_X) \\ H(\vartheta_Y) , & H(i_Y) \end{pmatrix}$ is, too. Consequently, $H(\vartheta_X)$ is a monomorphism and, choosing $X \neq \emptyset$, we see easily that every distinguished pair of H is regular. If H satisfies a), b), it clearly preserves finite intersections.

III.6. Lemma: If $f : X \to Y$ is an injection, $B \subset Y$, $A = f^{-1}(B)$, then every functor H preserving finite products satisfies $H(A)_X = [H(f)]^{-1}(H(B)_Y)$.

Proof: It is evident.

III.7. Proposition: The following properties of a functor H are equivalent:

(i) H preserves preimages;

(ii) H preserves finite intersections and if $f : X \to Y$ is a surjection, $B \subset Y$, then $H(f^{-1}(B))_X = [H(f)]^{-1}(H(B)_Y)$;

(iii) all $H(\vartheta_X)$ are monomorphisms and if $f : X \to Y$ is a mapping, $B \subset Y$, then $H(f^{-1}(B))_X = [H(f)]^{-1}(H(B)_Y)$.

Proof: is easy. Use the well known fact that a diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ A \arrow[ur, "\varkappa"] \arrow[dr, "g"'] & & Y \\ & B \arrow[ur, "b"'] & \end{tikzcd}$$

where s is an injection, is a pullback diagram iff κ is an injection and $\kappa(A) = f^{-1}(s(B))$.

III.8. <u>Definition</u>: Let m be an infinite cardinal, H a functor. We shall say that H <u>preserves intersections up to</u> m if all $H(\vartheta_X)$ are monomorphisms and $H(Y)_X = \bigcap_{\alpha \in A} H(X_\alpha)_X$ whenever $X_\alpha \subset X$ for all $\alpha \in A$, $Y = \bigcap_{\alpha \in A} X_\alpha$, $\operatorname{card} A < m$.

III.9. <u>Definition</u>: A couple $\langle X; \{\varphi_\alpha; \alpha \in A\}\rangle$ is called a <u>separating system</u> if all φ_α are mappings with domain X and they are collectionwise monomorphic, i.e. for every $x, y \in X$, $x \neq y$ there exists $\alpha \in A$ such that $\varphi_\alpha(x) \neq \varphi_\alpha(y)$.

<u>Definition</u>: Let m be an infinite cardinal. We shall say that a functor H <u>preserves separating systems</u> (or products) <u>up to</u> m if $\langle H(X); \{H(\varphi_\alpha); \alpha \in A\}\rangle$ is a separating system (or product) whenever $\operatorname{card} A < m$ and $\langle X; \{\varphi_\alpha; \alpha \in A\}\rangle$ is a separating system (or product, respectively).

III.10. <u>Note</u>: 1) Evidently, if H preserves products up to m, it preserves separating systems up to m .

2) The preserving of separating systems differs from the preserving of subdirect products only in the value of H at $\emptyset$ (see [10], Note 5,4).

3) We say that H preserves finite (or countable) products or separating systems instead of saying that it preserves them up to $\aleph_0$ (or up to $\aleph_1$, respectively).

III.11. Proposition: If a functor preserves finite intersections and finite separating systems, it preserves equalizers.

Proof: Let H be a functor which preserves finite intersections and finite separating systems. Let $f, g : X \to Y$ be mappings, $m = eq(f, g)$. Let $\langle X \times Y, \{\pi_X, \pi_Y\} \rangle$ be the product of X and Y. Let $\bar{f}, \bar{g} : X \to X \times Y$ be the mappings with $\pi_X \circ \bar{f} = id_X$, $\pi_Y \circ \bar{f} = f$, $\pi_X \circ \bar{g} = id_X$, $\pi_Y \circ \bar{g} = g$. Since $\begin{pmatrix} m, & \bar{f} \\ m, & \bar{g} \end{pmatrix}$ is a pull-back diagram and $\bar{f}, \bar{g}$ are injections, $\begin{pmatrix} H(m), & H(\bar{f}) \\ H(m), & H(\bar{g}) \end{pmatrix}$ is also a pull-back diagram. Thus $H(m) = eq(H(\bar{f}), H(\bar{g}))$ and, since $\langle H(X \times Y); \{H(\pi_X), H(\pi_Y)\} \rangle$ is a separating system, $H(m) = eq(H(f), H(g))$.

III.12. Proposition: Let $\mathfrak{m}$ be an infinite cardinal. Let a functor H preserve finite intersections and separating systems up to $\mathfrak{m}$. Then H preserves intersections up to $\mathfrak{m}$.

Proof: Let $\mathfrak{X} \subset \exp X$, $\operatorname{card} \mathfrak{X} < \mathfrak{m}$. If $Y \in \mathfrak{X}$, denote by $i_Y : Y \to X$ the inclusion. Put $L = \bigcap_{Y \in \mathfrak{X}} Y$ and denote by $i_L : L \to X$ the inclusion. Choose mappings $f_Y, g_Y : X \to M_Y$ with $i_Y = eq(f_Y, g_Y)$. Let $\langle M; \{\pi_Y; Y \in \mathfrak{X}\} \rangle$ be a product of the collection $\{M_Y; Y \in \mathfrak{X}\}$; denote by $f, g : X \to M$ the mappings with $\pi_Y \circ f = f_Y$, $\pi_Y \circ g = g_Y$ for all $Y \in \mathfrak{X}$. Then $i_L = eq(f, g)$. Since H preserves equalizers (see III.11), $H(i_L) = eq(H(f), H(g))$,

$H(i_Y) = eq(H(f_Y), H(g_Y))$. Since $\langle H(M); \{H(\pi_Y); Y \in \mathfrak{X}\}\rangle$ is a separating system, $H(i_L)$ is an equalizer of the collection $\{\langle H(f_Y), H(g_Y)\rangle; Y \in \mathfrak{X}\}$. Consequently $H(L)_X = \bigcap_{Y \in \mathfrak{X}} H(Y)_X$.

IV.

