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On projective limits of probability spaces

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Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
The aim of this paper is to correct some results in the interesting paper of V.M. Rao [3].

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1. Pure Probabilities

1.1. Definition (see [3], 4.1). Let \( P : \mathcal{A} \to [0,1] \) be a finitely additive set function on an algebra \( \mathcal{A} \subset \mathcal{P} X \).

A ring \( \mathcal{R} \subset \mathcal{A} \) is called \( P \)-pure if

(i) \( \lambda_n \in \mathcal{R} \) for \( n \in \mathbb{N} \) (\( \mathbb{N} \) is the set of all non-negative integers), \( \lambda_n \searrow \emptyset \) imply \( P[\lambda_{n_0}] = 0 \) for some \( n_0 \),

(ii) \( P[A] = \inf \{ \sum_{n \in \mathbb{N}} P[A_n] | A_n \in \mathcal{R} \} \) and \( \bigcup_{n \in \mathbb{N}} A_n = A \).

for each \( A \in \mathcal{A} \).

If there exists a \( P \)-pure ring then \( P \) is said to be pure.

Remark. Any pure \( P \) is \( \sigma \)-additive ([3], 4.2) but the converse is not true as it will be shown below (beforehand, David Preiss constructed another counter-example).

1.2. Lemma (cf.[2], 7(ii)). Let \( P : \mathcal{A} \to [0,1] \) be a non-atomic probability, let \( \mathcal{R} \subset \mathcal{A} \) be a \( P \)-pure ring, \( E \in \mathcal{R} \),
There exist \( E_1, E_2 \in \mathcal{R} \) such that
\[ E_1 \cup E_2 \subseteq E, \ E_1 \cap E_2 = \emptyset \quad \text{and} \quad \frac{1}{4} P[E_1] + P[E_2] > 0 \]
for \( i = 1, 2 \).

**Proof.** As \( P \) is non-atomic there are \( A_1, A_2 \in \mathcal{R} \) such that
\[ A_1 \cup A_2 \subseteq E, \ A_1 \cap A_2 = \emptyset, \ P[A_1] = P[A_2] = \frac{1}{2} P[E]. \]
Then exist \( B_1^j \in \mathcal{R} \ (j = 1, 2; \ j \in \mathbb{N}) \) such that
\[ \bigcup_{j \in \mathbb{N}} B_1^j = A_1 \]
and \( P\left[ \bigcup_{j \in \mathbb{N}} B_1^j \right] < \frac{1}{4} P[E] \)
for \( j = 1, 2 \). Obviously
\[ P[B_2^j \setminus E] > 0 \]
for some \( j \in \mathbb{N} \). As \( \bigcup_{j \in \mathbb{N}} (B_1^j \setminus E) \subseteq B_2^j \)
\[ A_1 \setminus \bigcup_{j \in \mathbb{N}} B_2^j = A_1 \setminus \bigcup_{j \in \mathbb{N}} B_1^j \setminus A_2 \]
and \( P[A_1] = \frac{1}{2} P[E], \)
\[ P\left[ \bigcup_{j \in \mathbb{N}} B_2^j \setminus A_2 \right] < \frac{1}{4} P[E] \]
one has \( P\left[ \bigcup_{j \in \mathbb{N}} (B_2^j \setminus E) \right] = 0 \).

Hence \( P\left[ \bigcup_{j \in \mathbb{N}} (B_2^j \setminus E) \right] = 0 \)
for some \( j \in \mathbb{N} \). The sets
\[ E_1 = (B_1^j \setminus E) \setminus B_2^j \]
and \( E_2 = B_2^j \setminus E \)
have the required properties.

### Proposition (cf. [2], 7(iii)).
Let \( P : \mathcal{A} \rightarrow [0, 1] \)
be a non-atomic probability (on a \( \sigma \)-algebra \( \mathcal{R} \)) and let
\( \mathcal{R} \subset \mathcal{A} \)
be a \( P \)-pure ring, \( E \in \mathcal{R}, \ P[E] > 0 \).
Then there exists \( A \in \mathcal{A} \)
such that \( A \subseteq E \), and \( A = \text{sup}_0 \).

