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Introduction. Universal algebras of a given type $\Delta = \{\varkappa_\lambda \mid \lambda < \beta\}$ (Δ is a family of ordinal numbers indexed by ordinal numbers) form the category $A(\Delta)$ whose objects are the pairs $(X, \{\omega_\lambda^X \mid \lambda < \beta\})$ where X is a set and ω_λ^X are mappings $\omega_\lambda^X : X^{\varkappa_\lambda} \longrightarrow X$ and morphisms from $(X, \{\omega_\lambda^X\})$ to $(Y, \{\omega_\lambda^Y\})$ are mappings $f : X \longrightarrow Y$ such that $\omega_\lambda^Y \circ f^{\varkappa_\lambda} = f \circ \omega_\lambda^X$ for every λ, $\lambda < \beta$, where $f^{\varkappa_\lambda} : X^{\varkappa_\lambda} \longrightarrow Y^{\varkappa_\lambda}$ is f acting coordinate-wise on $\varkappa_\lambda$-tuples from $X^{\varkappa_\lambda}$.

Now, let this device work in a general situation. Given two functors F and G of the covariant variance from sets to sets, we can define the generalized algebraic category as follows: objects are again pairs $(X, \{\omega_\lambda^X\})$ but operations ω_λ^X range over $F(X)$ and take values in $G(X)$ (so they are mappings $\omega_\lambda^X : F(X)^{\varkappa_\lambda} \longrightarrow G(X)$) and morphisms are mappings $f : X \longrightarrow Y$ such that $\omega_\lambda^Y \circ F(f)^{\varkappa_\lambda} = G(f) \circ \omega_\lambda^X$ for every λ, $\lambda < \beta$.

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It is known that $\mathcal{A}(\Delta) = \mathcal{A}(I, I, \Delta)$ always has limits. The general problem of the existence of limits in categories $\mathcal{A}(F, G, \Delta)$ is not so clear. Some results are known ([2],[4]).

The subject of the present paper is the study of equalizers in $\mathcal{A}(F, G, \Delta)$.

The first part of the paper gives some basic definitions and results. In the first paragraph, we prove that the existence of such equalizers in $\mathcal{A}(F, G, 1)$ that the natural forgetful functor $\mathfrak{X}$ preserves them is, roughly speaking, equivalent to the fact that the functor G preserves equalizers. In the second paragraph, we shall give up the requirement for the equalizers to be preserved by the functor $\mathfrak{X}$. The essential part here is whether the functor F preserves unions. The theorems 2.1, 2.2 give the necessary and sufficient condition for the existence of equalizers.

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0. Basic definitions, facts and notation

1. An ordinal number α is the set of all ordinal numbers β, $\beta < \alpha$.

2. All functors throughout this paper will be covariant functors from the category $\mathcal{S}$ of all sets and all their mappings into itself. Natural equivalence of functors will be denoted by $\simeq$.

3. The identical functor will be denoted by I .

4. Let P, M be sets, $p : P \longrightarrow M$ a mapping. Then $C_{P,p,M}$ is the functor F given by formulas $F(\emptyset) = P$ and if $X \neq \emptyset$, then $F(\vartheta_X) = p$, $\vartheta_X : \emptyset \to X$, $F(X) = M$, $F(f) = id_M$ whenever $f : X \to Y$, id_M is the identical mapping. If $P \subset M$ and p is the inclusion, we write simply $C_{P,M}$.

5. Q_M denotes a hom-functor from the set M, i.e. $Q_M(X) = \mathcal{H}om(M, X)$.

6. The current set-theoretic notation, e.g. $(\subset, \cup, \cap, \times, \vee, \circ)$ will be used for functors, too. So, if two functors F_1, F_2 are given, then $F_1 \cup F_2$ denotes the functor F (provided that it exists) such that $F(X) = F_1(X) \cup F_2(X)$ for every set X and F_1, F_2 are the subfunctors of F. The functors $F_1 \times F_2, F_1 \vee F_2$ always exist.

7. We shall write $F(X)_Y = [F(i)]F(X)$, where F is a functor, $X \subset Y$ and $i : X \longrightarrow Y$ is the inclusion.

8. Recall that a functor F preserves union if, whenever Y is a set and $\{Y_\alpha, \alpha \in J\}$ a collection of its subsets, then $F(\bigcup_{\alpha \in J} Y_\alpha)_Y = \bigcup_{\alpha \in J} F(Y_\alpha)_Y$.

9. A functor F preserves unions if and only if $F \simeq (I \times C_{P,p,M}) \vee C_{H,h,K}$ (see [5]).

10. An equalizer for two morphisms is defined as usual ([7]). The definition of a category having equalizers is evident. The definition of an equalizers-preserving functor and a

non-void equalizers-preserving functor is obvious, too.

1. Equalizers in the category $A(F, G, 1)$ such that the natural forgetful functor $\mathfrak{X}$ preserves them

We denote $\mathfrak{X}$ the forgetful functor from the category $A(F, G, \Delta)$ into the category S of all sets and their mappings, i.e. if $f: (Y, \{\omega_\lambda^Y\}) \longrightarrow (X, \{\omega_\lambda^X\})$ is a morphism of $A(F, G, \Delta)$, then

$\mathfrak{X}(Y, \{\omega_\lambda^Y\}) = Y$,

$\mathfrak{X}(f) = f$.

Lemma 1.1. Let the functor G preserve equalizers. Then for every functor F the category $A(F, G, 1)$ has equalizers and $\mathfrak{X}$ preserves them.