As recalled in I.15, every functor is a disjoint union of a separating functor and a functor without non-trivial separating subfunctor. Thus, the preserving properties may be considered separatedly for separating functors and for those functors without non-trivial separating subfunctor. The latter is given in the present part.

IV.1. <u>Lemma</u>: Let $\nu: I^* \longrightarrow H^*$ be an epitransformation, which is not a natural equivalence. Then $H^* \simeq C_1^*$.

<u>Proof</u>: It is evident.

IV.2. <u>Lemma</u>: Let a functor H have no non-trivial separating subfunctor. Let there be an epitransformation $\nu: Q_X^* \longrightarrow H^*$. If H preserves either preimages or equalizers or finite separating systems, then $H^* \simeq C_1^*$.

Proof: If $card\ X \leqq 1$, the statement is evident. Let $card\ X > 1$, put $a = \nu_X(id_X)$. By Lemma IV.1 if $f, g: X \longrightarrow Y$ are constant mappings, then $\nu_Y(f) = \nu_Y(g) = b_Y$. Clearly, it is sufficient to prove $a = b_X$.

a) Let H preserve preimages: let $h: X \longrightarrow 2$ be the constant mapping onto $1 \in 2$. Put $B = \{0\} \subset 2$. Then $h^{-1}(B) = \emptyset$, $H(\emptyset)_X = [H(h)]^{-1}(H(B)_2)$. However,

$H(B)_2 = \{ b_2 \}$ and $b_2 = \nu_2(h) = \nu_2(h \circ id_X) =$
$= \nu_2([Q_X^*(h)](id_X)) = [H(h)](a)$ and consequently
$a \in H(\emptyset)_X \subset \{ b_X \}$.

b) Let H preserve equalizers: let $f_0, f_1 : X \to 2$ be the constant mappings on 0 or 1, respectively. Then $\vartheta_X = eq(f_0, f_1)$, consequently $H(\vartheta_X) = eq(H(f_0), H(f_1))$. But $[H(f_0)](a) = \nu_2(f_0) = b_2 = \nu_2(f_1) = [H(f_1)](a)$ and consequently $a \in H(\emptyset)_X \subset \{ b_X \}$.

c) Let H preserve finite separating systems: put $Y = X \times X$, let $\pi_1, \pi_2 : Y \longrightarrow X$ be the projections. Choose $x_1, x_2 \in X$, $x_1 \neq x_2$. Let $f_1, f_2 : X \longrightarrow Y$ be the mappings with $\pi_2 \circ f_1 = \pi_2 \circ f_2 = id_X$, let $\pi_1 \circ f_1$ or $\pi_1 \circ f_2$ be the constant mapping onto x_1 or x_2, respectively. Put $c_1 = \nu_Y(f_1)$, $c_2 = \nu_Y(f_2)$. Since $[H(\pi_1)](c_1) = \nu_X(\pi_1 \circ f_1) = b_X = \nu_X(\pi_1 \circ f_2) = [H(\pi_1)](c_2)$ and $[H(\pi_2)](c_1) = [H(\pi_2)](c_2)$, then necessarily $c_1 = c_2$. Let $\ell : Y \longrightarrow X$ be a mapping such that $\ell \circ f_1$ is a constant, $\ell \circ f_2 = id_X$. Then

$$a = \nu_X(id_X) = \nu_X(\ell \circ f_2) = [H(\ell)](c_2) =$$
$$= [H(\ell)](c_1) = \nu_X(\ell \circ f_1) = b_X .$$

IV.3. <u>Note</u>: The statement is false for functors, preserving sets of fixed points only.

IV.4. <u>Proposition</u>: Let H have no non-trivial separating subfunctor.

If H preserves either preimages or equalizers, then $H \simeq C_M$.

If H preserves finite separating systems, then $H \simeq C_{p,m}$.

If H preserves finite products, then either $H = C_0$ or $H \simeq C_{0,1}$ or $H \simeq C_1$.

Proof: follows easily by IV.2.

V.

V.1. Now we recall a proposition from [9], needed later (Lemma 3.1 in [9]):

Proposition: Let G, H be functors, $\mu : G \rightarrow H$ a monotransformation, $f : X \rightarrow Y$ a mapping. If either $X \neq \emptyset$ or G is regular, then no $x \in H(X)$ satisfies

$(*)$ $[H(f)](x) \in \mu_Y(G(Y)) - [H(f)](\mu_X(G(X)))$.

An easy proof is given in [9].

V.2. Proposition: Let a regular functor G be a subfunctor of a functor H . If H preserves either a) equalizers or b) sets of fixed points or c) preimages or d) intersections up to m or e) separating systems up to m , then G also preserves them.

Proof: Let $\mu : G \rightarrow H$ be a monotramsformation. For shortness we shall suppose that all μ_X are inclusions. All $G(\vartheta_X)$ are monomorphisms since all $H(\vartheta_X)$ are monomorphisms. a) b) will be proved together: if $m = eq(f, g)$ (or, moreover, f is a monomorphism, respectively), $f, g : X \rightarrow Y$, put $A = \{x \in X;\ f(x) = g(x)\}$, $B = \{z \in G(X);\ [G(f)](z) = [G(g)](z)\}$. Then $G(A)_X \subset B \subset G(X) \cap H(A)_X$. If $x \in G(X) \cap H(A)_X$ then $x \in G(A)_X$ by $(*)$.

c) Let $f : X \rightarrow Y$ be a mapping, $B \subset A$, $A = f^{-1}(B)$. $(*)$ yields easily that $G(A)_X = G(X) \cap H(A)_X$,

$G(B)_Y = G(Y) \cap H(B)_Y$. Thus $G(A)_X = [G(f)]^{-1}(G(B)_Y)$.
d) is also easy.
e) is trivial, the regularity of G need not be required.

V.3. Proposition: Let H be a functor. If every $H_{\langle \varkappa , X\rangle}$ preserves either equalizers or sets of fixed points or preimages or intersections up to m, then H also preserves them.