**Proof.** Will be only sketched here (it is essentially
the same as the proof of 7(iii) in [2]): by means of Lemma
1.2 one can (inductively) construct the sets
\[ E(a_0, a_1, \ldots, a_m), \ m \in \mathbb{N}, \ a_i = 0,1 \text{ for } i = 0,1, \ldots, m, \]
such that \( P[E(a_0, a_1, \ldots, a_m)] > 0, \)
\[ E(a_0, a_1, \ldots, a_m, 0) \cap E(a_0, a_1, \ldots, a_m, 1) = \emptyset, \]
\[ E = E(a_0, a_1, \ldots, a_m) \cap E(a_0, a_1, \ldots, a_m + 1), \]

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and put $A = \bigcap_{n=1}^{m} E_{n}$ where $E_{n} = \bigcup \{ E(a_{0}, a_{1}, \ldots, a_{m}) | a_{i} = 0,1 \}$ for $0 \leq i \leq m$.

Remarks. Sierpiński proved (supposing continuum-hypothesis) that there exists a non-atomic probability space all null-sets of which are at most countable (see e.g. [4]); such a probability is not pure due to 1.3 (cf. [2], 7(iv)). The properties of pure probabilities are very similar to those of compact ones (for definition of compact measure see [2]), e.g. indirect product of pure probabilities is pure. It is even pretty possible that these two notions (compact, pure) are not really distinct; this is the case for countably-generated (in the sense of Carathéodory) probabilities; the proofs will soon be published.

2. Projective Limits

M.M. Rao gave conditions for $\sigma$-additivity of projective limits in terms of extensions of given probabilities ([3], 4.5 - 4.7). However, some of them are not correctly formulated (see 2.3).

2.0. Notations. Below, $D$ is a set directed by the relation $\leq$ (i.e. $R \circ R = R$, $R \cap R^{-1} =$ diagonal, $R \circ R^{-1} = D \times D$ where $R \subseteq D \times D$ realizes $\leq$), $\{ F_{\alpha} | \alpha \in D \}$ is a family of $\sigma$-algebras $\exp X$ such that $F_{\alpha} \subseteq F_{\beta}$ for $\alpha \leq \beta$; $\mathcal{F} = \bigcup_{\alpha \in D} F_{\alpha}$, $\sigma \mathcal{F}$ is the $\sigma$-algebra generated by $\mathcal{F}$. Given probabilities $P_{\alpha} : F_{\alpha} \rightarrow [0,1]$ for $\alpha \in D$ such that $P_{\alpha}[E] = P_{\beta}[E]$ for $E \in F_{\alpha} \cap F_{\beta}$, $P : \mathcal{F} \rightarrow [0,1]$ is the
finitely additive set function such that $P(\emptyset) = P_\alpha(\emptyset)$ for $\emptyset \in \mathcal{F}_\alpha$.

2.1. Proposition (see 2.0). The following conditions are equivalent:

(i) $P$ is $\sigma$-additive;

(ii) for any $\alpha \in D$ there exists a probability $P_\alpha: \sigma \mathcal{F} \rightarrow [0,1]$ that extends $P_\alpha$ and for every such extensions the following statement holds:

for every $A_m \in \mathcal{F} (m \in N)$, $A_m \not\in \mathcal{F}$ and $\varepsilon > 0$ there are $\alpha_0 \in D$, $m_0 \in N$ such that $P_\alpha[A_m] < \varepsilon$ for $\alpha \geq \alpha_0$, $m \geq m_0$ ($= $ mapping $\langle \alpha, m \rangle \mapsto P_\alpha[A_m]$ is continuous on $D \times N$);

(iii) for any $\alpha \in D$ there exists a probability $P_\alpha: \sigma \mathcal{F} \rightarrow [0,1]$ that extends $P_\alpha$ and $\lim_{n \to \infty} \lim_{m \to \infty} P_\alpha[A_m] = 0$ for every $A_m \in \mathcal{F}$ $(m \in N)$ with $A_m \not\in \mathcal{F}$ ($= $ mapping $\langle m \rightarrow P_\alpha[A_m] \rangle$ is continuous on $N$ uniformly for all $\alpha \in D$).

Proof. Implications (ii) $\implies$ (i) and (iii) $\implies$ (i) are immediate. (i) $\implies$ (ii) and (i) $\implies$ (iii): to show the existence of the required extensions one can use for $\overline{P}_\alpha$ the (unique) extension of $P$ on $\sigma \mathcal{F}$. If $\overline{P}_\alpha$'s are arbitrary extensions of $P_\infty$'s and $A_m \in \mathcal{F}$, $A_m \not\in \mathcal{F}$ then $P[A_m] < \varepsilon$ for some $m_0$ and $A_m \not\in \mathcal{F}_{\alpha_0}$ for some $\alpha_0$.