Lemma 1.2. If $F(\emptyset) = \emptyset$ and G preserves non-void equalizers, then $A(F, G, 1)$ has equalizers and $\mathfrak{X}$ preserves them.

Lemma 1.3. If G does not preserve non-void equalizers, then $A(F, G, 1)$ has not equalizers such that $\mathfrak{X}$ preserves them.

Lemma 1.4. If G does not preserve equalizers and $F(\emptyset) \neq \emptyset$, then $A(F, G, 1)$ has not equalizers such that $\mathfrak{X}$ preserves them.

Proofs of these lemmas are easy.

Theorem 1.1. Let $F(\emptyset) = \emptyset$. Then the category $A(F, G, 1)$ has equalizers such that $\mathfrak{X}$ preserves them if and only if G preserves non-void equalizers.

Theorem 1.2. Let $F(\emptyset) \neq \emptyset$. Then the category $A(F, G, 1)$ has equalizers such that $\mathfrak{Z}$ preserves them if and only if G preserves equalizers.

Proofs are evident.

2. Equalizers in the category $A(F, G, \Delta)$

Lemma 2.1. If G does not preserve equalizers and $F(\emptyset) \neq \emptyset$, then $A(F, G, 1)$ has not equalizers.

Proof is evident.

Lemma 2.2. If $F(\emptyset) = \emptyset$ and F preserves unions, then $A(F, G, 1)$ has equalizers.

Proof. Let $f, g : (X, \omega) \longrightarrow (X', \omega')$ be morphisms, $i : Z \rightarrow X$ an equalizer of mappings f, g . Let $\mathfrak{S}$ be the system of all $Y \subset Z$ such that

$(\forall x \in F(Y)_Z)\ (\exists y \in G(Y)_Z)[\omega \circ F(i)(x) = G(i)(y)]$,

put $S = \cup \mathfrak{S}$. One can see that $S \in \mathfrak{S}$ and it is easy to define $\sigma : F(S) \longrightarrow G(S)$ such that (S, σ) is a domain of an equalizer of f, g in $A(F, G, 1)$.

Statement 2.1. Let F, G be two functors, F do not preserve unions, G do not preserve non-void equalizers. Then there exist $f, g : Y \longrightarrow Y'$ such that $G(i) \neq$ $\neq eq(G(f), G(g))$, where $i : T \longrightarrow Y$ is an equalizer of f, g and $F(T) - \bigcup_{t \in T} F\{t\}_T \neq \emptyset$.

Proof. Take a set M such that $F(M) - \bigcup_{m \in M} F\{m\}_M \neq \emptyset$ and $\tilde{f}, \tilde{g} : X \longrightarrow X'$ such that $G(\tilde{i}) \neq eq(G(\tilde{f}), G(\tilde{g}))$, where $\tilde{i} = eq(\tilde{f}, \tilde{g})$, $\tilde{i} : Z \longrightarrow X$, $Z \neq \emptyset$

Put $Y = X \vee M$, $Y' = X' \vee M$, $f, g : Y \longrightarrow Y'$ such that $f(x) = \tilde{f}(x)$, $g(x) = \tilde{g}(x)$ for $x \in X$, $f(m) = g(m) = m$ for $m \in M$. It is easy to see that f, g have the required properties.

Lemma 2.3. If F does not preserve unions and G does not preserve non-void equalizers, then the category $A(F, G, 1)$ has not equalizers.

Proof. Let f, g have the properties from Statement 2.1 with respect to F, G. We can choose $\bar{y} \in G(Y) - G(T)_Y$, where $i : T \to Y$, $i = eq(f, g)$, $G(f)(\bar{y}) = G(g)(\bar{y})$. Put $y_i = G(\lambda_i)(\bar{y})$, where $\lambda_i : Y \longrightarrow Y \times \{1, 2\}$ is the mapping $\lambda_i(y) = (y, i)$, $i = 1, 2$. Choose $\bar{z} \in F(T)_Y - \bigcup_{t \in T} F\{t\}_Y$. Put $z = F(\lambda_1)(\bar{z})$, $Y'' = Y \times \{1, 2\}$. Define $\hat{f}, \hat{g} : Y'' \to Y'$ as follows:

$\hat{f}(y, 1) = f(y) = \hat{f}(y, 2)$, $\hat{g}(y, 1) = g(y)$, $\hat{g}(y, 2) = f(y)$.

Now, if we define ω'', ω' as follows: $\omega''(y) = y_2$ for all $y \neq z$, $y \in F(Y'')$, $\omega''(z) = y_1$, $\omega'(x) = G(\hat{f})(y_1)$ for all $x \in F(Y')$, so $\hat{f}, \hat{g} : (Y'', \omega'') \longrightarrow (Y', \omega')$ are morphisms of the category $A(F, G, 1)$ and one can see that they have no equalizer.

Theorem 2.1. Let $F(\emptyset) = \emptyset$. Then the category $A(F, G, 1)$ has equalizers if and only if either G preserves non-void equalizers or $F \simeq (I \times C_{P,n,M}) \vee C_{\emptyset,K}$.

Theorem 2.2. Let $F(\emptyset) \neq \emptyset$. Then the category $A(F, G, 1)$ has equalizers if and only if G preserves equalizers.

Proofs are evident.

<u>Statement 2.2</u>. The categories $\mathcal{A}(F, G, \Delta)$ and $\mathcal{A}(\bigvee_{\varkappa_\lambda \in \Delta} Q_{\varkappa_\lambda} \circ F, G, 1)$ are isomorphic.

Proof is evident.

In this way we can "translate" our results into the general case $\mathcal{A}(F, G, \Delta)$.

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