Proof: If every $H_{\langle \varkappa , X\rangle}$ preserves equalizers, then all $H(\vartheta)_X$ are monomorphisms. For, if $[H(\vartheta_X)](a) = c = [H(\vartheta)_X](b)$ for some $a, b \in H(\emptyset)$ then, since $H_{\langle c, X\rangle}(\vartheta_X)$ is a monomorphism, necessarily $a = b$. Now let $f, g : X \longrightarrow Y$ be mappings, $A = \{x \in X ; f(x) = g(x)\}$, $B = \{z \in H(X) ; [H(f)](z) = [H(g)](z)\}$. Then obviously $H(A)_X \subset B$. If $z \in B$, put $G = H_{\langle z, X\rangle}$. Then $z \in \{x \in G(X) ; [G(f)](x) = [G(g)](x)\} = G(A)_X \subset H(A)_X$. The proofs concerning the preservation of sets of fixed points or preimages or intersections up to m are quite analogous.

Note: An analogous statement on separating system does not hold.

VI.

In this part, some special functors will be investigated.

VI.1. First we define the category $\mathbb{F}$ of filters:

The category $\mathbb{F}'$: Objects are all pairs $\langle M, \mathcal{F}\rangle$, where either $\langle M, \mathcal{F}\rangle = \langle \emptyset, \{\emptyset\}\rangle$ or $\mathcal{F}$ is a filter on a non-void set M ; morphisms from $\langle M, \mathcal{F}\rangle$ to $\langle N, \mathcal{G}\rangle$ are all mappings $f : M \longrightarrow N$ with $f^{-1}(G) \in \mathcal{F}$ for all $G \in \mathcal{G}$.

The category $\mathbb{F}$ is a factor category of $\mathbb{F}'$: $\mathbb{F}'^{\sigma} = \mathbb{F}^{\sigma}$ and $f, g \in \mathbb{F}'(\langle M, \mathcal{F}\rangle, \langle N, \mathcal{G}\rangle)$ determine the same morphism of $\mathbb{F}$ (denoted by f^+ or g^+ respectively) iff $f/_F = g/_F$ for some $F \in \mathcal{F}$.

The category $\mathbb{F}$ is studied in [4], where its concreteness is proved. The following proposition is also given in [4]:

Proposition: A morphism $f^+ \in \mathbb{F}(\langle M, \mathcal{F}\rangle, \langle N, \mathcal{G}\rangle)$ is an epimorphism (or a monomorphism) of $\mathbb{F}$ iff $f(F) \in \mathcal{G}$ for all $F \in \mathcal{F}$ (or iff there is $F \in \mathcal{F}$ such that $f/_F$ is an injection, respectively).

VI.2. Definition: Let $\mathcal{E} : \mathbf{S} \longrightarrow \mathbb{F}$ be the full embedding with $\mathcal{E}(X) = \langle X, \{X\}\rangle$ for every set $X \in \mathbf{S}^{\sigma}$. Let $\langle M, \mathcal{F}\rangle \in \mathbb{F}^{\sigma}$, $M \neq \emptyset$. Denote by $Q_{M,\mathcal{F}} : \mathbf{S} \rightarrow \mathbf{S}$ the functor $Q_{M,\mathcal{F}}(-) = \mathbb{F}(\langle M, \mathcal{F}\rangle, \mathcal{E}(-))$.

VI.3. Proposition: There is a 1-1-correspondence between transformations from $Q_{N,\mathcal{G}}$ to $Q_{M,\mathcal{F}}$ and elements of $\mathbb{F}(\langle M, \mathcal{F}\rangle, \langle N, \mathcal{G}\rangle)$. Monotransformations correspond to epimorphisms, epitransformations to monomorphisms.

Proof: If $\varphi : Q_{N,\mathcal{G}} \longrightarrow Q_{M,\mathcal{F}}$ is a transformation, take the mapping $\lambda : M \longrightarrow N$ with $\varphi(id_N^+) = \lambda^+$. It is easy to see that $\lambda^{-1}(G) \in \mathcal{F}$ for all $G \in \mathcal{G}$.

VI.4. Proposition: Every $Q_{M,\mathcal{F}}$ preserves equalizers.

Proof: is easy.

VI.5. Let $\mathfrak{m}$ be an infinite cardinal. We recall that a filter $\mathcal{F}$ is said to be $\mathfrak{m}$-complete if $\bigcap_{X \in \mathfrak{X}} X \in \mathcal{F}$ whenever all X are in $\mathcal{F}$ and $\mathrm{card}\ \mathfrak{X} < \mathfrak{m}$.

Proposition: The following properties of a functor $Q_{M,\mathcal{F}}$ are equivalent:

(i) $Q_{M,\mathcal{F}}$ preserves products up to $\mathfrak{m}$;

(ii) $Q_{M,\mathcal{F}}$ preserves separating systems up to $\mathfrak{m}$;

(iii) $Q_{M,\mathcal{F}}$ preserves intersections up to $\mathfrak{m}$;

(iv) the filter $\mathcal{F}$ is $\mathfrak{m}$ -complete.

Proof: (i) $\Longrightarrow$ (ii) is trivial, (ii) $\Longrightarrow$ (iii) follows from III.12. (iii) $\Longrightarrow$ (iv): Denote $H = Q_{M,\mathcal{F}}$. Let $\mathfrak{X} \subset \mathcal{F}$, $\mathfrak{X} \neq \emptyset$, $\mathrm{card}\ \mathfrak{X} < \mathfrak{m}$. Put $Y = \bigcap_{X \in \mathfrak{X}} X$, denote by $\iota_Y : Y \rightarrow M$ the inclusion. Obviously, $(id_M)^+ \in H(X)_M$ for all $X \in \mathfrak{X}$, consequently $(id_M)^+ \in H(Y)_M$. Then necessarily $id_M/_F = \iota_Y \circ \varkappa/_F$ for some $F \in \mathcal{F}$ and $\varkappa : M \rightarrow Y$. Thus, $F \subset Y$ and consequently $Y \in \mathcal{F}$.

(iv) $\Longrightarrow$ (i) is evident.

Corollary: Every $Q_{M,\mathcal{F}}$ preserves limits of finite diagrams.