Hence $P_\alpha[A_m] = P[A_m] = P[A_m] < \varepsilon$ for $\alpha \geq \alpha_0$ and $P_\alpha[A_m] \leq P[A_m]$ for $m \geq m_0$.

Remark. The condition in 2.1(iii) can be reformulated like this:
\{F_\alpha \mid \alpha \in \mathbb{D}\} \subset ca(X, \sigma \mathcal{F}) \) is weakly sequentially compact (see [11, IV.9.1]) or like this:

\( F_\alpha \)'s are uniformly \( \lambda \)-continuous for some \( \lambda \in ca(X, \sigma \mathcal{F}) \) (see [11, IV.9.2]). But these conditions need not hold for every family \( \{F_\alpha\} \) of extensions (see 2.3).

2.2. Proposition (see 2.0). Let \( \mathcal{D} = \mathbb{N} \) (\( \mathbb{N} \) naturally ordered). The following conditions are equivalent:

(i) \( \mathcal{P} \) is \( \sigma \)-additive;

(iv) for any \( k \in \mathbb{N} \) there exists a probability \( F_k : \sigma \mathcal{F} \rightarrow [0,1] \) that extends \( F_k \) and for every such extensions and for every \( A_n \in \mathcal{F} \) \( (m \in \mathbb{N}) \), \( A_n \not\subseteq \emptyset \) it holds

\[
\lim_{n \to \infty} \left( \sup_{k \in \mathbb{N}} F_k[A_n] \right) = 0 \quad (= \text{mapping } m \mapsto F_k[A_n] \text{ is continuous on } \mathbb{N} \text{ uniformly for all } k \in \mathbb{N} \text{).}
\]

(v) for any \( k \in \mathbb{N} \) there exists a probability \( F_k : \sigma \mathcal{F} \rightarrow [0,1] \) that extends \( F_k \) and such that \( \lim_{k \to \infty} F_k[A] \) exists for any \( A \in \sigma \mathcal{F} \) (\( (= \text{mapping } k \mapsto F_k[A] \text{ is continuous on } \mathbb{N} \text{ for any } A ) \).

Proof. Implication (iv) \( \implies \) (i) is clear, implication (v) \( \implies \) (i) is the theorem of Nikodym (see [11, III.7.4]). (i) \( \implies \) (iv) and (i) \( \implies \) (v): the existence of extensions \( F_k \) is obvious as in the proof of 2.1.

Let \( F_k \)'s be arbitrary extensions of \( F \)'s, \( A_n \in \mathcal{F}, A_n \not\subseteq \emptyset, \varepsilon > 0 \). For some \( m_1 \) it holds \( P[A_{m_1}] < \varepsilon \), for some \( m_1 \) it holds \( A_{m_1} \in \mathcal{G}_{m_1} \), hence \( F_k[A_{m_1}] = P_k[A_{m_1}] = P[A_{m_1}] < \varepsilon \) for \( k \geq m_1 \). For \( l = 0, 1, \ldots, m_1 - 1 \)
there are $e_i$ such that $P_i(A_{e_i}) < \varepsilon$; put $n_o = \max\{n_1, n_2, n_3, \ldots, n_{m-1}\}$; then $P_0(A_{n_o}) < \varepsilon$ for any $n \in \mathbb{N}$.

2.3. **Examples.** (a) The condition in 4.5 of [3] does not necessarily hold for arbitrary extensions $P_\alpha :$ Lebesgue probability on $[0,1]$ is the projective limit of all its restrictions to finite subalgebras and any such restriction can be extended as convex combination of Dirac measures. The family $\{P_\alpha\}$ containing all these extensions works very wildly and does not satisfy any expected condition.

(b) This example shows (for $D = \mathbb{N}$) that a family $\{P_\alpha\}$ of extensions need not be terminally uniformly $\lambda$-continuous for any finite measure $\lambda$ on $\mathcal{F}$:

For $n \in \mathbb{N}$, $\mathcal{A}_n \subset \exp [0,1]$ is the algebra of all the finite unions of intervals with end-points $\frac{k}{2^n}$, $n = 0, 1, \ldots, 2^n$, $P_\alpha$ is the restriction of the Lebesgue probability on $[0,1]$ to $\mathcal{A}_n$, $P_\alpha = \frac{1}{2^n} \sum_{k=1}^{2^n} \delta_{\alpha(x, \beta)}$ where $x(\alpha, \beta) = \frac{p-1}{2^{n+1}}$ and $\delta_\alpha$ is the Dirac measure supported by $\alpha$.

**References**


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