VI.6. The following functors are considered, e.g., in [7], [9],[12]:

The functor $\mathbb{N}$: $\mathbb{N}(X) = \{ Z \subset X ;\ Z \neq \emptyset \}$; if $f : X \rightarrow Y$ is a mapping, $\mathbb{N}(f) : \mathbb{N}(X) \longrightarrow \mathbb{N}(Y)$ is the mapping with $[\mathbb{N}(f)](Z) = f(Z)$.

The functor $\mathbb{N}_{\mathfrak{m}}$: If $\mathfrak{m} > 2$ is a cardinal, $\mathbb{N}_{\mathfrak{m}}$ is a subfunctor of $\mathbb{N}$ with $\mathbb{N}_{\mathfrak{m}}(X) = \{ Z \subset X ;\ Z \neq \emptyset ,\ \mathrm{card}\ Z < \mathfrak{m} \}$.

The functor Φ : If X is a set, $\Phi(X)$ is the set of all filters on X ; if $f : X \rightarrow Y$ is a mapping, $\mathcal{F} \in \Phi(X)$, $[\Phi(f)](\mathcal{F}) = \{ Z \subset Y ;\ f^{-1}(Z) \in \mathcal{F} \}$ or, equivalently,

$[\Phi(f)](\mathcal{F})$ is the filter on Y with the base $\{f(F)\,;\ F \in \mathcal{F}\}$.

The functor β : It is a subfunctor of Φ such that $\beta(X)$ is the set of all ultrafilters on X.

VI.7. Proposition: The functors $\mathbb{N}$, $\mathbb{N}_m$ preserve intersections and preimages. They do not preserve sets of fixed points.

Proof: is easy.

VI.8. Proposition: The functor β preserves preimages and sets of fixed points.

Proof: β evidently preserves preimages. The preserving of sets of fixed points follows easily from the following theorem, proved in [2],[3]: if $f : X \longrightarrow X$ is a mapping, then $X = X_0 \cup X_1 \cup X_2 \cup X_3$, where X_i $(i = 0,\dots,3)$ are disjoint, $X_0 = \{x \in X\,;\ f(x) = x\}$ and $f(X_i) \cap X_i = \emptyset$ for $i = 1, 2, 3$.

VI.9. Lemma: Let N be the set of all natural numbers, $P = N \times N - \{\langle n, n\rangle\,;\ n \in N\}$, $A_i, B_i \subset N$, $A_i \cap B_i = \emptyset$, $i = 1, 2, \dots, k$.
Then $P - \bigcup_{i=1}^{k} (A_i \times B_i) \neq \emptyset$.

Proof: Suppose $B_i = N - A_i$. Put $T = P - \bigcup_{i=1}^{k} (A \times B_i)$. For every $n \in N$ put $K_n = \{i\,;\ n \in A_i\}$. Since $K_n \subset \{1, 2, \dots, k\}$ there are $p, q \in N$, $p \neq q$ such that $K_p = K_q$. Thus, $\langle p, q\rangle \in T$.

VI.10. Proposition: The functor β does not preserve countable intersections and equalizers.

Proof: β evidently does not preserve countable intersections. We prove that β does not preserve equalizers. Let N be the set of all natural numbers, $P = N \times N - \{\langle n, n\rangle ; n \in N\}$, $f, g : P \to N$, $f(\langle m, n\rangle) = m$, $g(\langle m, n\rangle) = n$. Then $\vartheta_P = eq(f, g)$. We show that $\vartheta_{\beta(P)} \neq eq(\beta(f), \beta(g))$. Let $\mathcal{F}$ be an ultrafilter containing all sets $P - \bigcup_{i=1}^{k} (A_i \times B_i)$, where A_i, $B_i \subset N$, $A_i \cap B_i = \emptyset$. It is easy to see that $[\beta(f)](\mathcal{F}) = [\beta(g)](\mathcal{F})$.

VI.11. *Proposition*: The functor Φ preserves preimages. It does not preserve countable intersections and sets of fixed points.

Proof: is easy.

VII.

Here we give a characterization of functors preserving preimages or equalizers. The connections between preserving of pullback diagrams, preimages, finite products and equalizers, sets of fixed points are clarified.

VII.1. *Definition*: Let H be a functor, $\langle x, X\rangle$ be not distinguished. Put $H^{x,X} = \{A \subset X ; x \in H(A)_X\}$.

VII.2. *Proposition*: $H^{x,X}$ is a filter.

Proof: If $A, B \in H^{x,X}$, then $A \cap B \neq \emptyset$ since $\langle x, X\rangle$ is not distinguished (see II.4). Then $A \cap B \in H^{x,X}$, since $H(A \cap B)_X = H(A)_X \cap H(B)_X$.

VII.3. *Proposition*: If $f : X \to Y$, $[H(f)](x) = y$,

$\langle x, X\rangle, \langle y, Y\rangle$ are not distinguished, then $f(A) \in H^{y,Y}$ for every $A \in H^{x,X}$.

Proof: is evident.

VII.4. Proposition: A functor H preserves intersections up to m iff H is regular, all $H(\vartheta_X)$ are monomorphisms and if $H^{x,X}$ is m -complete for every non-distinguished $\langle x, X\rangle$.

Proof: is easy.

VII.5. Proposition: The following properties of a separating functor H are equivalent:

(i) H preserves preimages;

(ii) if $f: X \to Y$, $[H(f)](x) = y$, then $[\Phi(f)](H^{x,X}) = H^{y,Y}$;

(iii) the mappings $\varphi_X : H(X) \to \Phi(X)$, $\varphi_X(x) = H^{x,X}$ form a natural transformation $\varphi : H \to \Phi$;

(iv) if $f, g : X \to Y$, $[H(f)](x) = [H(g)](x)$, then $[\Phi(f)](H^{x,X}) = [\Phi(g)](H^{x,X})$.

Proof: If H is separating, then all $H(\vartheta_X)$ are monomorphisms and H preserves finite intersections.

(i) $\Longrightarrow$ (ii): Let $f: X \to Y$ be a mapping with $[H(f)](x) = y$. We have to prove α) if $B \in H^{y,Y}$, then $f^{-1}(B) \in H^{x,X}$; β) if $B \subset Y$, $f^{-1}(B) \in H^{x,X}$, then $B \in H^{y,Y}$.

α) is an easy consequence of the fact that H preserves preimages, β) follows from VII.3.

(ii) $\Longrightarrow$ (iii) is evident.

(iii) $\Longrightarrow$ (iv) is evident.

(iv) $\Longrightarrow$ (i): Let H do not preserve preimages. Then there is (see III.7) a surjection $f: X \to Y$, a set $B \subset$ $\subset Y$ and a point $a \in [H(f)]^{-1}(H(B)_Y) - H(A)_X$, where $A = f^{-1}(B)$. Choose an injection $\ell : Y \to X$ with $f \circ \ell = id_Y$. Put $b = [H(f)](a)$, $c = [H(\ell)](b)$. Then $[H(\ell \circ f)](a) = c$, $[H(f)](a) = [H(f)](c) = b$ and, since $\ell(B) \subset A$, c is an element of $H(A)_X$. Let $i : A \to X$ be the inclusion, $\kappa : X \to A$ be a mapping with $\kappa \circ i = id_A$. Since $c \in H(A)_X$, we have $[H(i \circ \kappa)](c) = c$. We have $[H(\bar{g})](c) = c$ for $\bar{g} = i \circ \kappa \circ \ell \circ f$. Put $g = f \circ \bar{g}$. Then $[H(f)](a) = b =$ $= [H(g)](a)$. But $[\Phi(f)](H^{a,X}) \neq [\Phi(g)](H^{a,X})$. For, $B \supset g(X)$ and hence $B \in [\Phi(g)](H^{a,X})$; but $B \notin [\Phi(f)](H^{a,X})$ because the converse implies $f^{-1}(B) \in H^{a,X}$, i.e. $a \in H(A)_X$ which is a contradiction.

<u>Corollary</u>: A functor G preserves preimages iff $G \simeq$ $\simeq C_M \vee H$, where H is separating and satisfies (ii) - (iv) from the proposition.

VII.6. <u>Proposition</u>: Let $\nu : Q_X \to H$ be a transformation, $\nu_X(id_X) = x$, $\langle x, X \rangle$ be not distinguished. Let $f, g : X \to Y$ be mappings with $f/_A = g/_A$ for some $A \in H^{x,X}$. Then $\nu_Y(f) = \nu_Y(g)$.

<u>Proof</u>: is evident.

VII.7. <u>Proposition</u>: The following properties of separating functors H are equivalent:

(i) H preserves equalizers;

(ii) if $f, g : X \to Y$, $[H(f)](x) = [H(g)](x)$ then $f/_A = g/_A$ for some $A \in H^{x,X}$.

(iii) $H_{\langle x, X \rangle} \simeq Q_{X, H^{x,X}}$ for every set X and every $x \in H(X)$;

(iv) $H = \bigcup_{\iota \in \mathcal{J}} G_\iota$, where $\mathcal{J}$ is a class and for every $\iota \in \mathcal{J}$ there is an $(M_\iota, \mathcal{F}_\iota) \in \mathbb{F}^\sigma$ with $G_\iota \simeq Q_{M_\iota, \mathcal{F}_\iota}$.

Proof: H is supposed to be separating, consequently all $H(\vartheta_X)$ are monomorphisms.

(i) $\Longrightarrow$ (ii): Put $A = \{x \in X;\ f(x) = g(x)\}$. Then $x \in \{x \in H(X);\ [H(f)](x) = [H(g)](x)\} = H(A)_X$ and consequently $A \in H^{x, X}$.

(ii) $\Longrightarrow$ (iii): The natural transformation $\nu: Q_{X, H^{x}, X} \rightarrow H_{\langle x, X \rangle}$ with $\nu_X(id_X^+) = x$ is obviously a natural equivalence.

(iii) $\Longrightarrow$ (iv) is evident.

(iv) $\Longrightarrow$ (i) follows by VI.4, V.2 and V.3.

Corollary: A functor G preserves equalizers iff $G \simeq C_M \vee H$ where H is separating and satisfies (ii) - (iv) from the proposition.

VII.8. Proposition: If a functor preserves equalizers then it preserves preimages.

Proof: If a functor G preserves equalizers, then $G \simeq C_M \vee H$ where H is separating and satisfies (ii) from VII.7. Consequently H satisfies (iv) from VII.5.

VII.9. Proposition: The following properties of H are equivalent:

(i) H preserves limits of finite diagrams;

(ii) H is connected and preserves pullback diagrams;

(iii) H preserves finite products and $H \not\simeq C_{0,1}$.

Proof: The implications (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii) are easy, (iii) $\Longrightarrow$ (i) follows from IV.4 and III.11.

VII.10. __Remark__: 1) Consequently the following assertions about a functor H are equivalent:

(i) H preserves pullback diagrams;

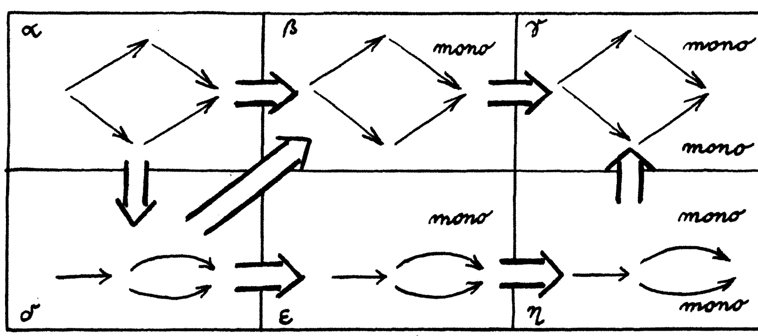
(ii) $H = \bigvee_{\iota \in H(1)} H$, every H_ι preserves finite products and $H_\iota \not\cong C_{0,1}$.

2) One can prove easily the equivalence of the following assertions:

(i) H preserves limits of all diagrams up to $\mathfrak{m}$;

(ii) H preserves products up to $\mathfrak{m}$ and $H \not\cong C_{0,1}$.

VII.11. The connection between preserving of pullback diagrams, preimages, equalizers etc. is indicated in the following picture:



where

α ... means preserving of pullback diagrams;

β ... " " " preimages;

γ ... " " " finite intersections;

δ ... " " " equalizers;

ε ... " " " sets of fixed points;

η ... " " " equalizers of pairs of monomorphisms.

The indicated implications and their compositions are true, and there are no others valid.

The implications $\alpha \Rightarrow \beta \Rightarrow \gamma$, $\delta \Rightarrow \varepsilon \Rightarrow \eta$ are trivial, $\alpha \Rightarrow \delta$ follows from VII.10 and VII.9, $\delta \Rightarrow \beta$ follows from VII.8. Now we prove $\eta \Rightarrow \gamma$: let a functor H preserve equalizers of pairs of monomorphisms and do not preserve finite intersections; there is necessarily a $c \in H(A)_X \cap H(B)_X - H(A \cap B)_X$ for some $A, B \subset X$. But then $A \cap B = \emptyset$. Choose monomorphisms $f, g : X \to Y$ with $\vartheta_X = eq(f, g)$. Then, since $\langle c, X \rangle$ is distinguished, $[H(f)](c) = [H(g)](c)$. Consequently $c \in$ $\in H(\emptyset)_X$, which is a contradiction.

VII.12. Examples:

$\delta \not\Rightarrow \alpha$ example: the factorfunctor of $Q_2^1 \vee Q_2^2$ (where Q_2^1 , Q_2^2 are two different copies of Q_2) given by the relation $\langle x, x \rangle^1 \sim \langle x, x \rangle^2$.

$\gamma, \beta \not\Rightarrow \eta, \varepsilon, \delta, \alpha$ example: all the functors $\mathbb{N}_m$, $\mathbb{N}$.

$\gamma, \varepsilon \not\Rightarrow \beta$, example: the factorfunctor of Q_3 given by the relation $\langle x, y, y \rangle \sim \langle z, y, y \rangle$.

$\varepsilon \not\Rightarrow \delta$, example: the functor β or the factorfunctor of Q_3 given by the relation $\langle x, y, y \rangle \sim \langle y, x, y \rangle$.

$\eta \not\Rightarrow \varepsilon$, example: the factorfunctor of Q_N given by the relation $\langle x_1, x_2, x_3, x_4, \dots \rangle \sim \langle x_1, x_1, x_2, x_3, x_4, \dots \rangle$.

VII.13. Some further implications are valid under certain assumptions, for example: if a functor preserves finite sums then it preserves preimages and sets of fixed points.

VII.14. We say that a functor H <u>is generated by finite sets</u> if $H(X) = \bigcup_{\substack{f: M \to X \\ M \text{ finite}}} [H(f)](H(M))$, or, equivalently, if H is a factorfunctor of some $\bigvee_{\iota \in \mathcal{J}} Q_{M_\iota}$, where $\mathcal{J}$ is a set and all M_ι are finite sets.

<u>Proposition</u>: Let H be a functor generated by finite sets. If H preserves equalizers of pairs of monomorphisms then it preserves sets of fixed points.

<u>Proof</u>: 1) Let H be a functor generated by finite sets and let H preserve equalizers of pairs of monomorphisms. Then all $H(\vartheta)_X$ are monomorphisms. If H does not preserve sets of fixed points, then there is $f: X \to X$ and $a \in H(X) - H(A)_X$ with $[H(f)](a) = a$, where $A = \{x \in X;\ f(x) = x\}$. Denote by $\iota_A : A \to X$ the inclusion. Choose M finite, $m \in H(M)$, $\varphi: M \to X$ with $[H(\varphi)](m) = a$. Denote by $\nu : Q_M \to H$ the transformation with $\nu_M(id_M) = m$. Clearly, if $\varphi' : M \to X$, $\nu_X(\varphi) = \nu_X(\varphi')$ then φ' does not factor through ι_A.

2) Put $R = \varphi(M) \cup f \circ \varphi(M)$, denote by $\iota_R : R \to X$ the inclusion. Choose a mapping $g: R \to R$ such that $g(x) = f(x)$ whenever $x \in \varphi(M)$, $g(x) \in \varphi(M) \cap f^{-1}(x)$ whenever $x \in f \circ \varphi(M) - \varphi(M)$. Denote by $\psi : M \to R$ the mapping defined by $\iota_R \circ \psi = \varphi$. Then $\iota_R \circ g \circ \psi = f \circ \varphi$, consequently $[H(\iota_R)](\nu_R(\psi)) = \nu_X(\varphi) = \nu_X(f \circ \varphi) = [H(\iota_R)](\nu_R(g \circ \psi))$. This yields $\nu_R(\psi) = \nu_R(g \circ \psi)$. Put $B = \{x \in R;\ g(x) = x\}$, let $\iota_B : B \to R$ be the inclusion. Clearly $B \subset A$, and consequently $\iota_R \circ \iota_B$ factors through ι_A.
If $\psi' : M \to R$, $\nu_R(\psi) = \nu_R(\psi')$, then ψ' cannot factor through ι_B because $\iota_R \circ \psi'$ does not factor

through ι_A . For, $\nu_X(\iota_R \circ \psi') = \nu_X(\iota_R \circ \psi) = \nu_X(\varphi)$.

3) Let C be the set of all points of all cycles of the mapping g (i.e. C is the greatest subset of R with $g(C) = C$), let $\iota_C : C \to R$ be the inclusion. Since R is finite, there is a natural number n such that $g^n \circ \iota_C = \iota_C$; then there is a mapping $\rho : R \to C$ with $g^n = \iota_C \circ \rho$. Let $h : C \to C$ be the mapping with $\iota_C \circ h = g \circ \iota_C$. Put $\chi = \rho \circ \psi$, consequently $[H(\iota_C)](\nu_C(h \circ \chi)) = \nu_R(\iota_C \circ h \circ \chi) = \nu_R(g \circ \iota_C \circ \chi) = \nu_R(g^{n+1} \circ \psi) = \nu_R(g^n \circ \psi) = \nu_R(\iota_C \circ \rho \circ \psi) = [H(\iota_C)](\nu_C(\chi))$ which implies $\nu_C(h \circ \chi) = \nu_C(\chi)$. Put $D = \{x \in C ; h(x) = x\}$, let $\iota_D : D \to C$ be the inclusion. Since $D \subset B$, $\iota_C \circ \iota_D$ factors through ι_B . The mapping h is not an identity because χ does not factor through ι_D . For, since $\nu_R(\psi) = \nu_R(g^n \circ \psi) = \nu_R(\iota_C \circ \chi)$, $\iota_C \circ \chi$ cannot factor through ι_B . But h is an injection and $[H(h)](\nu_C(\chi)) = \nu_C(\chi)$, $\nu_C(\chi) \in H(D)_C$, which is a contradiction.

VIII.

Now, we describe the functors preserving separating systems up to m or products up to m .

VIII.1. <u>Lemma</u>: Let $\sigma_i^+ : \langle M_i, \mathcal{F}_i \rangle \to \langle M, \mathcal{F} \rangle$, $i = 1, 2$, be epimorphisms in $\mathbb{F}$. Then there is a pullback-pushout diagram in $\mathbb{F}$, say

$$\begin{array}{ccccc} & & \langle M_1, \mathcal{F}_1 \rangle & & \\ & \overset{\lambda_1^+}{\nearrow} & & \overset{\sigma_1^+}{\searrow} & \\ \langle Z, \mathcal{L} \rangle & & & & \langle M, \mathcal{F} \rangle \\ & \underset{\lambda_2^+}{\searrow} & & \underset{\sigma_2^+}{\nearrow} & \\ & & \langle M_2, \mathcal{F}_2 \rangle & & \end{array}$$

λ_1^+ and λ_2^+ are epimorphisms. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are m-complete, so is $\mathcal{L}$.

Proof: Put $Z = \{\langle m_1, m_2\rangle \in M_1 \times M_2 ; \sigma_1(m_1) = \sigma_2(m_2)\}$, $\lambda_i(\langle m_1, m_2\rangle) = m_i$, $i = 1, 2$; let $\mathcal{L}$ be the filter with the base $\{\lambda_1^{-1}(F_1) \cap \lambda_2^{-1}(F_2) ; F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$. Then $\langle Z, \mathcal{L}\rangle ; \lambda_1^+, \lambda_2^+$ have the required properties.

VIII.2. Lemma: Let $H = G_1 \cup G_2$ be a functor such that $G_1 \simeq Q_{M_1,\mathcal{F}_1}$, $G_2 \simeq Q_{M_2,\mathcal{F}_2}$, $G_1 \cap G_2 \simeq Q_{M,\mathcal{F}}$. Then there exists a monotransformation of H into some $Q_{Z,\mathcal{L}}$. If $\mathcal{F}_1$, $\mathcal{F}_2$ are m-complete, so is $\mathcal{L}$.

Proof: follows easily from the previous lemma and VI.3.

VIII.3. Proposition: Let m be an infinite cardinal. The following properties of H are equivalent:

(i) H preserves separating systems up to m;

(ii) $H \simeq C_{p,q} \vee G$ where $G = \bigcup_{\iota \in \mathcal{J}} G_\iota$, $\mathcal{J}$ is a class and

a) for every $\iota \in \mathcal{J}$, $G_\iota \simeq Q_{M_\iota, \mathcal{F}_\iota}$ where $\mathcal{F}_\iota$ is m-complete;

b) for every $\iota_1, \iota_2 \in \mathcal{J}$, $G_{\iota_1} \cap G_{\iota_2} = \bigcup_{G_\iota \subset G_{\iota_1} \cap G_{\iota_2}} G_\iota$;

c) if $\mathcal{J}' \subset \mathcal{J}$, $\operatorname{card} \mathcal{J}' < m$, $\bigcup_{\iota \in \mathcal{J}'} G_\iota \subset G_{\iota_1} \cap G_{\iota_2}$, then there is a $\iota_3 \in \mathcal{J}$ with $\bigcup_{\iota \in \mathcal{J}'} G_\iota \subset G_{\iota_3} \subset G_{\iota_1} \cap G_{\iota_2}$.

Proof: (i) $\Longrightarrow$ (ii): If H preserves separating systems up to m, then $H \simeq C_{p,q} \vee G$, where G is separating, preserves equalizers and intersections up to m (see IV.4, III.11, III.12). Denote by $\mathcal{J}$ the class of all $\iota = \langle x, X\rangle$ where $x \in G(X)$ and put $G_\iota = G_{\langle x, X\rangle}$. Then $G_\iota \simeq Q_{X, \mathcal{F}_\iota}$ for the m-complete filter $\mathcal{F}_\iota = G^{x,X}$ (see V.2, VII.7, VI.5). One can verify that G_ι have all the required properties.

(ii) $\Longrightarrow$ (i): Let H satisfy the assumptions of (ii). It is sufficient to prove that G preserves separating systems up to $\mathfrak{m}$. Let $\langle X; \{\varphi_\alpha ; \alpha \in A\}$ be a separating system, $\operatorname{card} A < \mathfrak{m}$. Suppose there are $a_1, a_2 \in G(X)$ with $[G(\varphi_\alpha)](a_1) = b_\alpha = [G(\varphi_\alpha)](a_2)$ for all $\alpha \in A$. Choose $\iota_1, \iota_2 \in \mathcal{J}$ such that $a_1 \in G_{\iota_1}(X)$, $a_2 \in G_{\iota_2}(X)$. Then there is a $\iota_3 \in \mathcal{J}$ with $b_\alpha \in G_{\iota_3}(X)$ for all $\alpha \in A$ and $G_{\iota_3} \subset G_{\iota_1} \cap G_{\iota_2}$. Let $K = K_1 \cup K_2$ be a functor and $\nu : K \longrightarrow G_{\iota_1} \cup G_{\iota_2}$ be an epitransformation such that the domain-range-restrictions $\nu^1 : K_1 \longrightarrow G_{\iota_1}$, $\nu^2 : K_2 \longrightarrow G_{\iota_2}$, $\nu^3 : K_1 \cap K_2 \longrightarrow G_{\iota_3}$ are natural equivalences. Put $K_3 = K_1 \cap K_2$, choose $c_i \in K_i(X)$ with $\nu_X(c_i) = a_i$, $i = 1, 2$; choose $d_\alpha \in K_3(X)$ such that $\nu_X(d_\alpha) = b_\alpha$ for all $\alpha \in A$. Since K is embeddable into some $Q_{Z,\mathcal{L}}$ with $\mathcal{L}$ $\mathfrak{m}$-complete (see VIII.2), it preserves separating systems up to $\mathfrak{m}$. Consequently $c_1 = c_2$ because $[K(\varphi_\alpha)](c_1) = d_\alpha = [K(\varphi_\alpha)](c_2)$. Thus, $a_1 = a_2$.

VIII.4. <u>Note:</u> If the class $\mathcal{J}$ from VIII.3 is a set, then, of course, the functor G is small. The problem, whether there is a big functor which preserves separating systems up to $\mathfrak{m}$, is easy under the assumption of an existence of a proper class of measurable cardinals. (We recall that a cardinal $\mathfrak{p} > \aleph_0$ is called measurable if there is a non-trivial $\mathfrak{p}$-complete ultrafilter on the set $\mathfrak{p}$.) Then, take for every cardinal $\mathfrak{p} \geq \mathfrak{m}$ a couple $\langle P, \mathcal{P}\rangle$ where $\mathcal{P}$ is a non-trivial $\mathfrak{p}$-complete ultrafilter on a set P and put $G = \bigcup_{\mathfrak{p} \geq \mathfrak{m}} G_{\mathfrak{p}}$ where $G_{\mathfrak{p}} \simeq Q_{P,\mathcal{P}}$ and for every p, p', $p \neq p'$, the intersection $G_p \cap G_{p'}$ is naturally equivalent to I (i.e. all $Q_{P,\mathcal{P}}$ are glued along the diagonal).

Without any set-theoretical assumption: even an existence of a big equalizer-preserving functor seems to be unknown.

VIII.5. Proposition: Let m be an infinite cardinal. The following properties of H are equivalent:

(i) H preserves products up to m ;

(ii) either $H = C_0$ or $H \simeq C_{0,1}$ or $H \simeq C_1$ or $H = \bigcup_{\iota \in \mathcal{J}} H_\iota$ where $\mathcal{J}$ is a class and

a) for every $\iota \in \mathcal{J}$, $H_\iota \simeq Q_{M_\iota, \mathcal{F}_\iota}$ where $\mathcal{F}_\iota$ is m-complete;

b) if $\mathcal{J}' \subset \mathcal{J}$, $\mathit{card}\ \mathcal{J}' < m$, then there exists $\iota \in \mathcal{J}$ with $\bigcup_{\iota' \in \mathcal{J}'} H_{\iota'} \subset H_\iota$.

Proof: (i) $\Longrightarrow$ (ii): If H preserves products up to m, then either $H = C_0$ or $H \simeq C_{0,1}$ or $H \simeq C_1$ or H is separating, preserves equalizers and intersections up to m (see IV.4, III.11, III.12). Denote by $\mathcal{J}$ the class of all $\iota = \langle x, X \rangle$, $x \in H(X)$ and put $H_\iota = H_{\langle x, X \rangle}$.
(ii) $\Longrightarrow$ (i): Let $H = \bigcup_{\iota \in \mathcal{J}} H_\iota$ satisfy a) b) from (ii). Let $\{X_\alpha ; \alpha \in A\}$ be a collection of sets, $\langle X ; \{\pi_\alpha ; \alpha \in A\}\rangle$ its product, $\mathit{card}\ A < m$. Choose $x_\alpha \in H(X_\alpha)$ for all $\alpha \in A$. Then there exists a $\iota \in \mathcal{J}$ such that $x_\alpha \in G_\iota(X_\alpha)$ for all $\alpha \in A$. Since G_ι preserves products up to m, there exists an $x \in G_\iota(X)$ with $[G_\iota(\pi_\alpha)](x) = x_\alpha$. If $[H(\pi_\alpha)](a) = [H(\pi_\alpha)](b)$ for all $\alpha \in A$ and some $a, b \in H(X)$, one can choose $\iota \in \mathcal{J}$ such that

$a, b \in G_\iota(X)$. Then, necessarily, $a = b$.

R e f e r e n c e s

[1] P. FREYD: Abelian categories, New York,Harper and Row, 1964.

[2] M. KATĚTOV: A theorem on mappings, Comment.Math.Univ. Carolinae 8(1967),431-433.

[3] H. KENYON: Partition of a domain, *Advanced problems* ..., Amer.Math.Monthly 71(1964),219.

[4] V. KOUBEK, J. REITERMAN: On the category of filters, Comment.Math.Univ.Carolinae 11(1970),19-29.

[5] J.M. MARANDA: Some remarks on limits in categories, Canad.Math.Bull.5(1962),133-136.

[6] B. MITCHELL: Theory of Categories, New York,Acad.Press, 1965.

[7] A. PULTR: On selecting of morphisms among all mappings between underlying sets of objects in concrete categories and realisation of these, Comment. Math.Univ.Carolinae 8(1967),53-83.

[8] A. PULTR: Limits of functors and realisations of categories, Comment.Math.Univ.Carolinae 8(1967), 663-682.

[9] V. TRNKOVÁ, P. GORALČÍK: On products in generalized algebraic categories, Comment.Math.Univ.Carolinae 10(1969),49-89.

[10] V. TRNKOVÁ: Some properties of set functors, Comment. Math.Univ.Carolinae 10(1969),323-352.

[11] V. TRNKOVÁ: When the product-preserving functors preserve limits, Comment.Math.Univ.Carolinae 11 (1970),365-378.

[12] O. WYLER: Operational Categories, Proceedings of the Conference on Categorial Algebra, La Jolla 1965,295-316.

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